Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 7

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

The vertical epicycle frequency is defined by:

$$\nu^2(R) = \frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z}\right)_{(R,z=0)} \tag{1}$$

and the circular frequency is:

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)_{(R,z=0)} \tag{2}$$

where $\Phi(R, z)$ is an axi-symmetric potential. A spherical potential is a subclass of axi-symmetric potentials and may then be written as:

$$\Phi(r) = \Phi(R, z) = \Phi(R^2 + z^2). \tag{3}$$

Thus, for a spherical potential, derivatives with respect to R and z writes:

$$\frac{\partial \Phi}{\partial z} = 2\frac{\partial \Phi}{\partial r}z \tag{4}$$

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\frac{\partial^2 \Phi}{\partial r^2}z^2 + 2\frac{\partial \Phi}{\partial r} \tag{5}$$

$$\frac{\partial \Phi}{\partial R} = 2\frac{\partial \Phi}{\partial r}R. \tag{6}$$

Thus, with partial derivatives computed in R and z=0, we have:

$$\nu^2(R) = 2\frac{\partial \Phi}{\partial r} \tag{7}$$

and

$$\Omega^2(R) = 2\frac{\partial \Phi}{\partial r}. \tag{8}$$

Problem 2:

Stating that the azimuthal angle $\Delta\phi$ between successive pericenters lies in the range $\pi \leq \Delta\phi \leq 2\pi$ is equivalent to state that the radial epicycle frenquency κ is in the range $\Omega \leq \kappa \leq 2\Omega$, where Ω is the circular frequency.

We consider the two possible extreme cases of spherical mass distribution in which the density decreases outwards. (ii) a constant density, (i) a mass point.

The radial dependency of the circular velocity for (i), i.e., a Keplerian orbit is:

$$v_c \sim r^{-1/2} \tag{9}$$

and thus:

$$\Omega \sim r^{-3/2} \qquad \text{and thus} \qquad \Omega^2 \sim r^{-1/2}. \tag{10}$$

The gradient of Ω^2 is thus:

$$\frac{\partial\left(\Omega^2\right)}{\partial r} \sim 2\Omega\frac{\partial \Omega}{\partial r} \sim -3\,\frac{\Omega^2}{r} \tag{11}$$

Using:

$$\kappa^2 = r\frac{\partial\left(\Omega^2\right)}{\partial r} + 4\Omega^2, \tag{12}$$

we obtain:

$$\kappa = \Omega, \tag{13}$$

The radial dependency of the circular velocity for (ii) is:

$$v_c \sim r \qquad \text{and thus} \qquad \Omega = cte. \tag{14}$$

Using:

$$\kappa^2 = r\frac{\partial\left(\Omega^2\right)}{\partial r} + 4\Omega^2, \tag{15}$$

we obtain:

$$\kappa = 2\Omega, \tag{16}$$

As (i) and (ii) encompass any other spherical mass distribution in which the density is decreasing outwards, we reach the conclusion that $\Omega \leq \kappa \leq 2\Omega$ and thus $\pi \leq \Delta\phi \leq 2\pi$.

<u>Problem 3</u>:

The specific angular momentum of a circular orbit being decreasing outside write:

$$\frac{\partial\left(L_z\right)}{\partial R} < 0 \qquad \text{but thus also} \qquad \frac{\partial\left(L_z^2\right)}{\partial R} < 0. \tag{17}$$

As $L_z = V_c\, R$, we have:

$$\begin{aligned}
\frac{\partial\left(L_z^2\right)}{\partial R} &< 0 & (18)\\
\frac{\partial\left(V_c^2\, R^2\right)}{\partial R} &< 0 & (19)\\
2\, R\, V_c^2 + R^2\frac{\partial\left(V_c^2\right)}{\partial R} &< 0 & (20)\\
2\,\frac{V_c^2}{R^2} + \frac{1}{R}\frac{\partial\left(V_c^2\right)}{\partial R} &< 0 & (21)\\
\kappa^2 &< 0. & (22)
\end{aligned}$$

The latter inequalities is true only for a radial epicycle frequency being complex. As the radial motion obey the harmonic equation:

$$\ddot{x} = -\kappa^2 x, \tag{23}$$

this lead to a general solution of the form:

$$x(t) = A\, e^{\lambda t} + B\, e^{-\lambda t}, \tag{24}$$

where we have defined $\kappa = i\lambda$, with λ a real positive number. If we request that $x(t = -\infty) = 0$, i.e., initially the orbit coincide with the circular orbit, B must be 0. We are left with an exponential solution which means that the orbit will exponentially deviates from the circular orbit, thus being unstable.

