

Stellar orbits

3rd part

Outlines

Motions of stars in the Sun neighbourhood

- The Oort constants
- Epicycle frequencies

Surfaces of section

- Integral of motions
- Poincaré maps

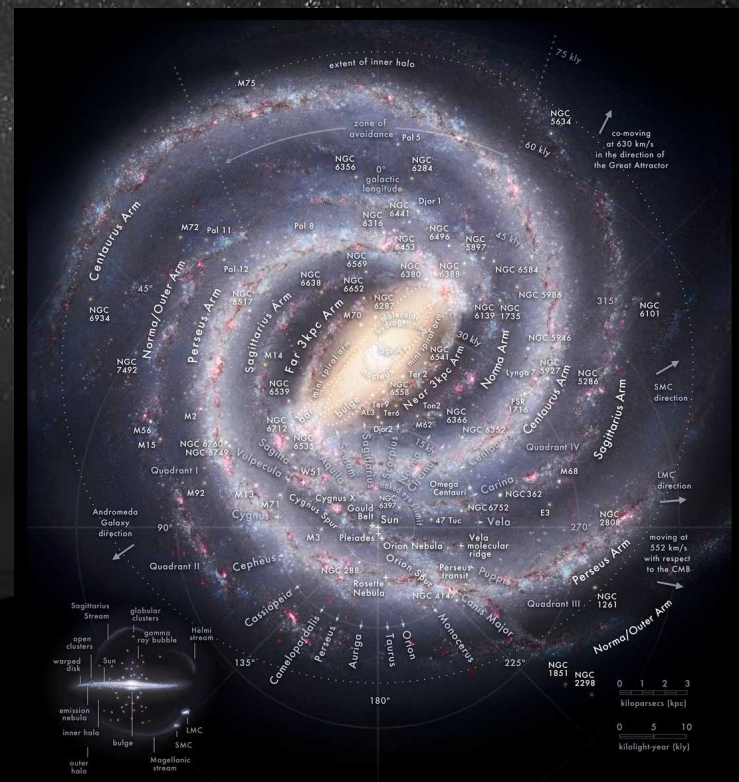
Stellar orbits

**Motions of stars in the Sun
neighbourhood**

The Oort constants

Motions of stars in the neighbourhood of the Sun

- How can we learn about the global motions of stars in the Milky Way?



Motions of stars in the neighbourhood of the Sun

- How can we learn about the global motions of stars in the Milky Way?
- Problem : we are living around the Sun, which is also moving around the Milky Way center...
- Solution :
 - describe in a general framework the motions of nearby stars
 - deduce from observations of nearby stars global motions of the Milky Way

Taylor expansion of the velocity field around $\vec{x}_0$

$$\tilde{V}(\vec{x}) = \underbrace{\vec{V}_0}_{\vec{V}(\vec{x}_0)} + \underbrace{\begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}}_{\text{Jacobian matrix}} (\vec{x} - \vec{x}_0)$$

Relative velocity field $\delta \tilde{V}(\vec{x}) = \tilde{V}(\vec{x}) - \vec{V}_0$

$$\begin{cases} \delta V_x = \frac{\partial V_x}{\partial x} (x - x_0) + \frac{\partial V_x}{\partial y} (y - y_0) \\ \delta V_y = \frac{\partial V_y}{\partial x} (x - x_0) + \frac{\partial V_y}{\partial y} (y - y_0) \end{cases}$$

$$\delta \vec{V} = \mathcal{J} \cdot \delta \vec{x}$$

$$\mathcal{J} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}$$

$$\delta \vec{x} = \vec{x} - \vec{x}_0$$

The matrix J can be decomposed on a basis

$$J = A \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + C \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + K \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Interpretation

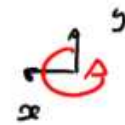
• "A" : shear

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



• "B" : vorticity

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



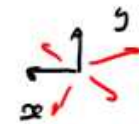
• "C" : shear

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

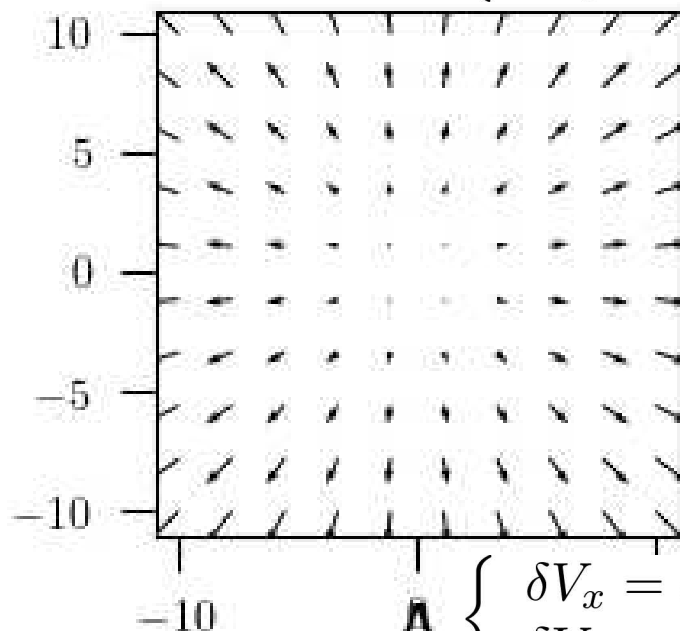


• "K" : divergeance

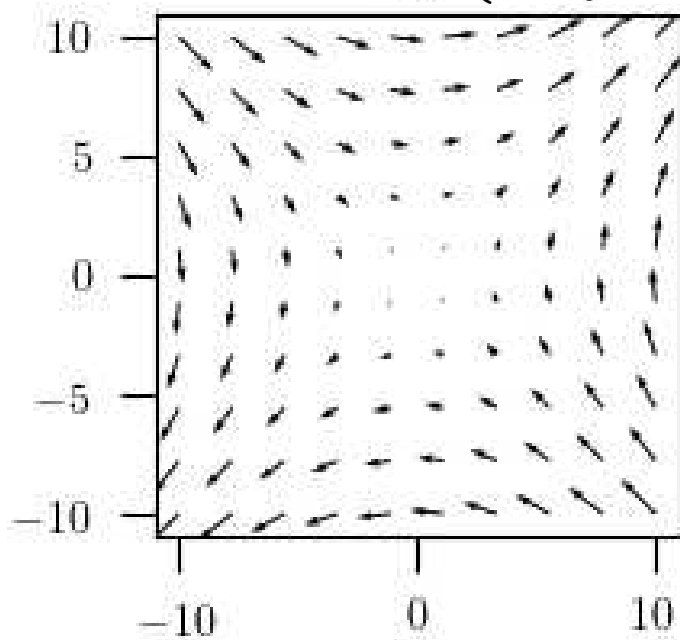
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



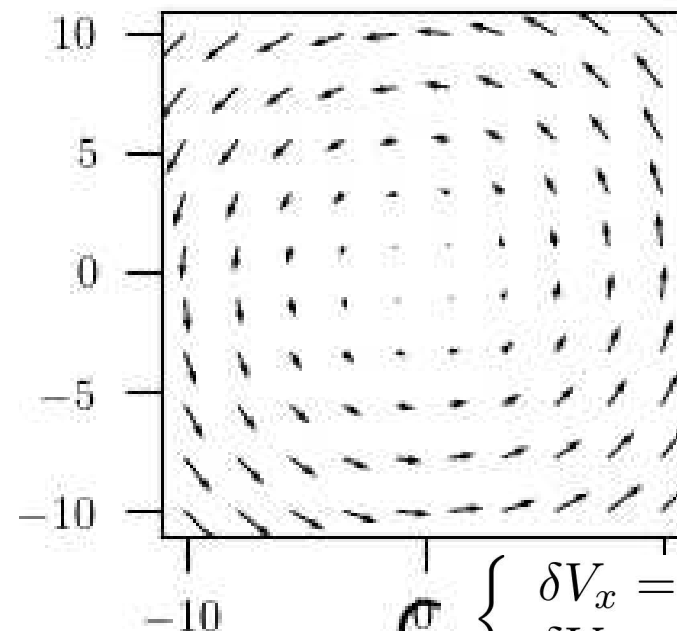
$$\mathbf{K} \begin{cases} \delta V_x = kx \\ \delta V_y = ky \end{cases}$$



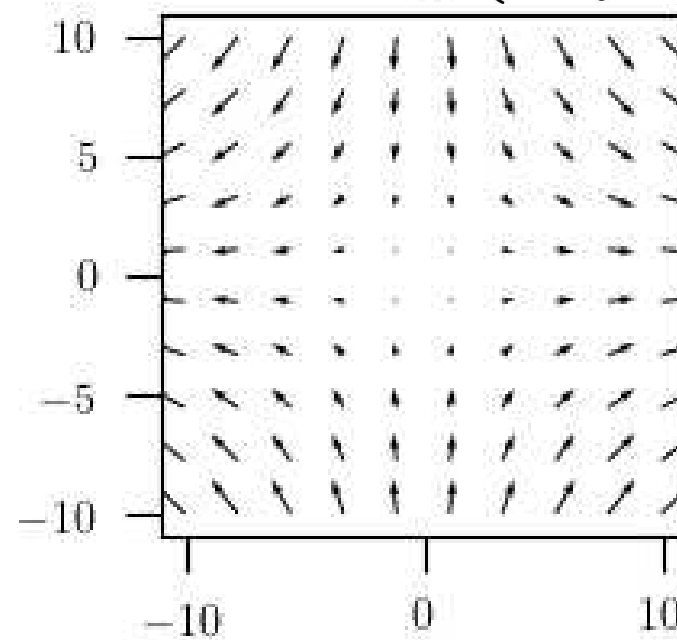
$$\mathbf{A} \begin{cases} \delta V_x = ay \\ \delta V_y = ax \end{cases}$$



$$\mathbf{B} \begin{cases} \delta V_x = -by \\ \delta V_y = bx \end{cases}$$



$$\mathbf{C} \begin{cases} \delta V_x = cx \\ \delta V_y = -cy \end{cases}$$



So, we have

$$\mathcal{J} = \begin{pmatrix} K + C & A - B \\ A + B & K - C \end{pmatrix} = \begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_x}{\partial y} \\ \frac{\partial V_y}{\partial x} & \frac{\partial V_y}{\partial y} \end{pmatrix}$$

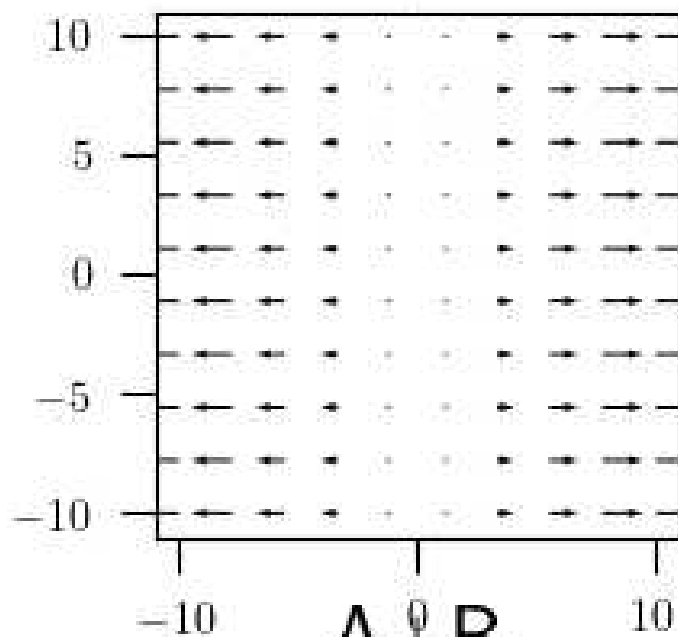
A, B, C, K

The Oort constants describe the local velocity field

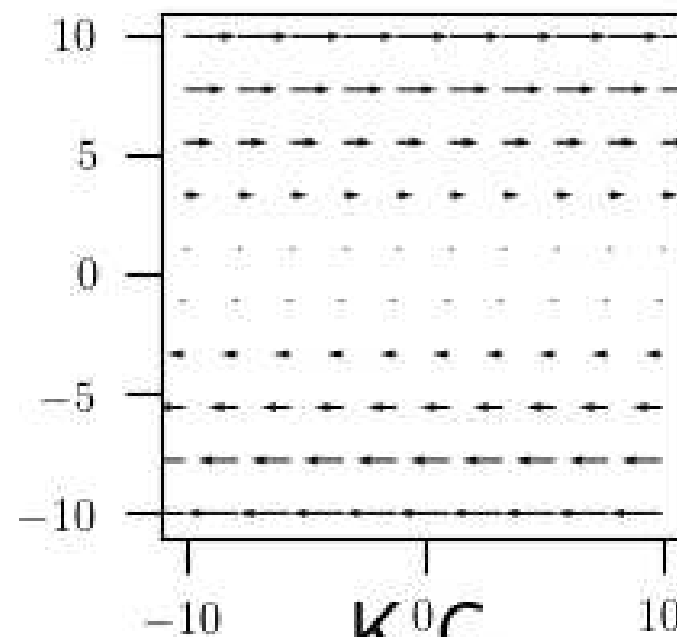
$$\begin{cases} A = \frac{1}{2} \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right) & K = \frac{1}{2} \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} \right) \\ B = \frac{1}{2} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) & C = \frac{1}{2} \left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \right) \end{cases}$$

