ENERGETIC COST OF QUANTUM COMPUTING FOR SUPERCONDUCTING NISQ DEVICES.

The possible advantage of quantum computers is always discussed with respect to the time complexity of algorithms. However there is also another interesting dimension to possible quantum advantage worth studying, namely that of energetic cost and power consumption. A holistic approach encompassing both algorithmic design as well as hardware aspects, is rarely discussed, and has been advocated by Alexia Auffèves in ref PRX Quantum 3, 020101 (2022); and has been dubbed the "Quantum energy initiative".

Here we consider the particular devices based on superconducting qubit platforms and follow the recent study PRX Quantum 4, 040319 (2022) "Optimizing Resource Efficiencies for scalable full-Stack Quantum Computers". We will refer to this paper as [PRX Q4 2022] in these notes.

We note here that the question of energy consumption was mentioned in the recent experiment on quantum supremacy by Google (on sampling from random circuits) which claimed a 10^5 order between a quantum processor with 50 qubit and classical one performing the same task. While we are not able to assess such claims here, the study of [PRX Q4 2022] is an important step forward.

In this lecture we follow the plan:

I) Power consumption for single qubit gates at chip level & low (milliKelvin) temperatures.

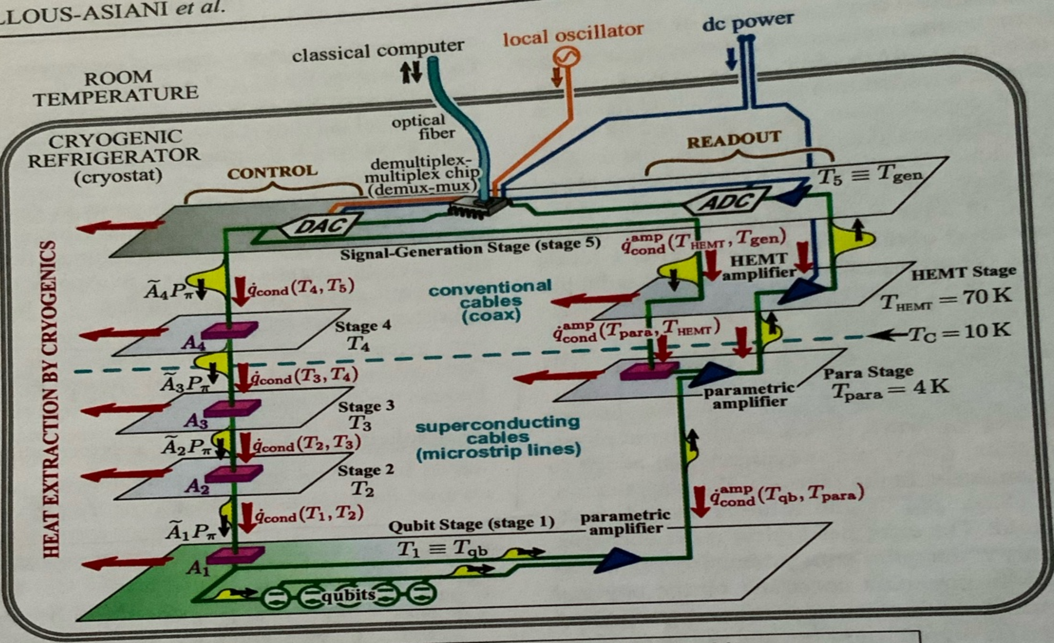
II) Power consumption at macroscopic level at room temperature (elementary simplified model)

III) Extension to noisy computations with circuits implemented on NISQ devices.

Our discussion makes use of a very simplified setting of the whole stack of electronics between room temperature & chip level low temperature. Moreover it does not take into account error correction that would be used in a future (possibly?) fault tolerant quantum computer. These two aspects are briefly touched upon here in the conclusion. We refer to [PRX QC 2022] for more details.

Maybe before starting with I), II), III and further more complicated questions it is a good idea to have a global view of the engineering system we are dealing with.

PRX QUANTUM 4, 040319 (2023)



Component	Ratio to qubits
optical fiber + demux-mux	1 for 270 qubits
local oscillator	1 wire (mm ⌀) for 10^6 qubits
dc (demux-mux+DAC+ADC)	1 wire (mm ⌀) for 10^6 qubits
dc (HEMT amplifiers)	1 wire (mm ⌀) for 10^8 qubits
Qubit control (1 & 2qb gates)	1 DAC+cable (with attenuators) for 25 qubits
Qubit readout (measurement)	1 cable+paramps+HEMT amp+ADC for 100 qubits

FIG. 9. A sketch of our model of the multistage cryogenics with all components. It is important to maximize the number of physical qubits per other component (using multiplexing etc.), and the table gives reasonable values for the ratio of the number of components to the number of physical qubits. The qubit control lines are particularly crucial to the energy consumption; we model them with four stages of attenuation (attenuators in purple), with conventional coaxial cables down 10 K and superconducting microstrip lines below that. The readout is less crucial to the energy consumption if one uses superconducting parametric amplifiers (paramps) at T_{qb} and $T_{amp} = 4$ K, with a third amplification stage using high-electron-mobility transistors (HEMTs) at $T_{HEMT} = 70$ K. The black arrows indicate the flow of information and/or signals, while the red arrows indicate heat conduction. The demux-mux, digital-to-analog converters (DACs), analog-to-digital converters (ADCs), attenuators, and amplifiers all also generate heat.

heat conduction, we assume all wiring to be superconducting below 10 K and thus to conduct vastly less heat than normal-metal wires. The heat-conduction properties of these control lines are given in Appendix B 2. As above, we assume that the cryogenics have Carnot efficiency and thus use the minimal possible power to evacuate heat as

2. Control electronics

A second improvement to bring us closer to experimental reality is that we now add the circuitry to read out the qubits (see the right-hand side of Fig. 9). The signal from the qubits has to be amplified significantly above the thermal noise level at the temperature stage

We will discuss in detail only the highly simplified model without the many attenuation & amplification stages for power.