<u>Problem 4</u>:

The radial component of the equations of motion of an orbit in a spherical potential is:

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{\partial}{\partial r}\Phi(r). \tag{25}$$

Thus,

$$r\frac{\partial}{\partial r}\Phi(r) = -r\ddot{r} + -r^2\,\dot{\theta}^2. \tag{26}$$

The time average of the latter equation is:

$$\frac{1}{T}\int_0^T dt\, r\frac{\partial}{\partial r}\Phi(r) = -\frac{1}{T}\int_0^T dt\, r\ddot{r} + \frac{1}{T}\int_0^T dt\, r^2\,\dot{\theta}^2. \tag{27}$$

We can integrate by part the first term of the right hand side:

$$\begin{aligned} \frac{1}{T}\int_0^T dt\, r\frac{\partial}{\partial r}\Phi(r) &= -\frac{1}{T}\left(r\dot{r}\Big|_0^T - \int_0^T dt\, \dot{r}^2 \right) + \frac{1}{T}\int_0^T dt\, r^2\,\dot{\theta}^2 & (28) \\ &= -\frac{1}{T}\left(r\dot{r}\Big|_0^T \right) + \int_0^T dt\, \left(\dot{r}^2 + r^2\,\dot{\theta}^2\right). & (29) \end{aligned}$$

If we decide to average over a large number of radial period and set $t = 0$ at the pericenter or apocenter, as at those points, $\dot{r} = 0$:

$$r\dot{r}\Big|_0^T = 0. \tag{30}$$

Moreover, the integrant of the last right hand side therm is the square of the velocity:

$$\dot{r}^2 + r^2\,\dot{\theta}^2 = v^2. \tag{31}$$

We thus reach the conclusion that:

$$\left\langle r\frac{\partial}{\partial r}\Phi(r) \right\rangle = \langle v^2 \rangle. \tag{32}$$

Problem 5:

The definition of the Oort constants are:

$$A = -\frac{1}{2}R\frac{\partial\Omega}{\partial R} \qquad \text{and} \qquad B = -\left(\Omega + \frac{1}{2}R\frac{\partial\Omega}{\partial R}\right). \tag{33}$$

Thus:

$$A^2 = -\frac{1}{4}R^2\left(\frac{\partial\Omega}{\partial R}\right)^2 \qquad \text{and} \qquad B^2 = \Omega^2 + \frac{1}{4}R^2\left(\frac{\partial\Omega}{\partial R}\right)^2 + \Omega R\left(\frac{\partial\Omega}{\partial R}\right), \tag{34}$$

which leads to:

$$2\left(A^2 - B^2\right) = -2\Omega^2 - 2\Omega R\left(\frac{\partial\Omega}{\partial R}\right). \tag{35}$$

The Poisson equation for an axi-symmetric potential $\Phi(R,z)$ writes:

$$\nabla^2\Phi(R,z) = 4\pi\rho(R,z), \tag{36}$$

where $\rho(R,z)$ is the corresponding density.

Using cylindrical coordinates, this gives:

$$\nabla^2\Phi(R,z) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2}, \tag{37}$$

and thus:

$$\frac{\partial^2\Phi}{\partial z^2} = 4\pi\rho(R,z) - \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right), \tag{38}$$

and in particular:

$$\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} = 4\pi\rho_0 - \frac{1}{R}\frac{\partial}{\partial R}\left.\left(R\frac{\partial\Phi}{\partial R}\right)\right|_{z=0}, \tag{39}$$

Using the circular frequency:

$$\Omega = R\left.\frac{\partial\Phi}{\partial R}\right|_{z=0}, \tag{40}$$

the previous equation writes:

$$\begin{aligned}
\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} &= 4\pi\rho_0 - \frac{1}{R}\frac{\partial}{\partial R}\left(\Omega^2R^2\right) && (41)\\
&= 4\pi\rho_0 - 2\Omega^2 - 2\Omega R\left(\frac{\partial\Omega}{\partial R}\right) && (42)\\
&= 4\pi\rho_0 + 2\left(A^2 - B^2\right). && (43)
\end{aligned}$$