Note: we could do all this in 3D

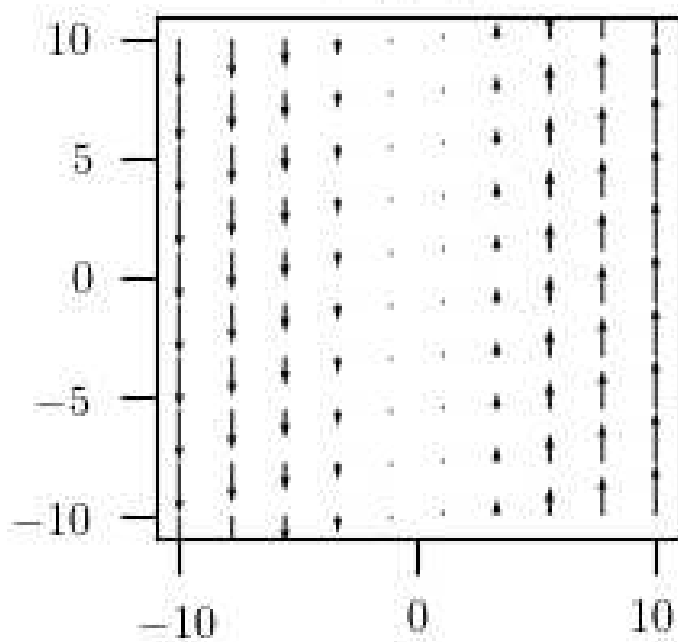
$K+C$



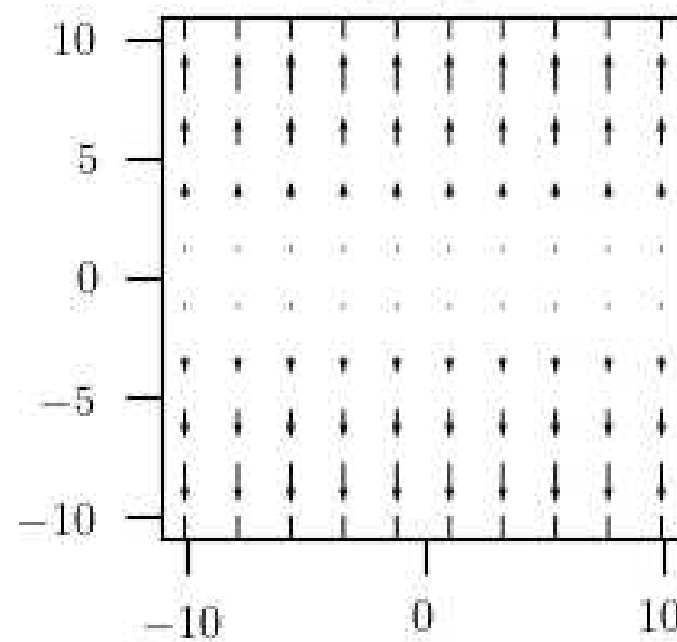
$A-B$



$A+B$



$K-C$



Motions of nearby stars

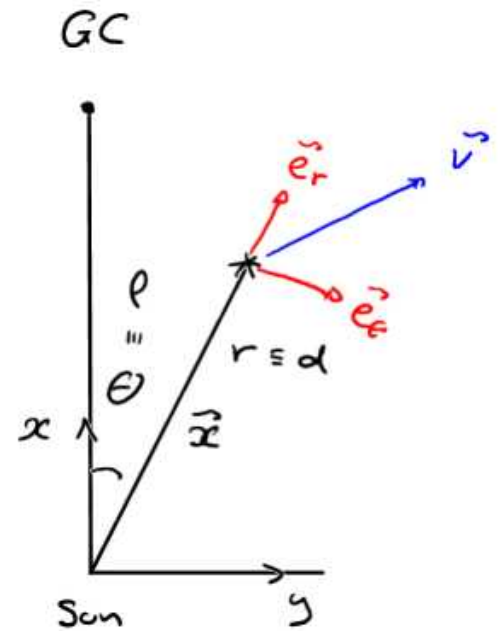
$$\vec{V}_0 = \vec{V}_0$$

$\vec{x}$: position of a nearby star

$r \equiv d$: distance to the star

$$\vec{x} = \begin{cases} x = d \cos \ell & \equiv r \cos \Theta \\ y = d \sin \ell & \equiv r \sin \Theta \end{cases}$$

galacto centric polar
coord coord



Radial and tangential velocities

$$\left\{ \begin{aligned} V_r &= \vec{v} \cdot \vec{e}_r = \vec{v} \cdot \frac{\vec{x}}{r} = \frac{1}{r} (x v_x + y v_y) \\ &= \cos \Theta v_{xc} + \sin \Theta v_y \\ V_\theta &= \left| \vec{v} \times \frac{\vec{x}}{r} \right| = \frac{1}{r} (x v_y - y v_x) \\ &= \cos \Theta v_y - \sin \Theta v_{xc} \end{aligned} \right.$$

Re-write the Jacobian matrix in galacto-centric coordinates
term of V_r and V_θ

$$\frac{\partial V_r}{\partial r} = \frac{\partial V_r}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial V_r}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial V_r}{\partial x} \cos \theta + \frac{\partial V_r}{\partial y} \sin \theta$$

$$\frac{\partial V_r}{\partial \theta} = \dots \quad \frac{\partial V_\theta}{\partial r} = \dots \quad \frac{\partial V_\theta}{\partial \theta} = \dots$$

$$A = \frac{1}{2} \left(-\frac{1}{r} \frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2} \left(\frac{1}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = \frac{1}{2} \left(-\frac{1}{r} \frac{\partial V_\theta}{\partial \theta} - \frac{V_r}{r} + \frac{\partial V_r}{\partial r} \right)$$

$$K = \frac{1}{2} \left(\frac{1}{r} \frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} + \frac{\partial V_r}{\partial r} \right)$$

See Shandrasekhar

Purely axisymmetric disk

(no radial motions)

$$\frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \quad \frac{\partial v_r}{\partial r} = 0$$

$$A = \frac{1}{2} \left(-\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$B = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$C = \frac{1}{2} \left(-\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r} + \frac{\partial v_r}{\partial r} \right)$$

$$K = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_r}{\partial r} \right)$$

$$A = \frac{1}{2} \left(\frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$B = \frac{1}{2} \left(-\frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$C = 0$$

$$K = 0$$

With $V_e \equiv V_c$ (circular velocity)

$$A(R) := \frac{1}{2} \left[\frac{V_c(R)}{R} - \frac{d}{dR} V_c(R) \right] \equiv -\frac{1}{2} R \frac{d\Omega(R)}{dR}$$

$$B(R) := -\frac{1}{2} \left[\frac{V_c(R)}{R} + \frac{d}{dR} V_c(R) \right] \equiv -\left(\Omega(R) + \frac{1}{2} R \frac{d\Omega(R)}{dR} \right)$$

We can express Ω and $\mathcal{R}$ from the Oort constants

$$\Omega = A - B$$

$$\mathcal{R}^2 = -4B(A - B) = -4B\Omega$$

Expressions of $\mathcal{R}$ and ν from the total potential

$$\begin{aligned} \mathcal{R}^2(R_S) &= \frac{\partial^2 \phi_{tot}}{\partial R^2} \Big|_{(R_S, 0)} = \frac{\partial^2 \phi}{\partial R^2} \Big|_{(R_S, 0)} + 3 \frac{L_z^2}{R_S^4} \quad L_z^2 = V_c^2 R_S^2 \\ &= \frac{\partial^2 \phi}{\partial R^2} \Big|_{(R_S, 0)} + \frac{3}{R_S} \frac{\partial \phi}{\partial R} \Big|_{(R_S, 0)} \\ &\quad \text{circ. frequency } \mathcal{R}^2 = \frac{\partial^2 \phi}{\partial R^2} \Big|_{(R_S, 0)} + 3 \mathcal{R}^2 \\ &\quad \text{" } \mathcal{R}^2 = \frac{V_c^2}{R^4} \Big|_{(R_S, 0)} = \left(R \frac{\partial(\Omega^2)}{\partial R} + 4 \mathcal{R}^2 \right) \Big|_{(R_S, 0)} \\ &= \left(\frac{1}{R} \frac{\partial(V_c^4)}{\partial R} + 2 \mathcal{R}^2 \right) \Big|_{(R_S, 0)} = \left(\frac{1}{R} \frac{\partial(V_c^4)}{\partial R} + 2 \frac{V_c^4}{R^4} \right) \Big|_{(R_S, 0)} \end{aligned}$$

Rigid rotation $\Omega = \text{cte}$

$$\frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \quad \frac{\partial v_r}{\partial r} = 0 \quad \frac{\partial \Omega}{\partial r} = 0$$

$$A = 0 \quad C = 0$$

$$B = -\Omega \quad K = 0$$

$$A = \frac{1}{2} \left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

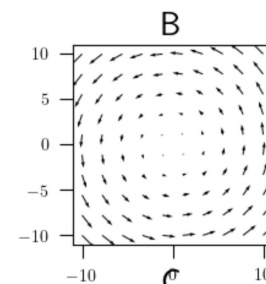
$$B = \frac{1}{2} \left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