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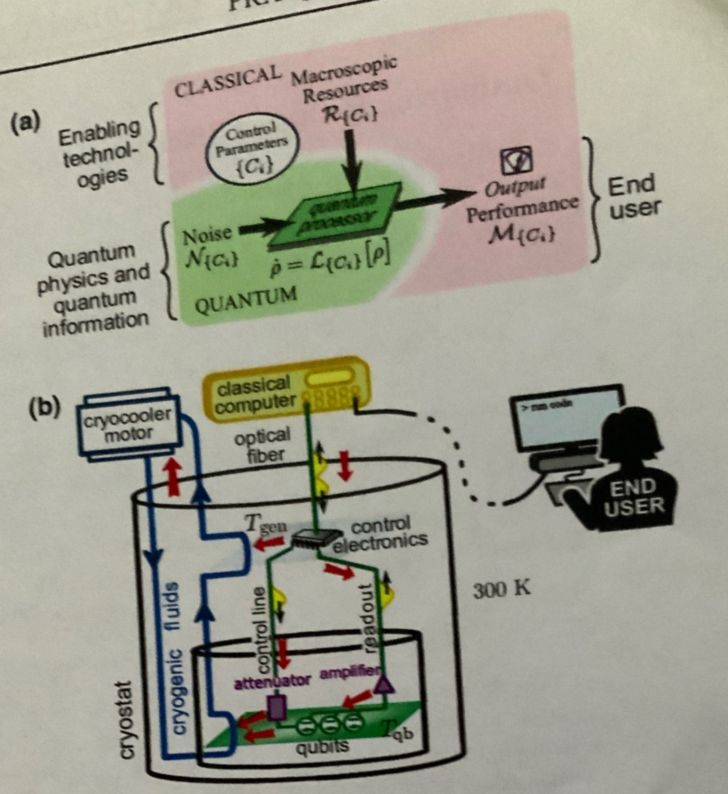
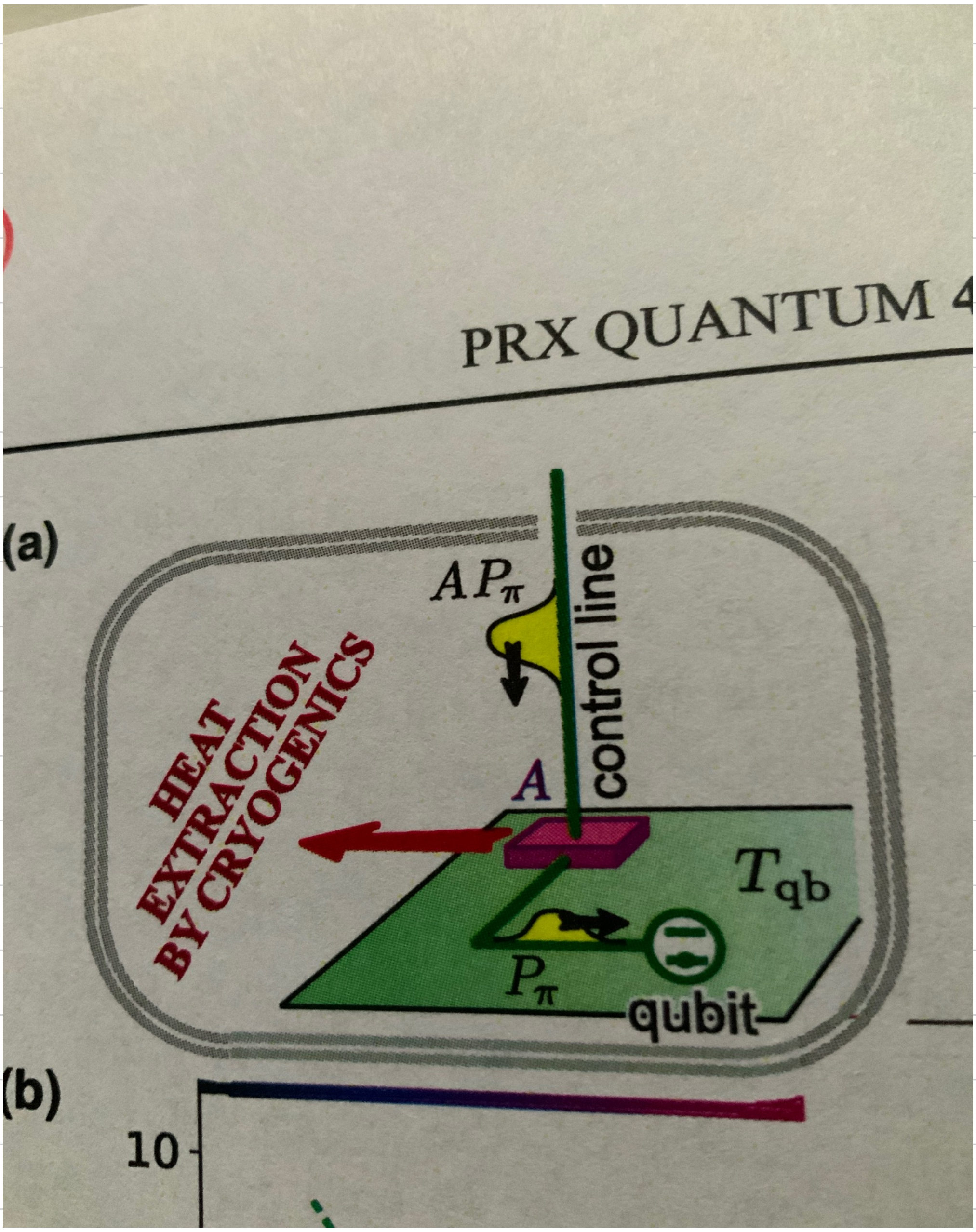


FIG. 1. (a) A schematic of our metric noise resource (MNR) methodology, which models the battle of control against noise. The control parameters affect the performance metric, the noise, and the resource consumption. Such parameters include the qubit temperature, the amount of error correction, etc. The performance metric can be improved by using resources to reduce the noise (e.g., cooling the qubits) or spending resources to make the metric less sensitive to existing noise (e.g., better error correction). (b) A simplified sketch of the physical elements in a typical full-stack superconducting quantum computer, with qubits at temperature T_{qb} and classical control electronics at temperature T_{gen}. The classical computer, at room temperature, compiles the user-specified algorithm and code into a sequence of physical gate operations, interprets detected errors in real time, and can modify the gate sequence to correct them. The black arrows indicate information flows. The red arrows indicate heat flows that bring noise that can cause gate errors (heat conducted by wiring, heat generated by attenuators and amplifiers, etc.); more details are shown in Fig. 9. We model the full stack by considering each physical element in (b) in terms of its effect on the metric, noise, and resource in (a).

This is further simplified in sections I) & II) to a discussion for a single qubit:



We assume a single NOT gate operates on a superconducting qubit at low temperature T_{qb} and consumes power P_π (π = "pi-pulse"). The power coming from higher temperature (say room temperature in a simplified model) is AP_π with $A \gg 1$. It is attenuated by an attenuator A which dissipates heat. The heat has to be extracted which is the main source of power cost for this simple setting.

I) Power consumption at chip level for single qubit level at low temperature.

In superconducting qubit platforms qubits are small LC circuits with a non-linear inductance L made of Josephson junction. In the low temperature quantum regime the LC circuit behaves like a quantum level system with two states $|0\rangle = |g\rangle = |\text{ground state}\rangle$ and $|1\rangle = |e\rangle = |\text{exited}\rangle$, with Hamiltonian

$$H_{qb} = -\frac{\hbar\omega_0}{2}\sigma_z \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\hbar\omega_0 = E_e - E_g$ = energy diff of $|e\rangle$ & $|g\rangle$.

The qubit is connected to microwave transmission lines for control and measurement.

The coupling between qubit and transmission line has two effects:

- driving qubit by pulses or incoming photons
- decoherence of qubit dissipating photons into the line.

Note that both the drive & decoherence are determined by the common coupling strength (called g below). So intuitively the stronger the drive, the stronger the dissipation and vice-versa

Model for drive: we use a semi-classical formulation for the hamiltonian

$$H_{drive} \propto g\, E \cos\omega t \, (\underbrace{\sigma_+ + \sigma_-}_{\sigma_x})$$

where $E \cos\frac{\omega t}{2}$ = electric field of microwave pulse of frequency ω. Following the theory of Rabi oscillations going to moving frame we get

$$H_{qb} + H_{drive} \longrightarrow \frac{\hbar\Delta}{2}\sigma_z + \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-)$$

with $\Delta = \omega_0 - \omega$ = detuning between qubit and drive frequencies and $\Omega \propto \frac{gE}{\hbar}$ = Rabi frequency.

A NOT gate is performed by applying the drive at $\Delta = 0$ (i.e $\omega = \omega_0$) for a time $\tau_\pi = \frac{\pi}{\Omega}$ (so called π-pulse).

For an Hadamard gate one applies the drive for a time $\tau_{\pi/2} = \frac{\pi}{2\Omega}$ (so called $\frac{\pi}{2}$-pulse).

Dissipation / decoherence : the qubit is coupled to a thermal bath of photons in the control line. The bath hamiltonian is of the form

$$H_{bath} \propto \sum_k \hbar \omega_k \, a_k^+ a_k$$

where a_k^+, a_k^- creation / annihilation operators of photons (osc harm) at frequency ω.