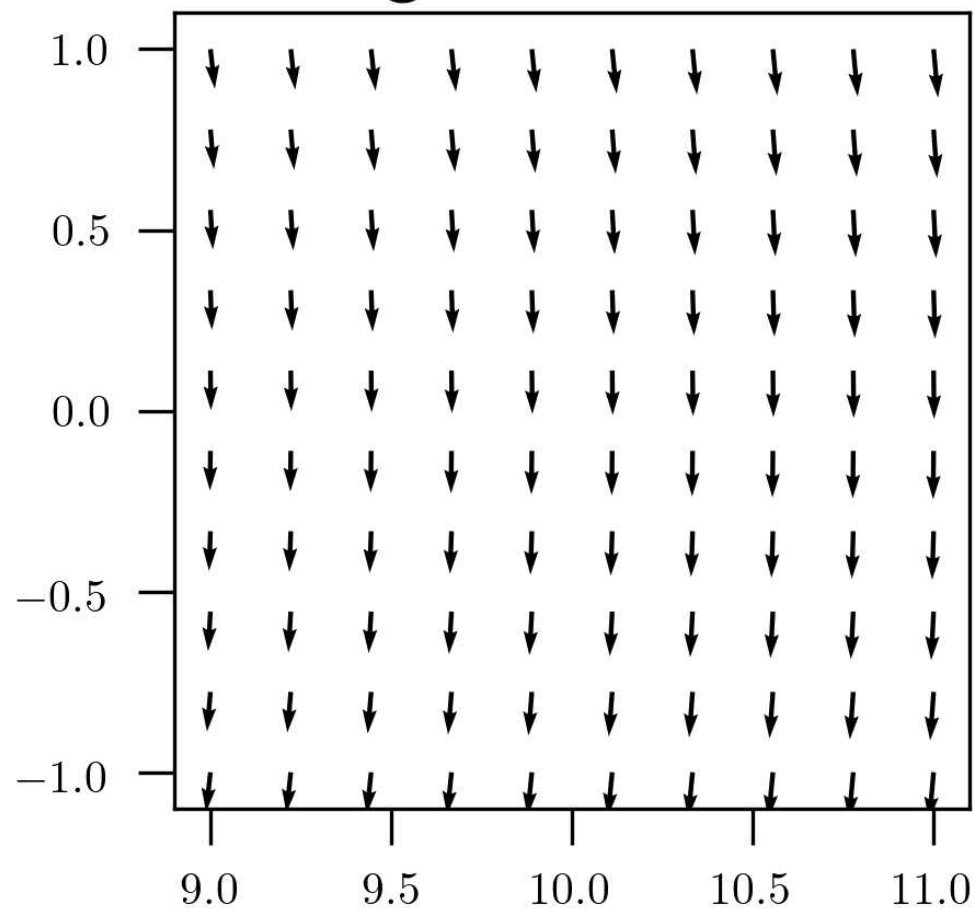
$$K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

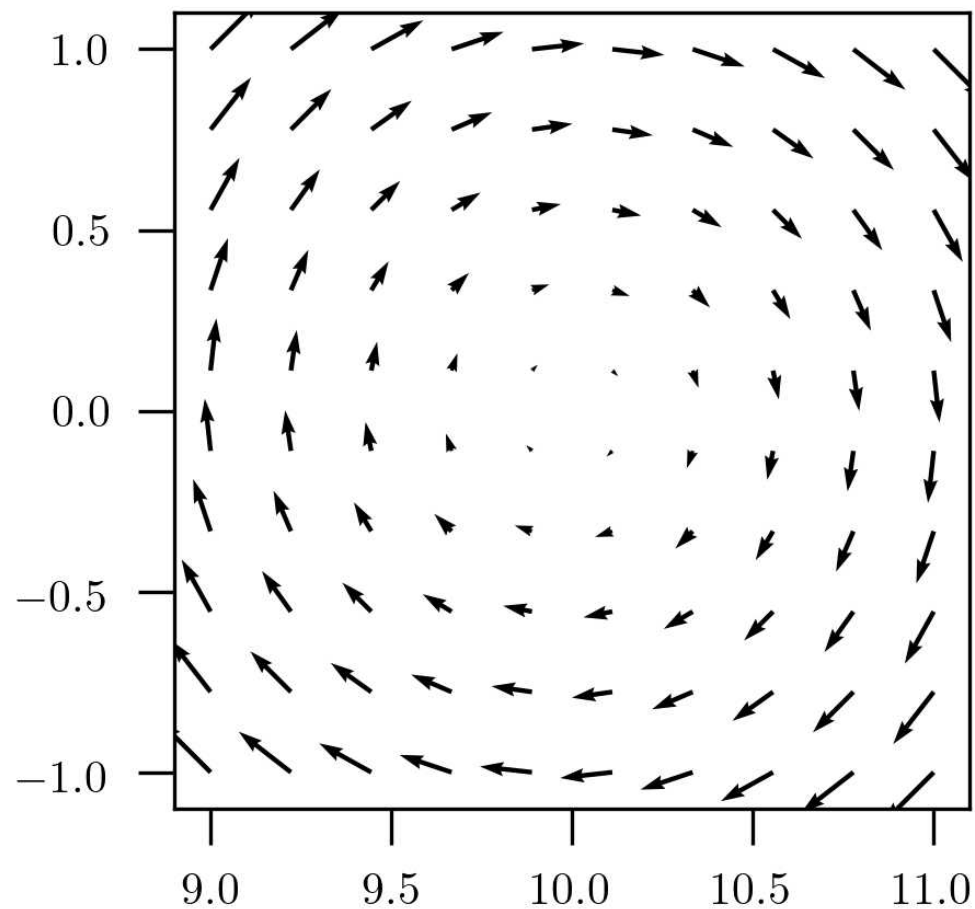
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



rigid rotation



differential velocities



Rigid rotation $\Omega = \text{cte}$

$$\frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \quad \frac{\partial v_r}{\partial r} = 0 \quad \frac{\partial \Omega}{\partial r} = 0$$

$$\begin{aligned} A &= 0 & C &= 0 \\ B &= -\Omega & K &= 0 \end{aligned}$$

$$A = \frac{1}{2} \left(\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$B = \frac{1}{2} \left(-\frac{V_\theta}{r} - \frac{\partial V_\theta}{\partial r} \right)$$

$$C = 0$$

$$K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

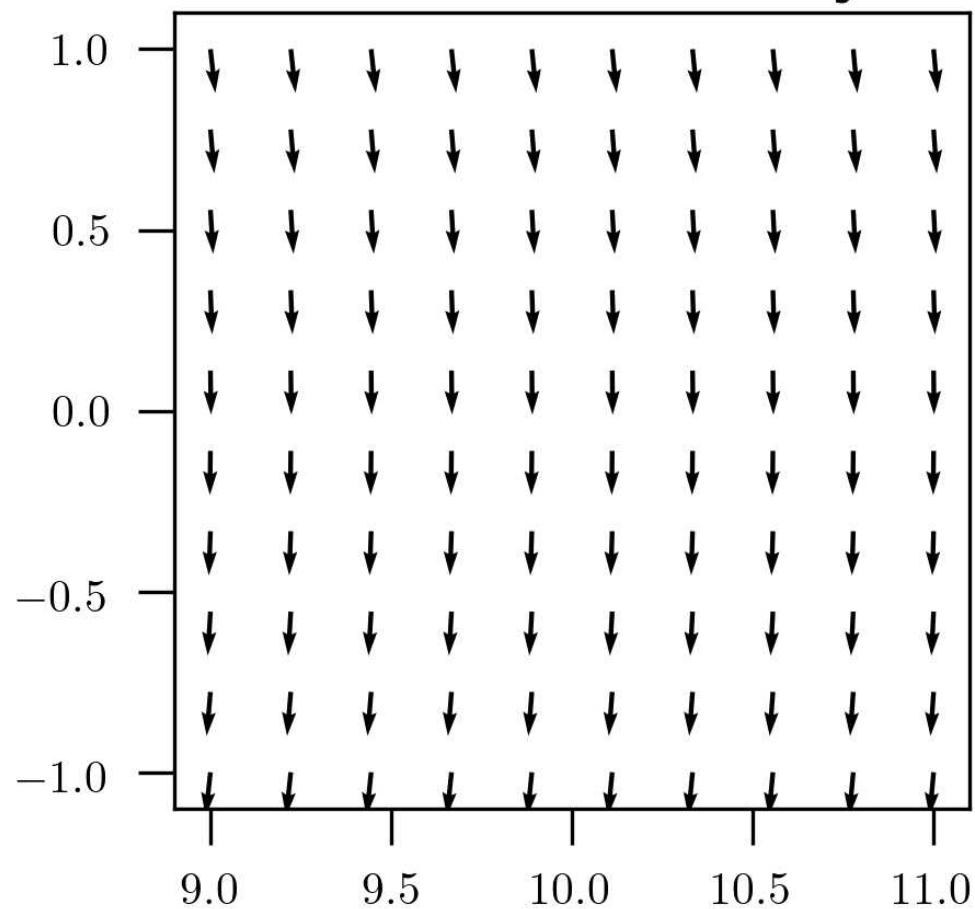
Constant rotation curve

$$\frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \quad \frac{\partial v_r}{\partial r} = 0 \quad \frac{\partial V_\theta}{\partial r} = 0$$

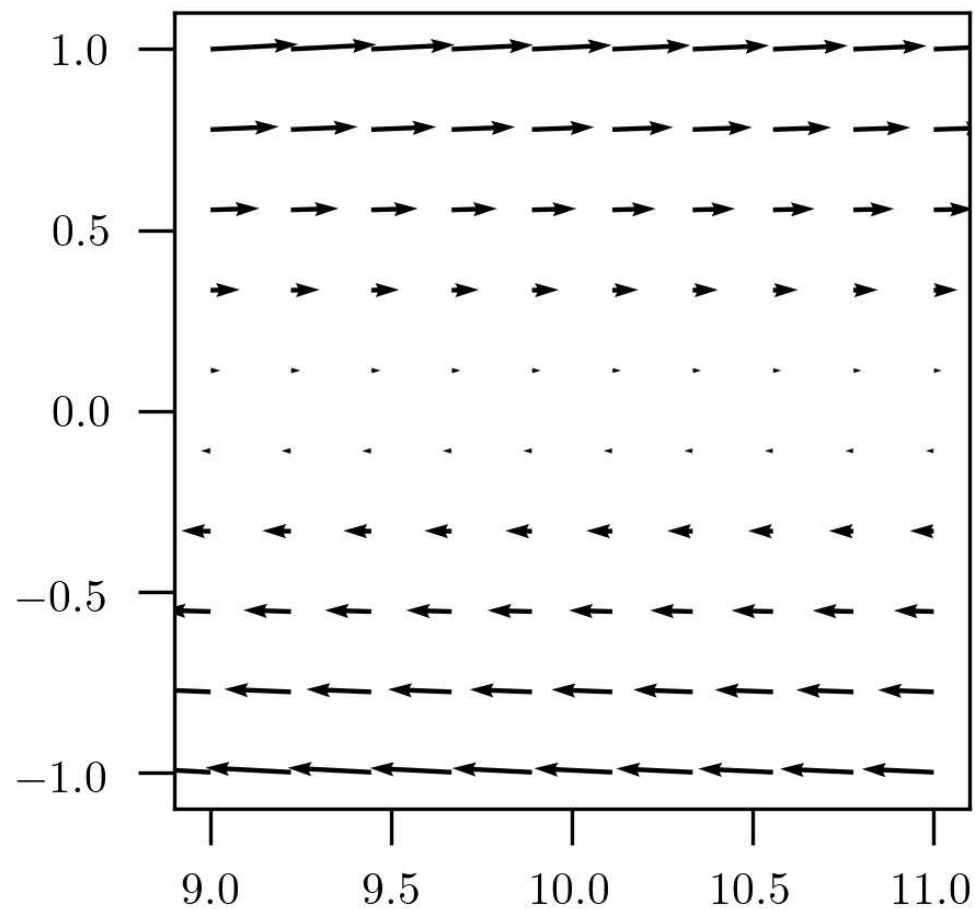
$$\begin{aligned} A &= \frac{1}{2} \Omega & C &= 0 \\ B &= -\frac{1}{2} \Omega & K &= 0 \end{aligned}$$

$$J = \begin{pmatrix} 0 & A-B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ 0 & 0 \end{pmatrix}$$