The interaction between bath & qubit is

$$H_{int} \propto \hbar \sum_k g_k \left(a_k^+ \sigma_- + a_k^- \sigma_+ \right)$$

and we will assume $g_k \simeq g$ = constant for the relevant modes

Let us now discuss the system qubit + bath (not looking at the drive) :

$$H = H_{qbit} + H_{bath} + H_{int}$$

$$\propto \frac{\hbar\omega_0}{2}\sigma_z + \sum_k \hbar\omega_k a_k^+ a_k^-$$
$$+ \hbar \sum_k g_k (a_k^+ \sigma^- + a_k^- \sigma^+) .$$

While the dynamics can be analyzed in detail, here we stay at a heuristic level. There are three effects:

- spontaneous emission of photon in transition $|e\rangle \to |g\rangle$ at rate γ $[s^{-1}]$.
- stimulated emission of photon in transition $|e\rangle \to |g\rangle$ at rate $\gamma\, n_{th}(\omega_0)$
- absorption of photon in transition $|g\rangle \to |e\rangle$ at rate $\gamma\, n_{th}(\omega_0)$.

where

$$n_{th}(\omega) = \frac{1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}$$

= average number of thermal photons at frequency ω.

What is the spontaneous emission rate?

A heuristic estimate comes from Fermi Golden rule $\Gamma_{i\to f} =$

$$\frac{2\pi}{\hbar} \left| \langle \text{final state} | H_{int} | \text{initial state} \rangle \right|^2 \delta(E_f - E_i)$$

One can see that this leads to the estimate:

$$\gamma \propto |g(\omega_0)|^2 = g^2$$

with appropriate proportionality factors.

Summarizing we have for the <u>Rabi frequency</u> (setting the duration of gate) ;

$$\Omega \propto g E$$

and for the <u>spontaneous emission rate</u>

$$\gamma \propto g^2 .$$

Thus $\Omega \propto \sqrt{\gamma}\, E$.

The electric field can be expressed in terms of the power of a microwave pulse as follows ;

$$P = \hbar\omega \, . \underbrace{(\text{Photon flux})}_{}$$

number of photons of frequency ω per unit time.

and $E \propto a_\omega^+ e^{i\omega t} + a_\omega^- e^{-i\omega t}$

$\Rightarrow \quad E^2 \propto a_\omega^+ a_\omega \Rightarrow \quad P \propto \hbar\omega E^2 .$

Therefore $\Omega \propto \sqrt{\dfrac{\gamma}{\hbar\omega_0}}\sqrt{P}$ or $P \propto \Omega^2 \dfrac{\hbar\omega_0}{\gamma}$

one can check that units are correct so the proportionality constant is numerical.

For a π-pulse realizing a NOT gate we arrive at the power:

$$\boxed{P_\pi \propto \pi^2 \frac{\hbar\omega_0}{\gamma\,\tau_\pi^2}}$$

The correct formula is $P_\pi = \dfrac{\pi^2}{4}\dfrac{\hbar\omega_0}{\gamma\,\tau_\pi^2}$.

Fidelity of the gate.

We discuss a heuristic estimate of the fidelity of the gate and relate it to P_π. Errors in the gate $|g\rangle \to |e\rangle = \sqrt{NOT}|g\rangle$ are due to

- spontaneous emission with:

$$\text{prob} = \gamma\, \tau_\pi$$

rate = prob per unit time

duration of gate

- stimulated emission with:

$$\text{prob} = \gamma\, n_{th}\, \tau_\pi \qquad ; \quad n_{th} = \frac{1}{e^{\frac{\hbar\omega_0}{kT}} - 1}.$$

with T = low temp at chip level $\simeq 10^{-3}$ K.

$$\Rightarrow \quad P_{\text{error}} = \gamma(n_{th}+1)\,\tau_\pi$$

$$\Rightarrow \quad \boxed{\text{Fidelity} = \mathcal{M}_{1qb} = 1 - \gamma(n_{th}+1)\,\tau_\pi}.$$

Relation between fidelity and Power.

At chip level we consume power

$$P_\pi = \frac{\pi^2 \hbar \omega_0}{4 \gamma \tau_\pi^2}$$

and the fidelity of the gate is

$$\mathcal{M}_{1qb} = 1 - \gamma (n_{th}+1) \tau_\pi$$

Eliminating τ_π we find:

$$\boxed{P_\pi = \frac{\pi^2 \hbar \omega_0}{4} \frac{\gamma (n_{th}+1)^2}{(1-\mathcal{M}_{1qb})^2}}$$

* The power $\nearrow$ as we increase fidelity $\mathcal{M}_{1qb} \nearrow 1$.
* The power $\nearrow$ as $\gamma \nearrow$. So low spontaneous emission has lower power cost.

Ref [PRX Q4 2022] defines an "bare efficiency" as

$$\eta_0 = \frac{\mathcal{M}_{1qs}}{P_\pi} = \frac{4}{\pi^2}\,\frac{\mathcal{M}_{1qs}(1-\mathcal{M}_{1qs})^2}{\gamma\hbar\omega_0\,(M_{th}+1)^2}$$

Here $M_{th}(T_{qs}) = \dfrac{1}{e^{\frac{\hbar\omega_0}{KT_{qs}}}-1} \ll 1$ can be neglected for $\dfrac{\hbar\omega_0}{KT_{qs}} \gg 1$.

II) Power consumption at macroscopic level at room temperature. (single gate case.)

Consider the FIG (the last one). The signal from the control line carries power $P' \equiv A P_\pi$ where $A \gg 1$ is an attenuation factor (i.e $P_\pi = \frac{1}{A} P'$ in fact and it is customary to measure A in dB).

Heuristically the attenuator works as follows. A certain number of photons at high temperature T_{ext} arrive at the attenuator. A "small" fraction $\frac{1}{A}$ of them goes through in the circuit; and a "large" fraction $(1 - \frac{1}{A})$ is replaced by low temperature photons produced by attenuator. The fraction $(1 - \frac{1}{A})$ is dissipated as heat (red arrow). Hence we assume that the M_{th} we used in the previous section I) should be replaced by

$$M_{Noise} = \frac{1}{A} M_{th}(T_{ext}) + \left(1-\frac{1}{A}\right) M_{th}(T_{qb})$$

$$= M_{th}(T_{qb}) + \frac{1}{A}\left(M_{th}(T_{ext}) - M_{th}(T_{qb})\right)$$

An approximation often used if $T_{ext} \gg T_{qb}$ (but not valid if temperature diff is not big enough) is $M_{Noise} \simeq M_{th}(T_{qb}) + \frac{1}{A} M_{th}(T_{ext})$. This equation says that to chip level noise $M_{th}(T_{qb})$ we must add what comes from high temp signal attenuated (recall $\frac{1}{A} \ll 1$).

Note $M_{th}(T) = \dfrac{1}{e^{\frac{\hbar\omega_0}{kT}} - 1}$ the Bose-Einstein distribution.

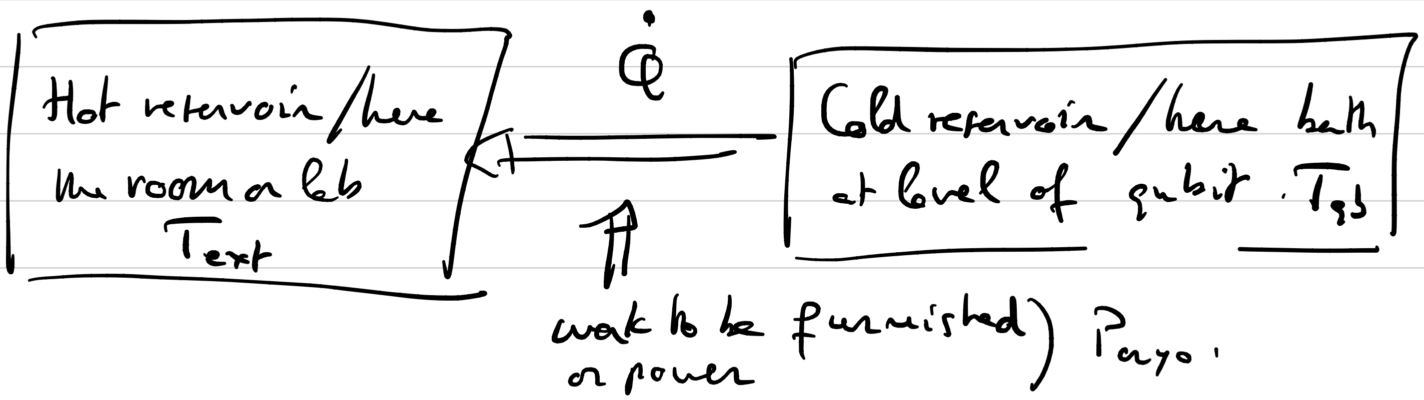
Heat flow extracted out of the system:

For $A \gg 1$ we may assume that almost all the incoming power $P' = A P_\pi$ is extracted as heat flux (heat per unit time). More exactly this is $\left(1 - \frac{1}{A}\right) A P_\pi \approx A P_\pi$.

$$\underbrace{\dot{Q}}_{\text{heat flux.}} \approx A P_\pi$$

Power consumed by cryogenics:

To extract this heat away from system we must furnish work to generate a flow towards room temperature bath. This is the work needed to operate the dilution fridge.



Using an optimal Carnot cycle just to get a rough estimate we have:

$$\frac{P_{cryo}}{\dot{Q}} = \frac{T_{ext} - T_{qb}}{T_{qb}} = \frac{T_{ext}}{T_{qb}} - 1 .$$

$$\Rightarrow \quad P_{cryo} = \left(\frac{T_{ext}}{T_{qb}} - 1 \right) A \, P_{\pi} .$$

To this we add the power $A P_{\pi}$ to control the qubit: and find for the "macro-power" to control 1 qb:

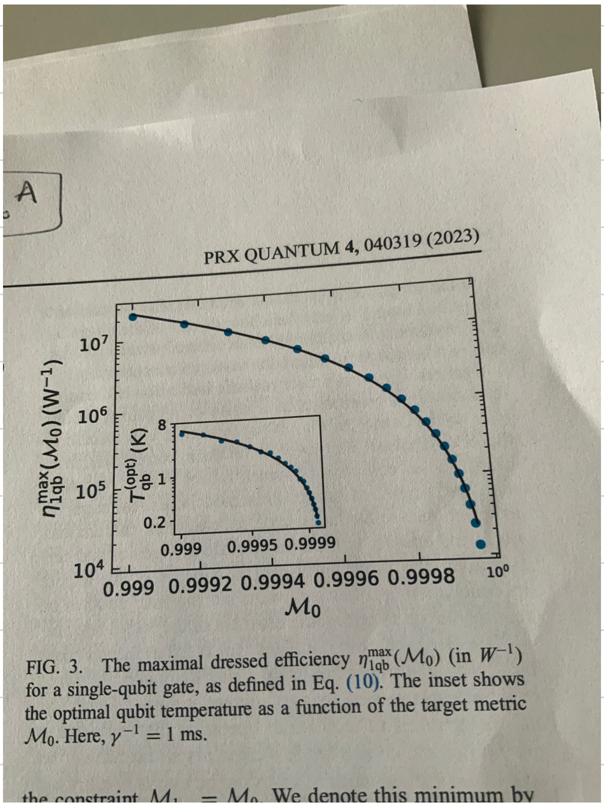
$$\boxed{\begin{aligned} P_{1qb}^{macro} &= \frac{T_{ext}}{T_{qb}} A \, P_{\pi} \\ &= \frac{T_{ext}}{T_{qb}} \cdot A \cdot \frac{\pi^2 \hbar \omega_0 \gamma}{4} \frac{(n_{Noise}+1)^2}{(1 - \mathcal{M}_{qb})^2} \\ \text{with} \quad n_{Noise} &= n_{th}(T_{qb}) + \frac{1}{A} \left(n_{th}(T_{ext}) - n_{th}(T_{qb}) \right) \end{aligned}}$$

Ref [PRX Q 4 2022] define the efficiency at macroscopic level as

$$\eta = \frac{\mathcal{M}_{1qb}}{P^{macro}_{1qb}} .$$

* Essentially w.r.t to chip level quantities the power is multiplied by $\frac{T_{ext}}{T_{qb}} A$ which can be $\simeq 10^4$. Efficiency is divided by this factor.

* Efficiency can be optimized over control parameters A & T_{qb} of hardware. Plot of optimized efficiency as fct of $\mathcal{M}_{1qb}$.

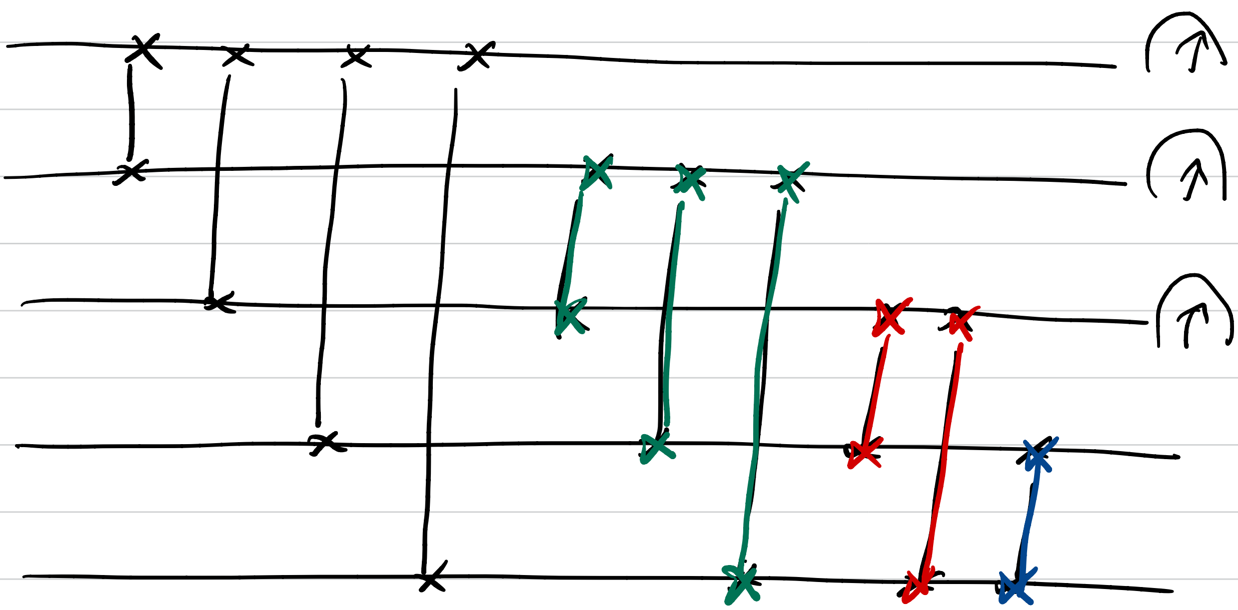


(here $\tau_{1qb} \simeq 25 \underbrace{\text{ns}}_{10^{-9}\,s}$; $\gamma^{-1} = \underbrace{\text{ms}}_{10^{-3}\,s}$)