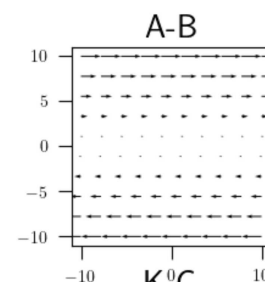
constant velocity



differential velocities



$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Keplerian decrease

$$\frac{\partial}{\partial \theta} = 0 \quad v_r = 0 \quad \frac{\partial v_r}{\partial r} = 0 \quad v_\theta \sim r^{-1/2}$$

$$A = \frac{1}{2} \left(\frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$B = \frac{1}{2} \left(-\frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r} \right)$$

$$C = 0$$

$$K = 0$$

$$A = \frac{3}{4} \Omega$$

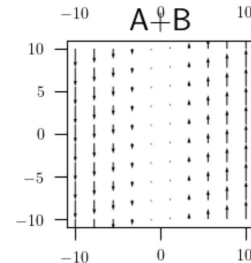
$$C = 0$$

$$B = -\frac{1}{4} \Omega$$

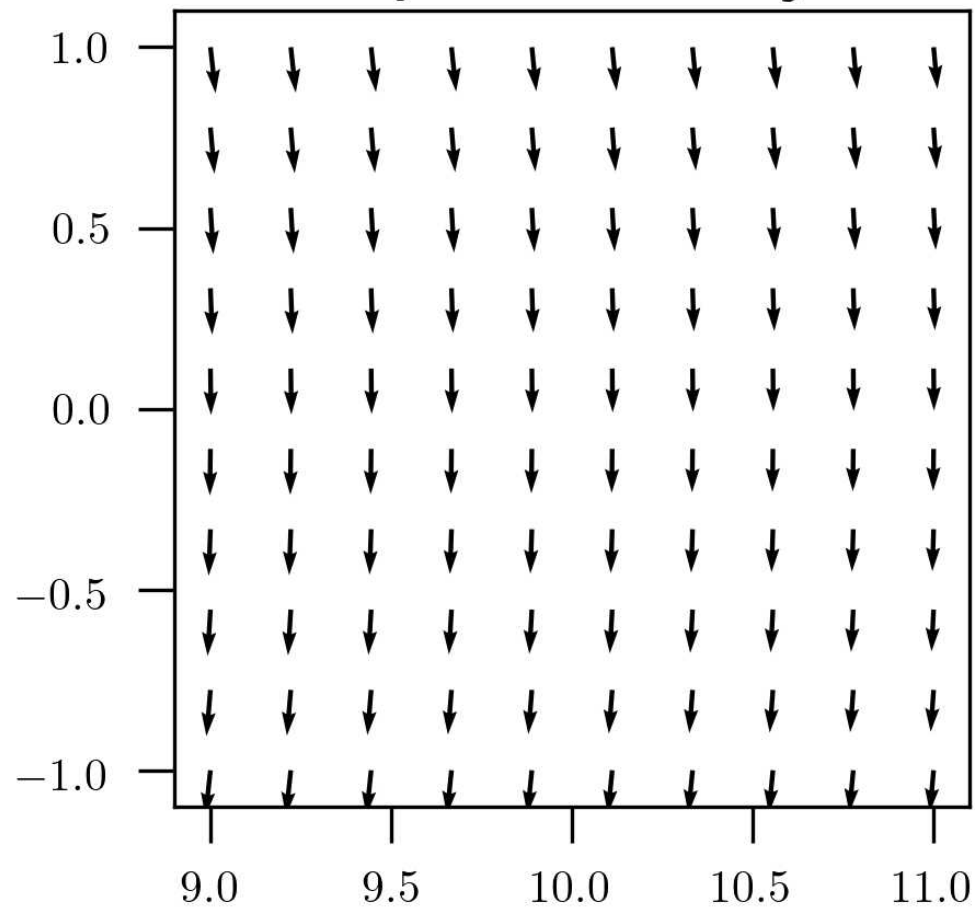
$$K = 0$$

$$J = \begin{pmatrix} 0 & A-B \\ A+B & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ \frac{1}{2}\Omega & 0 \end{pmatrix}$$

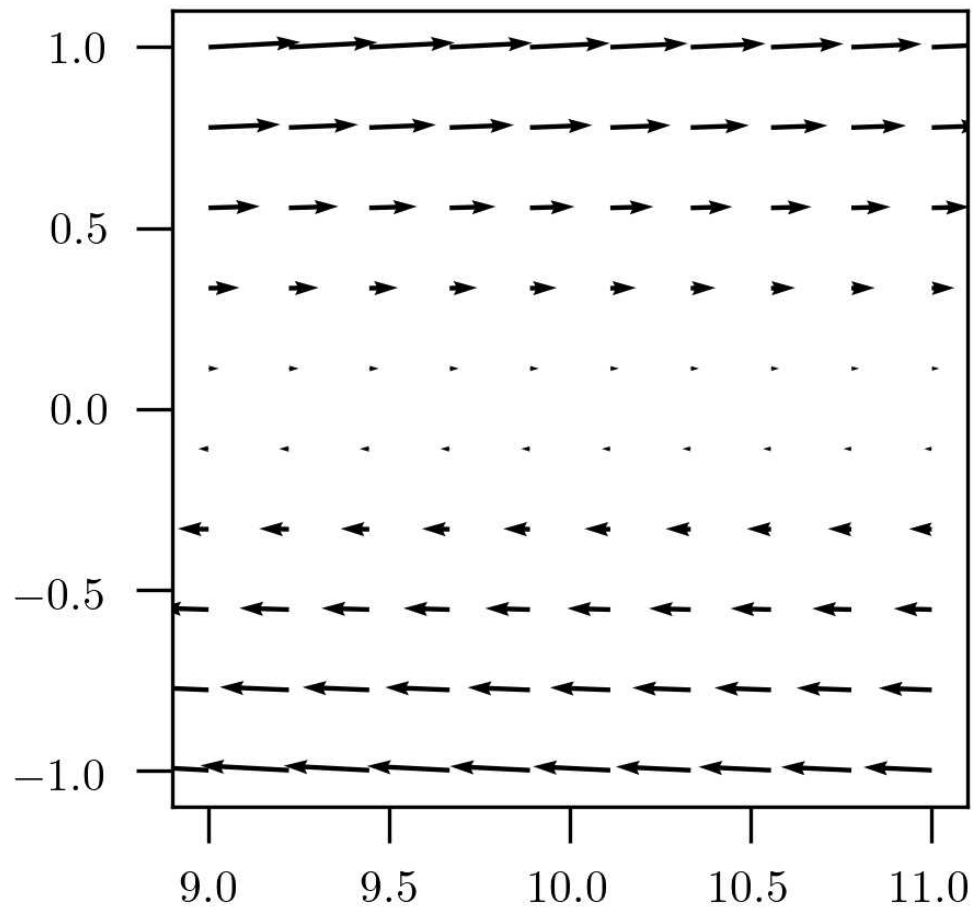
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



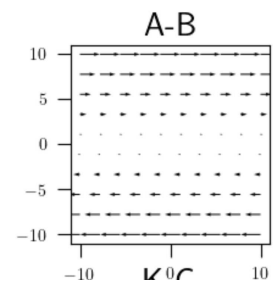
kepler velocity



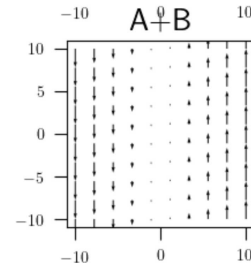
differential velocities



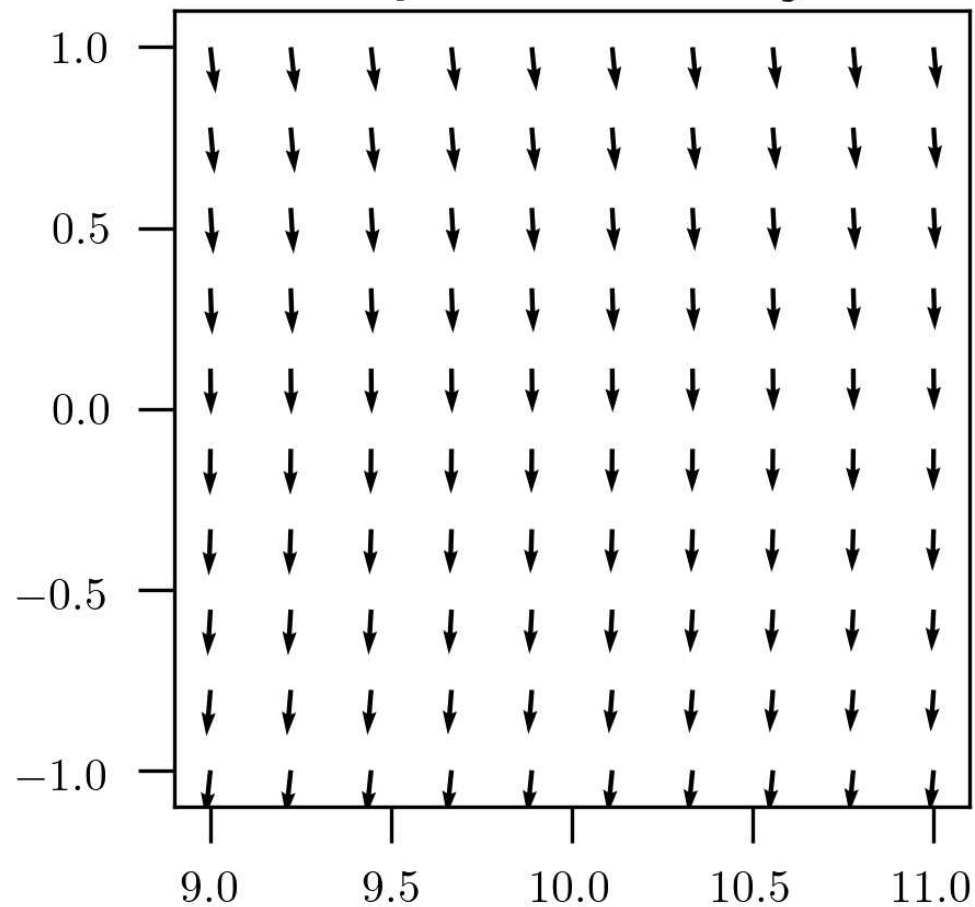
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



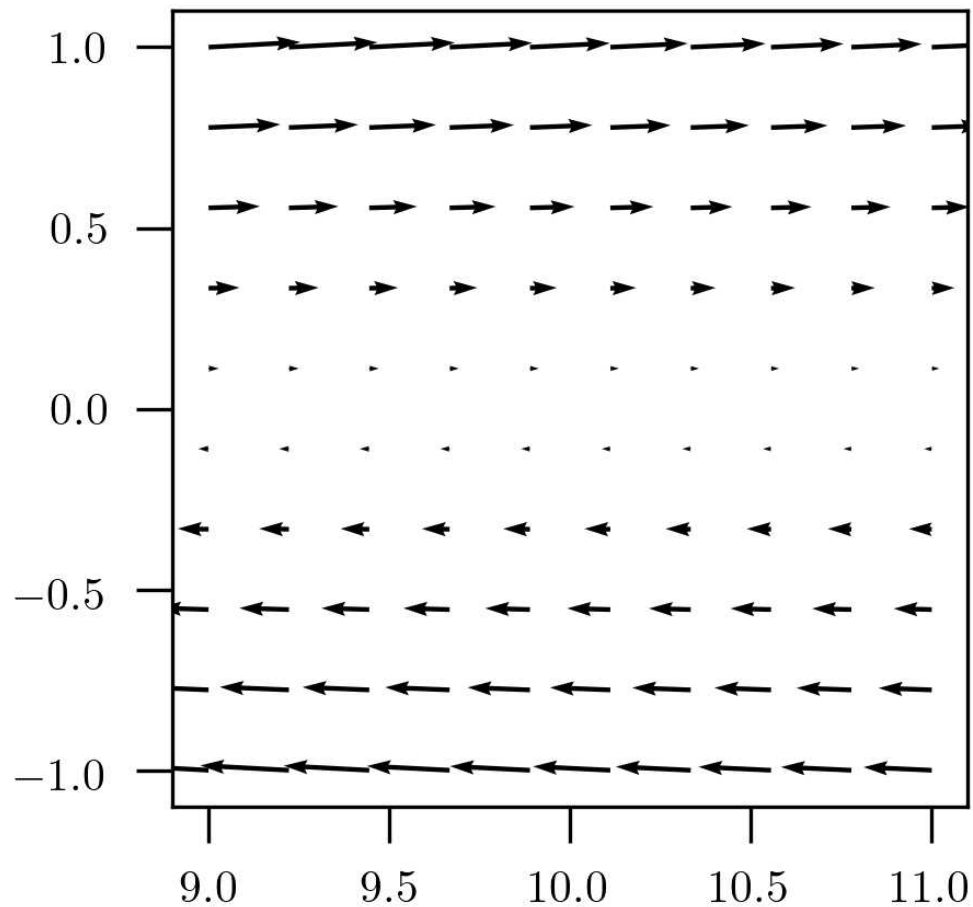
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



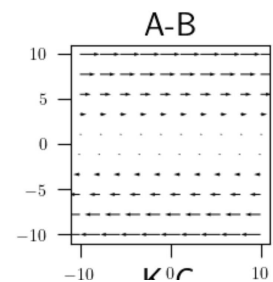
kepler velocity



differential velocities

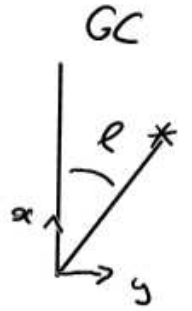


$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



Can we measure the Oort constants (at the Sun location) ?