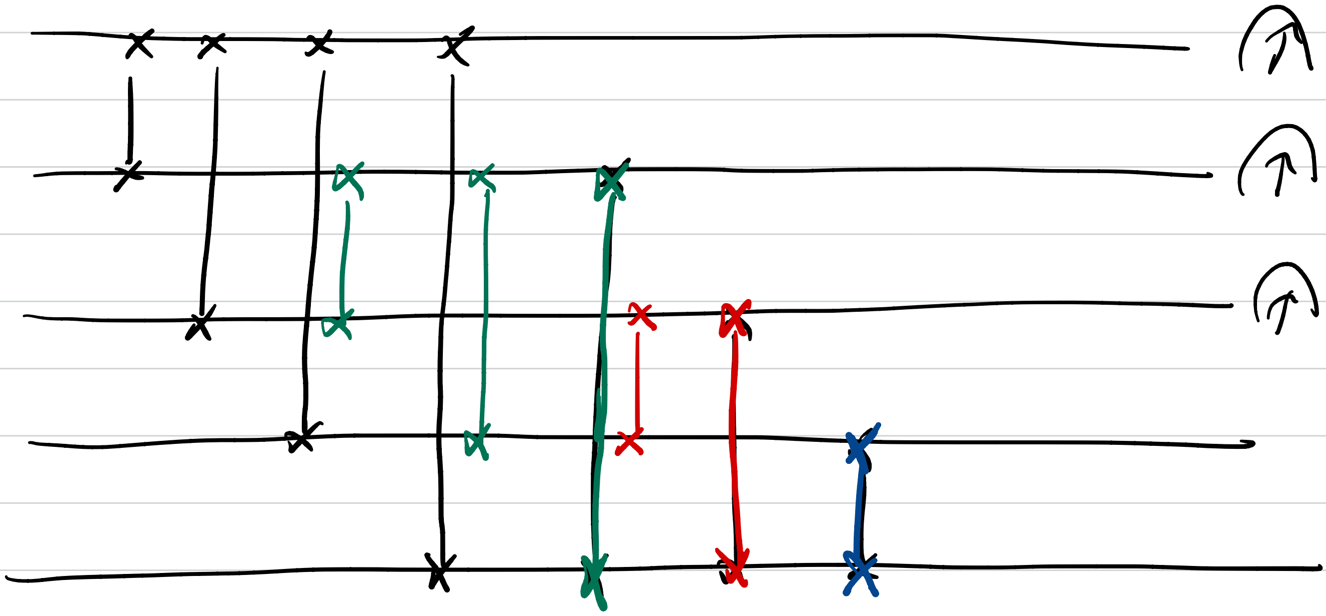
III) Extension to noisy computation on circuits implemented on NISQ devices.

We discuss the extension to whole circuits instead of single qubit gates. We also want to introduce in our control parameters the optimization of the circuit, besides the hardware quantities.

We take as a generic model circuit:



which can be optimized by a compression factor:



Between these two extremes, longest and shortest circuits, there are the intermediate ones. We will generically call $0 < \varepsilon < 1$ the "compression factor" with say $\varepsilon = 0$ No compression & $\varepsilon = 1$ full compression. More details in [PRX Q 2022].

We make the following suppositions:

* The elementary time step of the circuit is given by the slowest 2 qubit gates $\tau_{step} = \tau_{2qb}$ (we can think of 100 ms whereas $\tau_{1qb} \simeq 25$ ms). We take for the power at chip level for operating one 2 qbit gate

$$P_{2qb}^{\text{chip level}} = \frac{\hbar\omega_0 \pi^2}{4\gamma\, \tau_{2qb}^2}.$$

Then for operating a 1qb gate on average this power is multiplied by the fraction of time $\frac{\tau_{1qb}}{\tau_{2qb}} \simeq \frac{1}{4}$ and we assume now:

$$P_{1qb}^{\text{chip level}} = \frac{\hbar\omega_0 \pi^2}{4\gamma\, \tau_{2qb}^2} \cdot \frac{\tau_{1qb}}{\tau_{2qb}}.$$

These are the powers to operate single gates during one "time step" $\tau_{step} = \tau_{2qb}$.

* The prob of error for one time step is

$$p_{err} \simeq \gamma \, \tau_{step} \, (n_{Noise} + 1).$$

* The total "algorithmic fidelity" for the whole circuit is assumed to be (with indep errors):

$$\mathcal{M}_{algo} = (1 - p_{err})^{\mathcal{N}_g(\varepsilon)}$$

T_{2qb}

$$\simeq 1 - \mathcal{N}_g(\varepsilon) \, \gamma \, \tau_{step} \, (n_{Noise} + 1)$$

where $\mathcal{N}_g(\varepsilon)$ = total # of gates

$$= \mathcal{N}_{id\ gates}(\varepsilon) + \mathcal{N}_{1qb}(\varepsilon) + \mathcal{N}_{2qb}(\varepsilon)$$

as a function of the compression factor.

Again $n_{Noise} = n_{th}(T_{qb}) + \frac{1}{A}\left(n_{th}(T_{ext}) - n_{th}(T_{qb})\right).$

Total macro-power consumed :

We view this power as a function of the control parameters A, T_{qs}, ε :

$$P_{macro}(A, T_{qs}, \varepsilon) = P^{macro}_{1qs}(A, T_{qs})\, N_{1qs}(\varepsilon)$$
$$+ P^{macro}_{2qs}(A, T_{qs})\, N_{2qs}(\varepsilon)$$

where here $N_{1qs}(\varepsilon)$ & $N_{2qs}(\varepsilon)$ are the average number of 1qs and 2qs per time step. To be distinguished from the total numbers of one & two bit gates.

For example in the example circuits above for all ε ; $N_{2qs}(\varepsilon) = 10$ and for $\varepsilon = 0$ $N_{2qs}(0) = 10$ but for $\varepsilon = 1$ $N_{2qs}(1) = \frac{10}{7} \simeq 1{,}4$.

We have as before:

$$P^{Macro}_{1qb} = \frac{T_{ext}}{T_{qb}} A P^{chip}_{2qbit}$$

$$P^{Macro}_{2qb} = \frac{T_{ext}}{T_{qb}} A P^{chip}_{1qbit} = \frac{T_{ext}}{T_{qb}} A P^{chip}_{2qbit} \cdot \underbrace{\frac{\tau_{1qb}}{\tau_{2qb}}}_{\approx \frac{1}{4}.}$$

$$P^{chip}_{2qbit} = \frac{\pi^2 \hbar \omega_0}{4\gamma \tau^2_{2qb}}.$$

Like before we can express the power as a fct of a given fidelity essentially by eliminating τ_{2qb} from these equations. We find:

$$\boxed{P_{Macro}(A, T_{qb}, \varepsilon) = \frac{T_{ext}}{T_{qb}} \cdot A \cdot \frac{\mathcal{N}_g^2(\varepsilon)\,(n_{noise}+1)^2}{(1-\mathcal{M}_{algo})^2} \cdot \hbar \omega_0 \pi^2 \left[N_{2qb}(\varepsilon) + \frac{1}{4} N_{1qb}(\varepsilon) \right]}$$

As an illustrative example, we may optimize over A, T_{qs} (at fixed $\mathcal{M}_{algo}$) and find the minimum power as a fct of ε the compression factor.

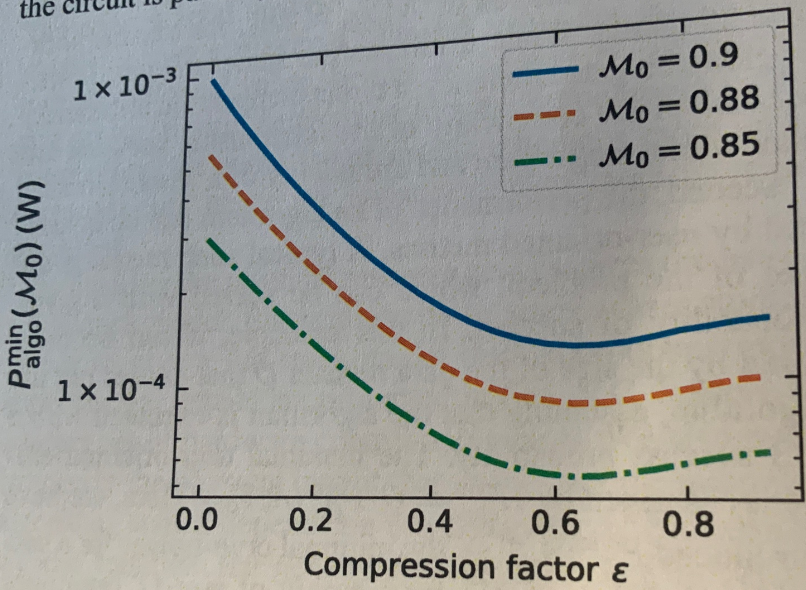


FIG. 5. The minimum power as a function of the compression ϵ of the circuit (see text). The circuit is a 25-qubit version of the one shown in Fig. 4. Here, $\gamma^{-1} = 10$ ms.