In Galactocentric coordinates (ℓ, b, d)



with
$$\begin{cases} x = d \cos \ell \\ y = d \sin \ell \end{cases} \quad \begin{cases} v_x = (\kappa + C)x + (A - B)y \\ v_y = (A + B)x + (\kappa - C)y \end{cases}$$

$$\bullet \quad v_r = \vec{v} \cdot \frac{\vec{x}}{r} = \frac{1}{r} (x v_x + y v_y) \quad \equiv \quad v_x \cos \ell + v_y \sin \ell$$

$$v_r = d [\kappa + C \cos(2\ell) + A \sin(2\ell)]$$

$$\bullet \quad v_t = \left| \vec{v} \times \frac{\vec{x}}{r} \right| = \frac{1}{r} (x v_y - y v_x) \quad \equiv \quad -v_x \sin \ell + v_y \cos \ell$$

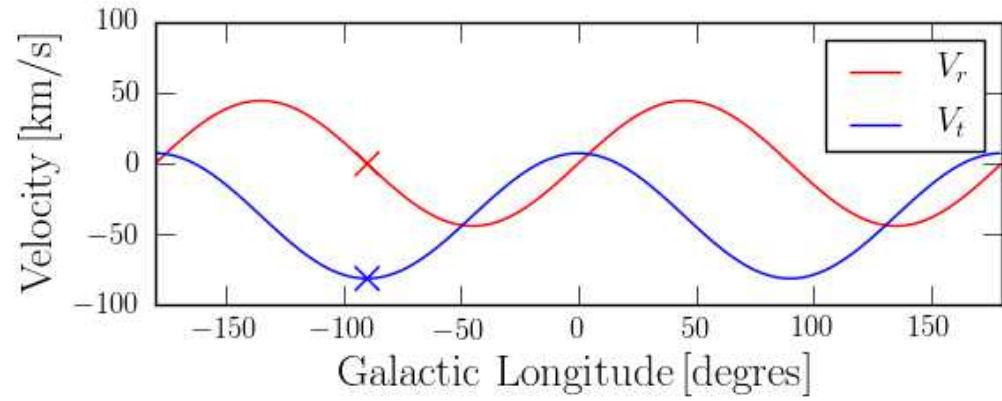
$$v_t = d [B + A \cos(2\ell) - C \sin(2\ell)]$$

In the axisymmetric case $C = \kappa = 0$
(purely circular orbits)

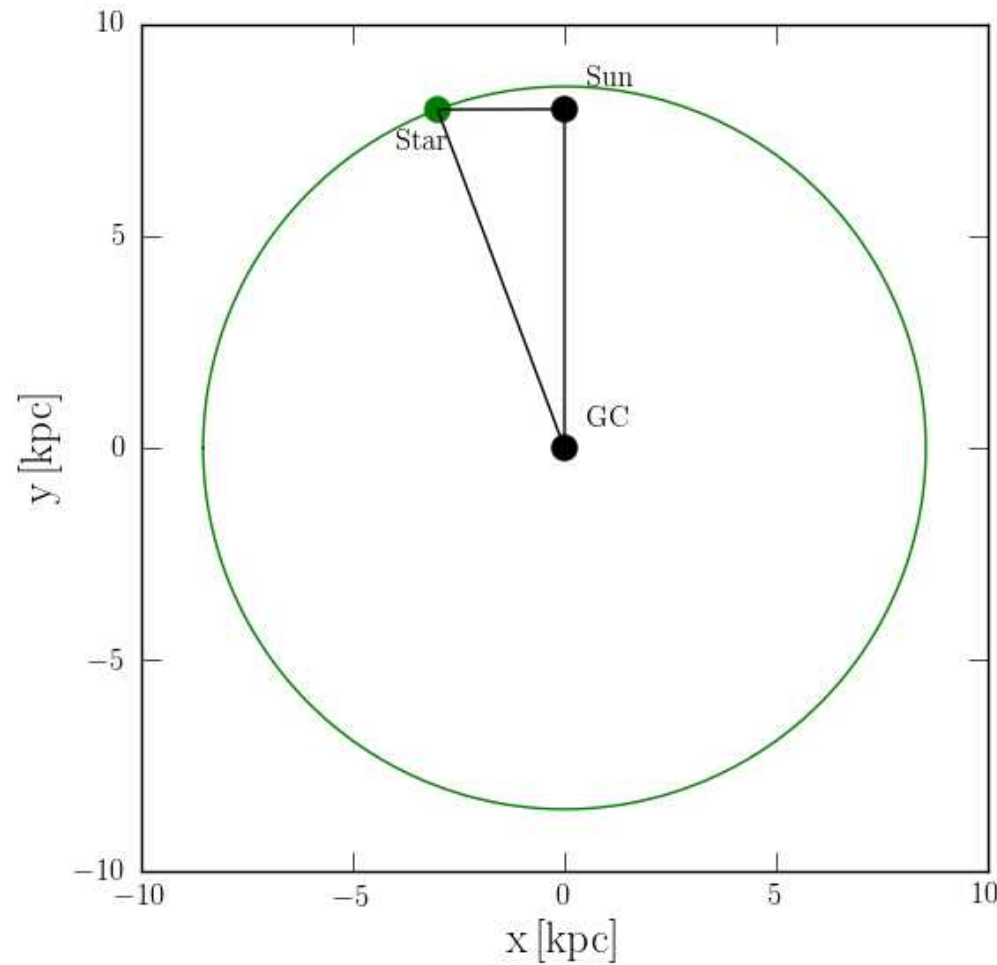
$$\begin{cases} V_r = A d \sin(2\ell) \\ V_t = A d \cos(2\ell) + B d \end{cases}$$

The Oort constants

$$l = -90$$



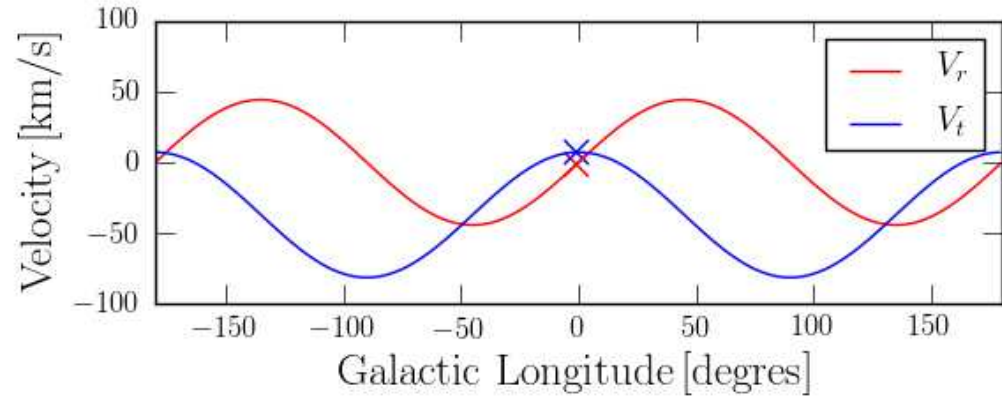
$$\begin{cases} V_r = Ad \sin(2l) \\ V_t = Ad \cos(2l) + Bd \end{cases}$$



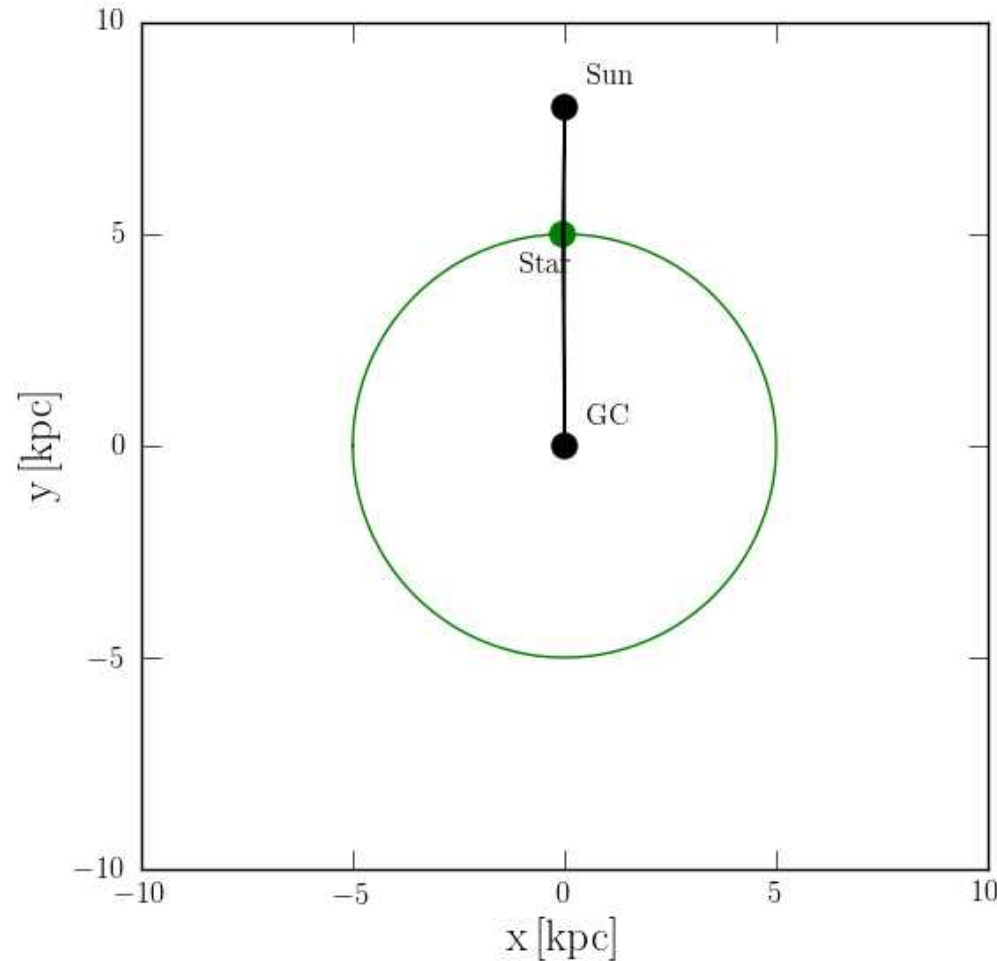
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

The Oort constants

$$l = 0$$



$$\begin{cases} V_r = Ad \sin(2l) \\ V_t = Ad \cos(2l) + Bd \end{cases}$$



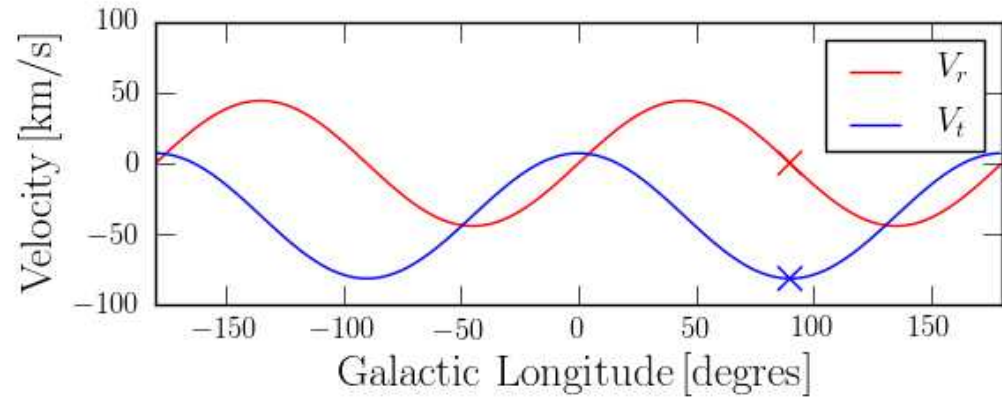
$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$

$$\Omega = \text{cte}$$

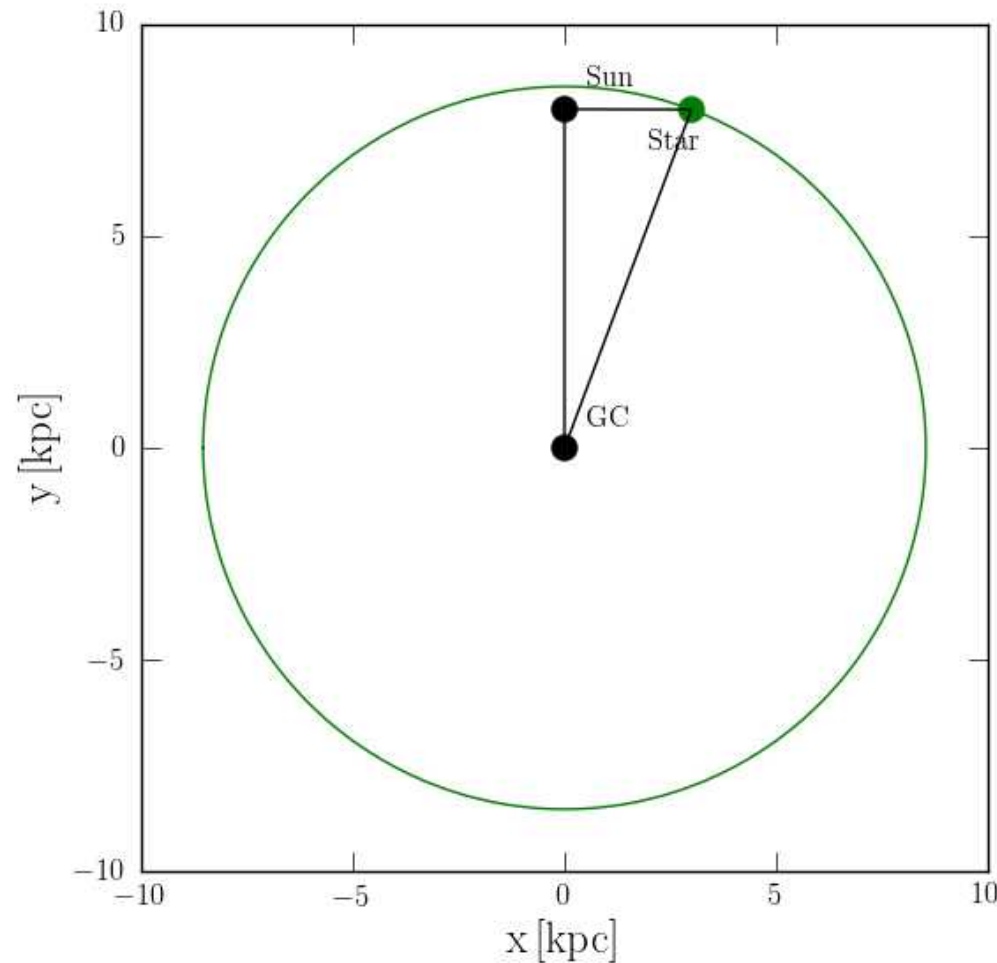
$$\begin{cases} V_r = 0 \\ V_t = -\Omega d \end{cases}$$

The Oort constants

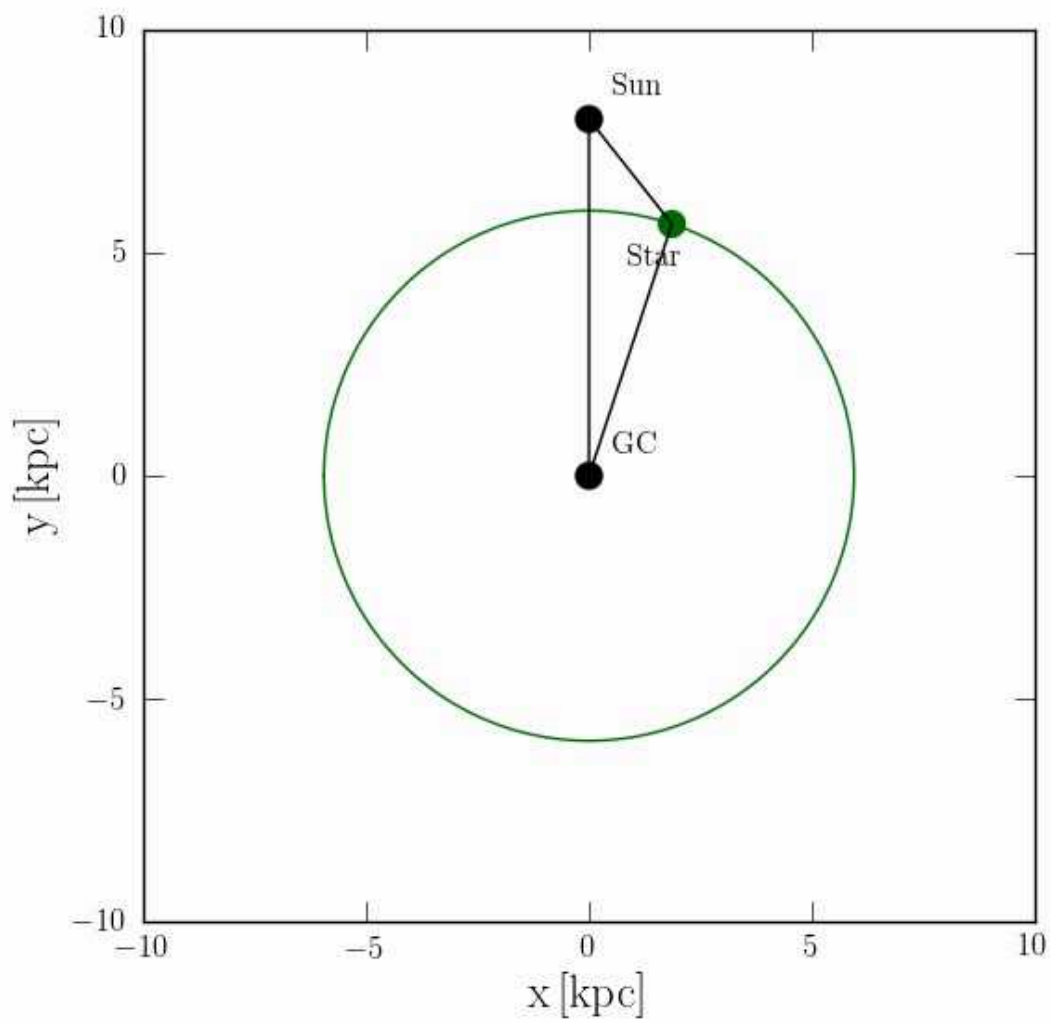
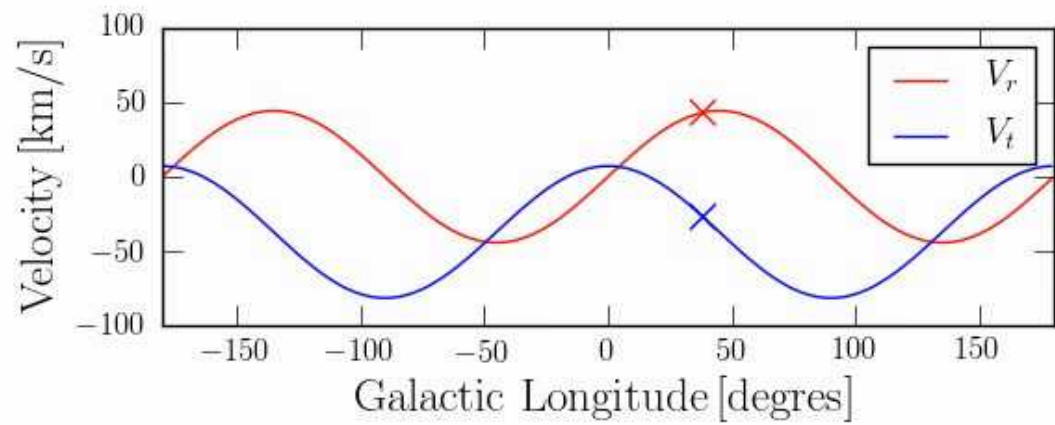
$$l = 90$$