Interestingly one observes a non trivial optimal intermediate compression factor, due to interplay of many factors.

CONCLUSIONS / DISCUSSION.

A) Full stack model on FIG (first figure).

An analysis is possible and discussed in [PRX Q4 2022].

One must generalize the formulas for P_{1qb}^{Macro}, P_{2qb}^{Macro} by taking into account full stack of attenuators, heat conduction in lines, as well as power to operate amplifiers and measurement gates. The formula for M_{Noise} must also be generalized to take into account the full stack of attenuators.

B) Fault tolerant computation.

To investigate envisioned future fault tolerant computation is perhaps a bit premature. However this can be done, and one has to investigate circuits operating on logical qubit (such as with a seven qubit code).

An interesting result discussed in (PRX Q4 2022) is given by this figure!

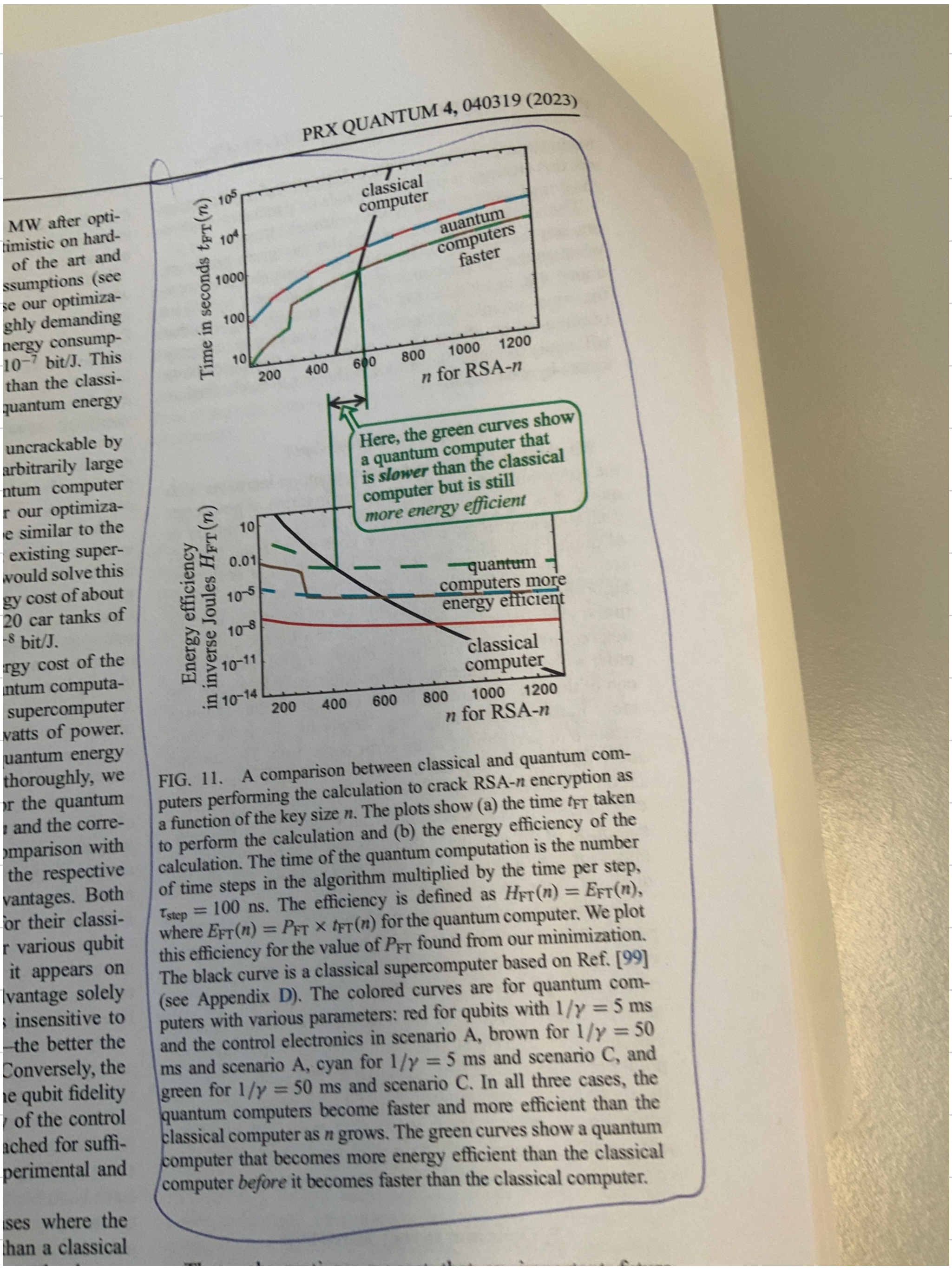


FIG. 11. A comparison between classical and quantum computers performing the calculation to crack RSA-n encryption as a function of the key size n. The plots show (a) the time t_{FT} taken to perform the calculation and (b) the energy efficiency of the calculation. The time of the quantum computation is the number of time steps in the algorithm multiplied by the time per step, $\tau_{\mathrm{step}} = 100$ ns. The efficiency is defined as $H_{\mathrm{FT}}(n) = E_{\mathrm{FT}}(n)$, where $E_{\mathrm{FT}}(n) = P_{\mathrm{FT}} \times t_{\mathrm{FT}}(n)$ for the quantum computer. We plot this efficiency for the value of P_{FT} found from our minimization. The black curve is a classical supercomputer based on Ref. [99] (see Appendix D). The colored curves are for quantum computers with various parameters: red for qubits with $1/\gamma = 5$ ms and the control electronics in scenario A, brown for $1/\gamma = 50$ ms and scenario A, cyan for $1/\gamma = 5$ ms and scenario C, and green for $1/\gamma = 50$ ms and scenario C. In all three cases, the quantum computers become faster and more efficient than the classical computer as n grows. The green curves show a quantum computer that becomes more energy efficient than the classical computer *before* it becomes faster than the classical computer.

This shows a regime of quantum power cost advantage even before a computation time advantage.

Appendix A; thermal state of photons (black body radiation).

The photon carries energy $\epsilon = \hbar\omega$, momentum $p = \hbar k$ where $\omega = ck$ (c = speed of light) and a polarization degree of freedom $p = 1, 2$ (linear, circular, ect..).

The Hamiltonian of a gas of photons can be viewed as that of a collection of harmonic oscillators (modes)

$$H = \sum_{p=1,2} \sum_{k} \hbar\omega_k \, a^+_{k,p} \, a_{k,p}$$

where a^+, a are creation and annihilation operators. On occupation number basis they act as

$$a^+_{k,p} \, |m_{k,p}\rangle = \sqrt{(m_{k,p}+1)}\, |m_{k,p}+1\rangle$$

$$a_{k,p} \, |m_{k,p}\rangle = \sqrt{m_{k,p}} \, |m_{k,p}-1\rangle$$

with $m_{k,p} = 0, 1, 2, 3, \ldots$

Their commutation relation is

$$[a_{k,p}, a^+_{k',p'}] = \delta_{kk'}\delta_{pp'}.$$

One checks

$$a^+_{k,p}\, a_{k,p} \,|n_{k,p}\rangle = n_{k,p}\,|n_{k,p}\rangle$$

Thus $a^+_{k,p}\, a_{k,p}$ has the interpretation of the "number operator": $a^+_{k,p}\, a_{k,p} = \hat{n}_{k,p}$.

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Thermal state.