$$\begin{cases} V_r = Ad \sin(2l) \\ V_t = Ad \cos(2l) + Bd \end{cases}$$



$$\begin{cases} V_r = 0 \\ V_t = (A + B)d \end{cases}$$



Proper motions measurements with GAIA

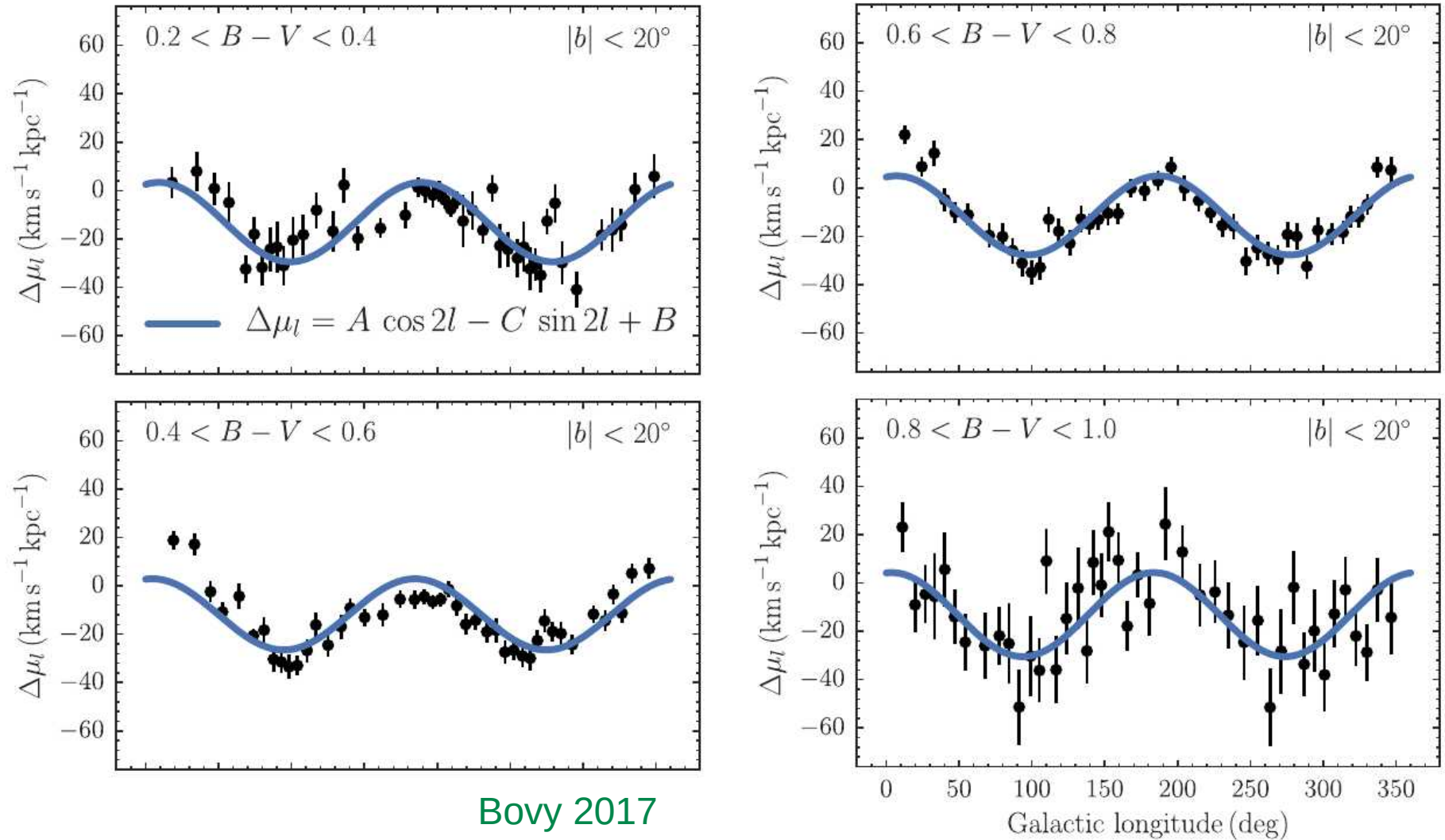


Figure 2. Comparison between the observed mean proper motion in Galactic longitude corrected for the solar motion (see equation 3) as a function of l and the best-fitting model for the four main colour bins used in the analysis. The data clearly display the expected signatures due to the differential rotation of the Galactic disc. The agreement between the model and the data is good.

Galactic rotation in *Gaia* DR1

The Oort constants

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$$A = 15.3 \pm 0.4 \text{ km s}^{-1} \text{ kpc}^{-1} \quad B = -11.9 \pm 0.4 \text{ km s}^{-1} \text{ kpc}^{-1}$$

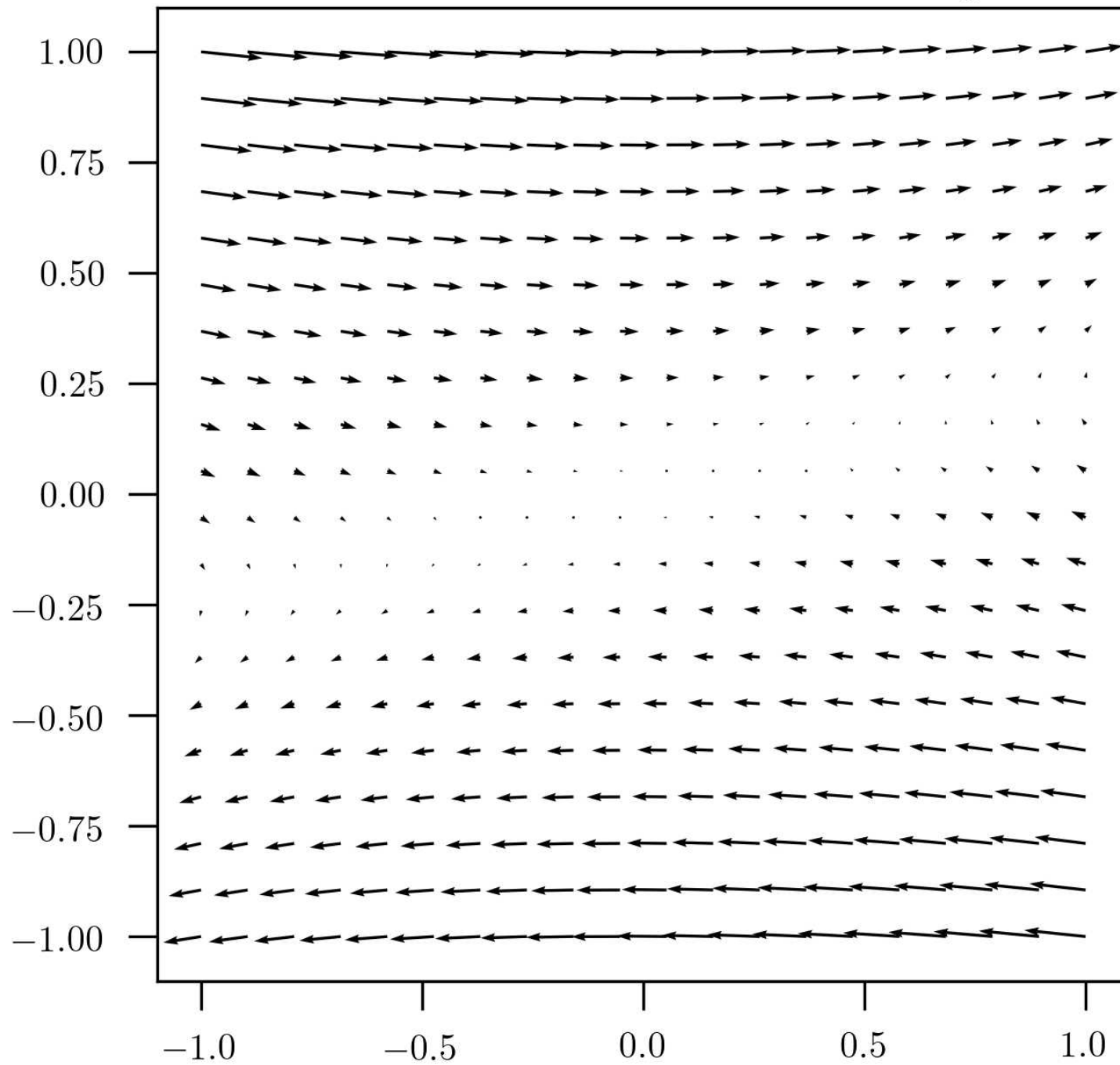
$$C = -3.2 \pm 0.4 \text{ km s}^{-1} \text{ kpc}^{-1} \quad K = -3.3 \pm 0.6 \text{ km s}^{-1} \text{ kpc}^{-1}$$

$$\text{using } \begin{cases} \Omega = A - B \\ \kappa^2 = -4B(A - B) = -4B\Omega \end{cases}$$

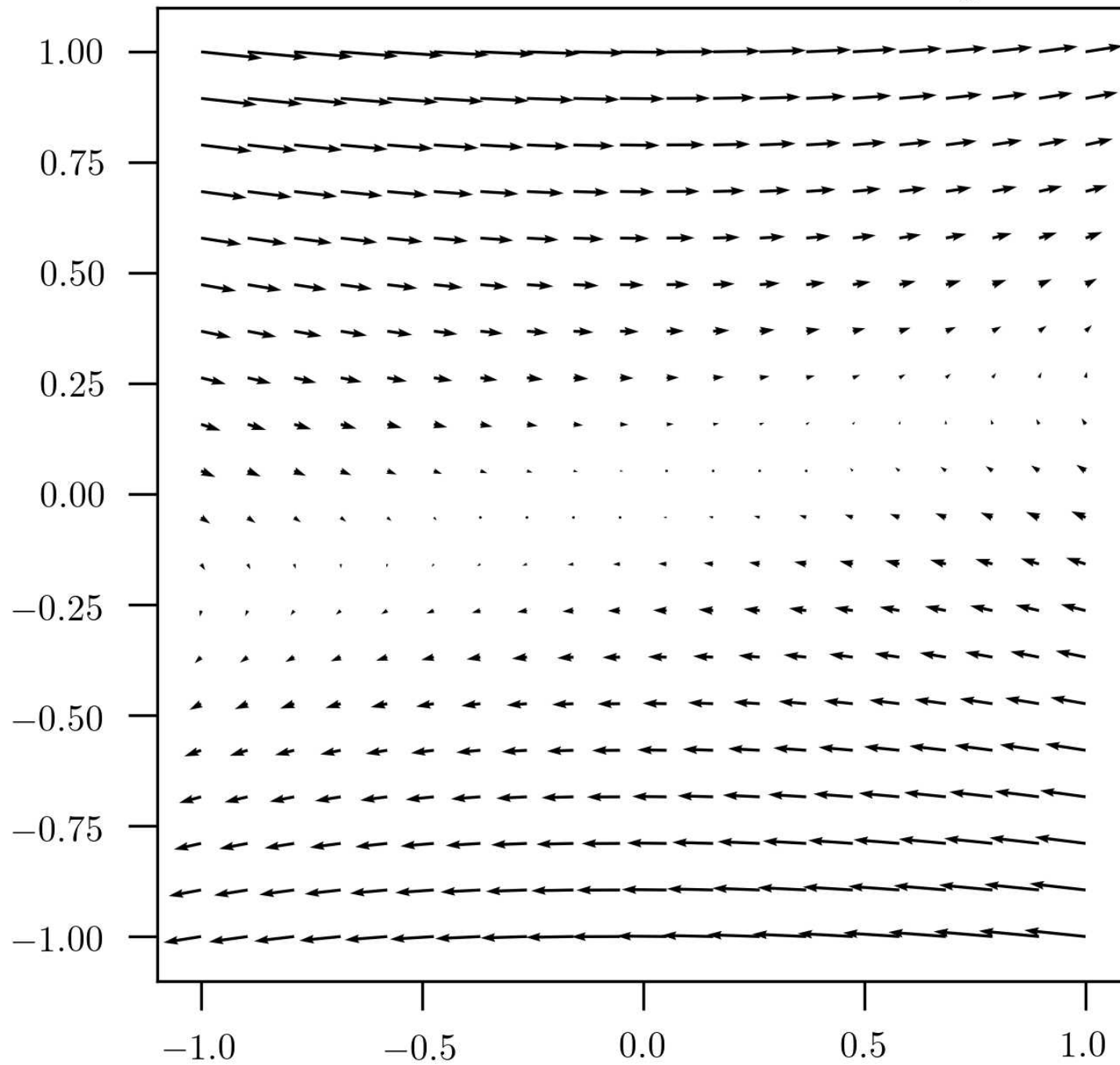
$$\begin{cases} \Omega_0 = 27 \pm 0.8 \text{ km s}^{-1} \text{ kpc}^{-1} \\ \kappa_0 = 35 \pm 0.2 \text{ km s}^{-1} \text{ kpc}^{-1} \end{cases} \quad \begin{cases} T_\phi \cong 227 \text{ Myr} \\ T_R \cong 175 \text{ Myr} \end{cases}$$

$$\frac{\kappa_0}{\Omega_0} = 2 \frac{-B}{A - B} \cong 1.29 \quad \kappa_0 > \Omega_0 \quad \text{as expected}$$

The local differential velocity field



The local differential velocity field



Stellar Orbits

ν

and the density relation

Galaxy properties from the vertical frequency ν

What can we learn from $\mathcal{R}$, Σ , ν ratios ?