Dropping the mode and polarisation index (we consider one mode & polarisation state) the density matrix for a thermal equilibrium state is

$$\rho = \frac{e^{-\beta\hbar\omega a^+ a}}{\mathrm{Tr}\, e^{-\beta\hbar\omega a^+ a}}$$

with $\beta = (k_B T)^{-1}$. We compute the average number of photons in this mode:

$$\langle \hat{n} \rangle = \mathrm{Tr}\,(a^\dagger a\, \rho)$$

$$= \frac{\mathrm{Tr}\; a^\dagger a\; e^{-\beta \hbar \omega a^\dagger a}}{\mathrm{Tr}\; e^{-\beta \hbar \omega a^\dagger a}} .$$

$$\mathrm{Tr}\; e^{-\beta \hbar \omega a^\dagger a} = \sum_{n=0}^{+\infty} \langle n | e^{-\beta \hbar \omega a^\dagger a} | n \rangle$$

$$= \sum_{n=0}^{+\infty} e^{-\beta \hbar \omega n}$$

$$= \frac{1}{1 - e^{-\beta \hbar \omega}} .$$

$$\mathrm{Tr}\; a^\dagger a\; e^{-\beta \hbar \omega a^\dagger a} = \sum_{n=0}^{+\infty} \langle n | a^\dagger a\, e^{-\beta \hbar \omega a^\dagger a} | n \rangle$$

$$= \sum_{n=0}^{+\infty} n\, e^{-\beta \hbar \omega n}$$

$$= -\frac{\partial}{\partial(\beta \hbar \omega)} \sum_{n=0}^{+\infty} e^{-\beta \hbar \omega}$$

A4

$$= -\frac{\partial}{\partial(\beta\hbar\omega)} \frac{1}{1-e^{-\beta\hbar\omega}}$$

$$= +\frac{e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^2}$$

2) $$\langle\hat{n}\rangle = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1}$$

So for one mode ω_k and one polarization state p:

$$\boxed{\langle\hat{n}_{k,p}\rangle = \frac{1}{e^{\beta\hbar\omega_k}-1} = \frac{1}{e^{\hbar\omega_k/k_B T}-1}}$$

This is the average number of photons of frequency ω_k (and given polarization) at temperature T.

(notation was $n_{thermal}$ or $\underbrace{n_{BE}}_{\text{Bose-Einstein}}(T)$ in notes)

Appendix B: emission and absorption rates

We use the Fermi Golden rule to give a heuristic derivation of emission & absorption rates.

Spontaneous emission rate:

We derive $\gamma \propto g^2$ up to prop factors.

In our case: $|i\rangle = |e\rangle \otimes |0 \text{ photon}\rangle$; $E_i = \hbar\omega_0$

$|f\rangle = |g\rangle \otimes |1 \text{ photon mode } k\rangle$;

$E_f = \hbar\omega_k$.

$$\Rightarrow \quad \Gamma^{\text{spont}}_{i\to f} = \frac{2\pi}{\hbar} |\langle g|\sigma^-|e\rangle|^2 |\langle 1_k| a^+ |0\rangle|^2 \hbar^2 |g(\omega_k)|^2 \delta(\hbar\omega_k - \hbar\omega_0)$$

$$= \frac{2\pi}{L} |g(\omega_k)|^2 \delta(\omega_k - \omega_0)$$

Total transition rate to any mode k (this is total probability per unit time; so we sum probabilities)

$$\gamma = \frac{2\pi}{L} \sum_k |g(\omega_k)|^2 \delta(\omega_k - \omega_0)$$

$$= \frac{2\pi}{L} \int \frac{L\,dk}{\pi} |g(\omega_k)|^2 \delta(\omega_k - \omega_0)$$

$\uparrow$

$\left(k = \frac{m\pi}{L} \text{ for a 1-d wave} \right)$

Using $\frac{d\omega_k}{dk} = v$ = group velocity assumed to be constant $\Rightarrow$ $\frac{dk}{d\omega_k} = \frac{1}{v}$ = density of states per unit frequency:

$$\gamma = 2|g(\omega_k)|^2 \frac{1}{v}.$$

$\blacksquare$

General emission rate :

Now $|i\rangle = |e\rangle \otimes |n_k\rangle$ (n_k photon state)

$|f\rangle = |g\rangle \otimes |n_k+1\rangle$ (n_k+1 photon state)

$$\Gamma^{\text{emission}}_{i\to f} = \frac{2\pi}{\hbar}\left|\langle g|\sigma^-|e\rangle\langle n+1|a_k^\dagger|n\rangle\right|^2 \cdot \hbar^2|g(\omega_k)|^2\,\delta(\hbar\omega_k-\hbar\omega_0)$$

we have $a_k^\dagger|n_k\rangle = \sqrt{n_k+1}\,|n_k\rangle \Rightarrow \langle n+1|a_k^\dagger|n\rangle = \sqrt{n+1}$

$$\Gamma^{\text{emission}}_{i\to f} = \frac{2\pi}{\hbar}(n_k+1)|g(\omega_0)|^2\,\delta(\omega_k-\omega_0).$$

"stimulated emission"

"spontaneous emission".

Averaging over thermal modes gives

$$\boxed{\langle\Gamma^{\text{emission}}\rangle \propto (n_{th}+1)\,g^2 \quad \text{with } n_{th} = \frac{1}{e^{\frac{\hbar\omega_0}{kT}}-1}.}$$

Absorption rate :

Let $|i\rangle = |g\rangle \otimes |n_k\rangle$

$|f\rangle = |e\rangle \otimes |n_k - 1\rangle$.

$$\Gamma^{abs}_{i\to f} = \frac{2\pi}{\hbar} \left| \langle e|\sigma^+|g\rangle \langle n_k - 1| a_k |n_k\rangle \right|^2 \hbar^2 |g(\omega_k)|^2 \delta(\hbar\omega_0 - \hbar\omega_k)$$

Since $a|n\rangle = \sqrt{n}\,|n\rangle$

$\Rightarrow \langle n_{k-1}| a_k |n_k\rangle = \sqrt{n_k}$.

$$\Rightarrow \Gamma^{abs}_{i\to f} = \frac{2\pi}{\hbar} n_k\, g(\omega_k)^2\, \delta(\omega_0 - \omega_k).$$

Averaging over thermal modes we find

$$\boxed{\langle \Gamma^{abs} \rangle \propto n_{th}\, g^2 \,, \quad n_{th} = \frac{1}{e^{\frac{\hbar\omega_0}{k_B T}} - 1}}$$

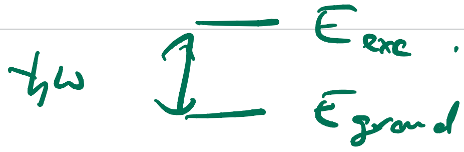
Appendix C: compatibility of emmission and absorption rates with thermal distribution.

The following argument is a simplification of Einstein's original derivation of the thermal state of photons.

We suppose that the interaction of radiation with atoms (or two level systems) proceeds via the three processes:

- spontaneous emission at rate γ.
- stimulated emission at rate γn_{th}
- absorption at rate γn_{th}

where n_{th} is the average number of photons of frequency $\hbar\omega = E_{exc} - E_{ground}$ of two level system



At thermal equilibrium between the atoms (or two level systems) and radiation with mode ω we should have detailed balance satisfied:

$$\frac{N_g}{N_g+N_e} q_{g\to e} = \frac{N_e}{N_g+N_e} q_{e\to g} \quad \text{(DB)}.$$

where $\frac{N_g}{N_g+N_e}$ = prob that atom is in $|g\rangle$

$\frac{N_e}{N_g+N_e}$ = prob that atom is in $|e\rangle$

and $q_{g\to e} = n_{th}\,\gamma\, dt$ = prob to have absorption in time dt

$q_{e\to g} = n_{th}\,\gamma\, dt + \gamma dt$ = prob to have emission in time dt.

(coming from stimulated or spontaneous emission).

Moreover at thermal equilibrium :

$$\frac{N_g}{N_g + N_e} = \frac{e^{-\beta E_g}}{e^{-\beta E_g} + e^{-\beta E_{exc}}}$$

$$\frac{N_e}{N_g + N_e} = \frac{e^{-\beta E_{exc}}}{e^{-\beta E_g} + e^{-\beta E_{exc}}}$$

or more simply:

$$\frac{N_{exc}}{N_g} = e^{-\beta(E_{exc} - E_g)} = e^{-\beta\hbar\omega}.$$

It then follows from (DB) condition:

$$n_{th} = (n_{th} + 1)\, e^{-\beta\hbar\omega}$$

$$n_{th}\left(1 - e^{-\beta\hbar\omega}\right) = e^{-\beta\hbar\omega}$$

$$\Rightarrow \boxed{n_{th} = \frac{1}{e^{\beta\hbar\omega} - 1}} \qquad \beta = (k_B T)^{-1}.$$

Remark: Einstein did not assume that stimulated emission & spontaneous emission have same rate but "proved it".

Appendix D: Master equation for open quantum system; Lindblad formalism.

A more basic derivation starts from the full Hamiltonian $H(t) = H_{qs} + H_{drive} + H_{bath} + H_{int}$ and the initial state $\rho = \rho_{qs} \otimes \rho_{bath}$ with say $\rho_{qs} = |g\rangle\langle g|$ and $\rho_{bath} = \dfrac{e^{-H_{bath}/kT}}{\mathrm{Tr}\, e^{-H_{bath}/kT}}$.

Then from the Heisenberg equation of motion

$$\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}\left[H(t), \rho(t)\right]$$

and eliminating the bath degrees of freedom (with appropriate assumptions) leads to

$$\frac{d}{dt}\rho_{qs}(t) = -\frac{i}{\hbar}\left[H_{qs} + H_{drive}, \rho_{qs}(t)\right]$$

$$+ \gamma(n_{th}+1)\,\mathcal{D}_{\sigma^-}(\rho_{qs}) + \gamma n_{th}\,\mathcal{D}_{\sigma^+}(\rho_{qs})$$

where the last two terms are the Lindbladians describing dissipative processes:

$$\mathcal{D}_A(\rho) = A\rho A^\dagger - \frac{1}{2}\left(A^\dagger A\rho + \rho A^\dagger A\right)$$

The fidelity is defined as

$$\mathcal{M} = \langle g|\, NOT\, \rho_{gb}(\tau_\pi)\, NOT\, |g\rangle$$

= prob to find qubit at time τ_π in correct state $NOT|g\rangle = |c\rangle$.

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Information theoretic derivation of Lindblad equation.

The decoherence processes can be viewed as a channel described by the map:

$$\rho(t) \to \rho(t+dt) = \sum_i E_i\, \rho(t)\, E_i^\dagger$$

D3

with $\sum_i E_i^\dagger E_i = 1$ (ensures $\mathrm{Tr}\,\rho(t+dt) = \mathrm{Tr}\,\rho(t) = 1$)

and $E_1 = \sqrt{\gamma\, n_{th}\, dt}\,\sigma^-$, $E_2 = \sqrt{\gamma(n_{th}+1)\,dt}\,\sigma^+$

We must have $E_0^\dagger E_0 = 1 - E_1^\dagger E_1 - E_2^\dagger E_2$

$$= 1 - \gamma n_{th}\, dt\, \sigma^+\sigma^- - \gamma(n_{th}+1)\,dt\,\sigma^-\sigma^+$$

$$\Rightarrow \quad E_0^\dagger = E_0 = \sqrt{1 - \gamma n_{th}\, dt\, \sigma^+\sigma^- - \gamma(n_{th}+1)\,dt\,\sigma^-\sigma^+}$$

$$\simeq 1 - \frac{1}{2}\gamma n_{th}\, dt\, \sigma^+\sigma^- - \frac{1}{2}\gamma(n_{th}+1)\,dt\,\sigma^-\sigma^+$$

(This is the analog of an Itô term in the theory of classical Brownian motion).

Thus

$$\rho(t+dt) \simeq E_0\,\rho(t)\,E_0^\dagger + E_1\,\rho(t)\,E_1^\dagger + E_2\,\rho(t)\,E_2^\dagger$$

$$\simeq \rho(t) - \frac{1}{2}\gamma n_{th}\, dt\,(\sigma^+\sigma^-\rho + \rho\sigma^+\sigma^-) - \frac{1}{2}\gamma(n_{th}+1)\,dt\,(\sigma^-\sigma^+\rho + \rho\sigma^-\sigma^+)$$

$$+ \gamma n_{th}\, dt\, \sigma^-\rho\sigma^+ + \gamma(n_{th}+1)\,dt\,\sigma^+\rho\sigma^- .$$

Lindblad's equation follows immediately.

D4

$$\frac{\rho(t+dt)-\rho(t)}{dt} = \frac{d\rho(t)}{dt}$$

$$= \gamma m_{th}\left[\sigma^- \rho(t)\sigma^+ - \frac{1}{2}\{\sigma^+\sigma^-, \rho\}\right]$$

$$+ \gamma(m_{th}+1)\left[\sigma^+\rho(t)\sigma^- - \frac{1}{2}\{\sigma^-\sigma^+, \rho\}\right]$$

(notation $\{A, B\} = AB + BA =$ "anticommutator").

<u>Generalization.</u>

The continuous time version of the Kraus operator formalism to describe maps:

$$\rho \rightarrow \mathcal{N}(\rho) = \sum_{k=0}^{R} E_k \rho E_k^+$$

with $\sum_k E_k^+ E_k = \mathbb{1}$ is obtained

by setting:

D5

$$E_k = L_k \sqrt{dt} \quad , \quad k = 1 \ldots R$$

$$E_0 = \mathbb{1} - \sum_k \frac{1}{2} L_k^\dagger L_k \, dt \, ,$$

Check to first order in dt :

$$\sum_{k=0}^{R} E_k^\dagger E_k = \mathbb{1} \, .$$

and we get

$$\rho(t+dt) = \mathcal{N}(\rho(t))$$

$$= \rho(t) + dt \left[\sum_{k=0}^{R} L_k \rho L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho \} \right]$$

$$\frac{d\rho}{dt} = \sum_{k=0}^{R} L_k \rho L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k ; \rho \}$$

If on top there is a usual unitary evolution

we have to add on the r.h.s a term

$$-\frac{i}{\hbar} [H, \rho] \, .$$

D6

In the Kraus formalism this corresponds to a map $\rho \to U \rho U^\dagger$ with

$$U = \exp\left(-\frac{i}{\hbar} t H\right).$$

For infinitesimal time dt this is

$$U_{dt} \simeq \mathbb{1} - \frac{i}{\hbar} dt\, H.$$

$$\rho(t+dt) = \left(\mathbb{1} - \frac{i}{\hbar} dt\, H\right) \rho(t) \left(\mathbb{1} + \frac{i}{\hbar} dt\, H\right)$$

$$= \rho(t) + \frac{i}{\hbar} dt\, \rho(t) H - \frac{i}{\hbar} dt\, H \rho(t)$$

$$\Rightarrow \quad \rho(t+dt) = \rho(t) - \frac{i}{\hbar} dt\, [H, \rho(t)]$$

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho(t)].$$

<u>Final remark</u>: In the derivation of the Lindblad master equation from the Kraus formalism there is a (hidden) Markovian assumption. For each dt the transition $\rho(t) \to \rho(t+dt) = \mathcal{L}'(\rho(t))$ only depend on the state at time t; i.e. there is no memory effect. This is why the master equ discussed here is purely local in time.

This Markovity assumption is also present in the more elaborate derivation from the full system + bath, when degrees of freedom of the bath are eliminated. If this assumption is not made one gets more complicated master equations which are non-local in time.

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