Poisson equation in cylindrical coordinates

$$\nabla^2 \phi(R, z) = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \phi}{\partial R} \right) + \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho(R, z)$$

$\downarrow z=0$

$$\nabla^2 \phi(R, z) = \frac{1}{R} \frac{\partial}{\partial R} (V_c^2) + \nu^2 = 4\pi G \rho(R, z=0)$$

① if $\rho(R, z) \sim S(z) \Sigma(R)$

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \phi}{\partial R} \right) \ll \frac{\partial^2 \phi}{\partial z^2}$$

② if $V_c = \text{cte}$,

$$\frac{1}{R} \frac{\partial}{\partial R} (V_c^2) = 0$$

so,

$$\nu^2 = 4\pi G \rho(R, z=0)$$

①

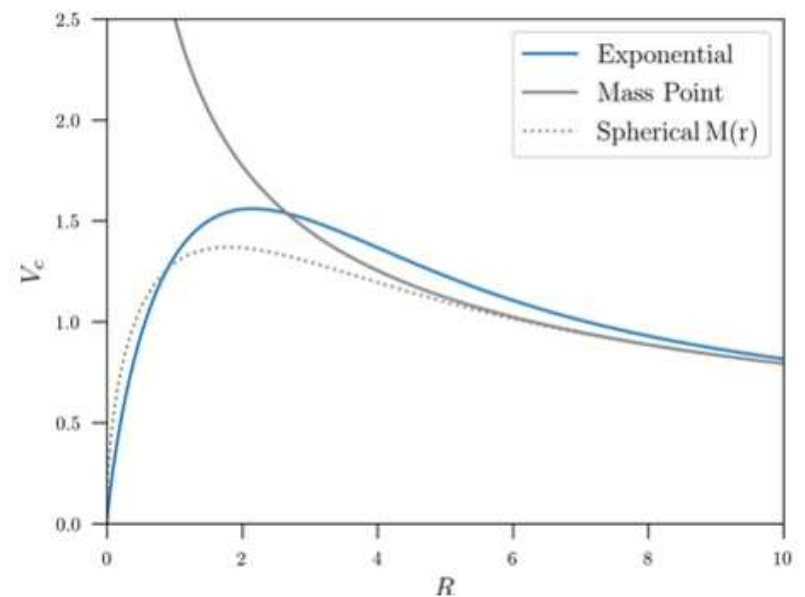
Expected relation between Ω and V ?

In spherical systems : $\Omega^2 = \frac{GM(r)}{r^3} = \frac{4}{3} \pi G \bar{\rho}$ $\bar{\rho} = \frac{M(r)}{\frac{4}{3} \pi r^3}$

As V_c for a cylindrical distribution is not so different than a spherical one if $M(r) = M(R)$ and $\Omega \sim \frac{1}{V_c}$

For an axisymmetric disk, we can estimate

$$\Omega^2 = \frac{GM(R)}{R^3} = \frac{4}{3} \pi G \bar{\rho} \quad (2)$$



For the flat rotation curve part, we have

$$\mathcal{L}^2 = \frac{1}{R} \frac{\partial}{\partial R} (V_c^2) + 2 \mathcal{L}^2 \quad \Rightarrow \quad \mathcal{L}^2 = 2 \mathcal{L}^2 \quad (3)$$

Combining (1) + (2) + (3)

$$\frac{V^2}{\mathcal{L}^2} = \frac{3}{2} \frac{\rho}{\bar{\rho}}$$

← density in the plane

← mean density computed
inside the radius

Estimation of the vertical frequency

From $\frac{\nu^2}{\kappa^2} = \frac{3}{2} \frac{\rho_d}{\bar{\rho}}$

$$\frac{4}{3}\pi G \bar{\rho} = \Omega_{\odot}^2 = \frac{V_{c,\odot}^2}{R_{\odot}^2} \quad \left\{ \begin{array}{l} V_{c,\odot} \cong 200 \text{ km/s} \\ R_{\odot}^2 \cong 8 \text{ kpc} \end{array} \right.$$

$$\Rightarrow \bar{\rho} \cong 0.039 \frac{M_{\odot}}{\text{pc}^3}$$

and with $\rho_d \cong 0.1 \frac{M_{\odot}}{\text{pc}^3}$

$$\left\{ \begin{array}{l} T_{\phi} \cong 227 \text{ Myr} \\ T_R \cong 175 \text{ Myr} \end{array} \right.$$

$$\frac{\nu}{\kappa} \cong 2$$

$$T_z = \frac{T_R}{2} \cong 87 \text{ Myr}$$

Stellar Orbits

**Integral of motion and
Surfaces of section**

Integrals of motion

A stellar orbit defines a path in the 6-D phase space ($x, y, z, \dot{x}, \dot{y}, \dot{z}$ in cartesian coordinates)

Definition :

An integral of motion $I [\mathbf{x}, \mathbf{v}]$ is any function of the phase-space coordinates alone that is constant along an orbit:

$$I [\mathbf{x}(t_1), \mathbf{v}(t_1)] = I [\mathbf{x}(t_2), \mathbf{v}(t_2)]$$

Examples :

- Hamiltonian

$$H(x, y, z, \dot{x}, \dot{y}, \dot{z}) = E$$

- Total angular momentum

$$\vec{L} : L_x = \text{cte}, L_y = \text{cte}, L_z = \text{cte}$$

- z-component of the angular momentum

$$L_z = \text{cte}$$

Remarks :

- Orbits may have between 0 to 5 integrals of motion.
- Integrals of motion may exist without an analytical form.

Integrals of motion

Interest of integrals of motion :

Restrict the study of an orbit to a subset of the phase space

Example I :

Orbit in spherical potentials

- 6-D 6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$
- Angular momentum conservation
3 integrals, 2 among the three

$\vec{n} = \vec{L}/|\vec{L}|$ defines a plane $\rightarrow$ 4-D

4 indep. variables $(r, \phi, \dot{r}, \dot{\phi})$

- Angular momentum conservation + energy

2 indep. variables (r, ϕ)

$$L = |\vec{L}| \quad E \rightarrow \text{2-D}$$

$$\dot{\phi} = \frac{L}{r^2}$$

defines a 2-D surface
in the phase space

$$\dot{r} = \pm \sqrt{2(E - \Phi(r)) - L^2/r^2}$$

Given $E, \vec{L}$ the position and velocities of a star (i.e. the position in the phase space)
is fully determined by providing two additional quantities, eg: r, ϕ

Integrals of motion

Is there a fifth integral ?

Example of the Keplerian potential :

We showed that:

$$r(\phi) = \frac{1}{C \cos(\phi - \phi_0) + \frac{GM}{L^2}}$$

with:

$$E = \frac{1}{2} (C L)^2 - \frac{1}{2} \left(\frac{GM}{L} \right)^2$$

we have then the new integral of motion:

$$\phi_0(r, \phi) = \phi - \arccos \left[\frac{1}{C(L, E)} \left(\frac{1}{r} - \frac{GM}{L^2} \right) \right]$$

→ **1-D**
a curve

1 indep. variable (r)

Given $E, \vec{L}, \phi_0$ the position and velocities of a star is fully determined by providing only one additional quantities, eg: r

Integrals of motion

Example II :

Orbit in axi-symmetric potentials

- **6-D** 6 indep. variables $(x, y, z, \dot{x}, \dot{y}, \dot{z})$
- z-component angular momentum conservation
1 integral $\dot{\theta} = \frac{L_z}{R^2}$ 5 indep. variables $(R, z, \dot{R}, \dot{z}, \theta)$
- Initial azimuth $\theta(t) = L_z \int_{t_0}^t \frac{1}{R^2(t')} dt' + \theta_0$ 4 indep. variables $(R, z, \dot{R}, \dot{z})$
4-D (meridional plane)
- Energy E not an integral, a constant ! 3 indep. variables $(R, z, \dot{R})$
3-D

Given E, L_z, θ_0 and t the position and velocities of a star (in the phase space) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Given E the position and velocities of a star (in the phase space of the meridional plane) is fully determined by providing three additional quantities, ex: $R, z, \dot{R}$

Is there a third integral ?

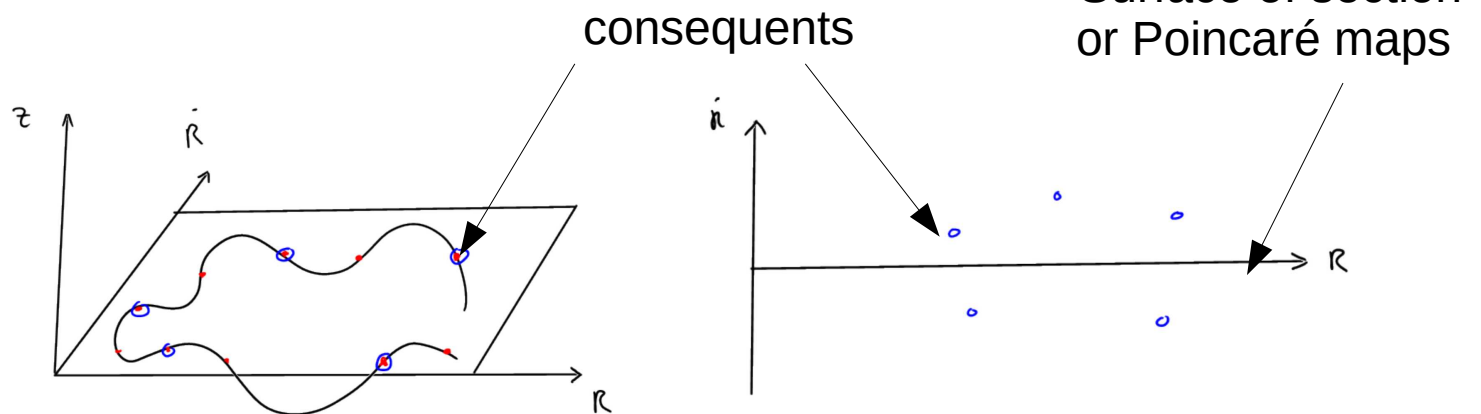
Surfaces of section

Can we visualize the phase space and check if an additional integral of motion exists ?

Idea :

We study the orbits in the meridional plane

- 4-D 4 indep. variables $(R, z, \dot{R}, \dot{z})$
- Energy E 3 indep. variables $(R, z, \dot{R})$
- 3-D
- Drawing a 3-D phase space is still not easy. Instead, we draw slices of the phase space. We plot only phase space points that:
 - cross the $z = 0$ plane
 - have $\dot{z} > 0$

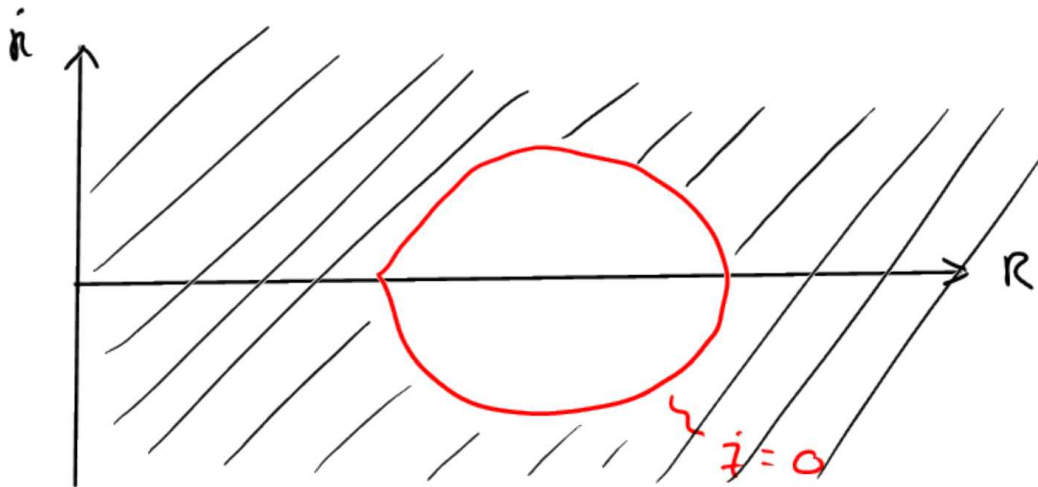


Surfaces of section

- A point in the surface of section (for a given E and L_z) defines an orbit as the three independent variables ($R, \dot{R}, z = 0$) are defined.
- Even if orbits have the same energy, they will never intersect in the plane (EoM are first order diff. equations).
- Zero velocity curve : curve defined by $\dot{z} = 0$

$$E = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \Phi_{\text{eff}}(R, z = 0) \quad \Rightarrow \quad \dot{R} \leq \pm \sqrt{2[E - \Phi_{\text{eff}}(R, z = 0)]}$$

$\dot{R}(R) = \pm \sqrt{2[E - \Phi_{\text{eff}}(R, z = 0)]}$ defines the accessible region of the phase space

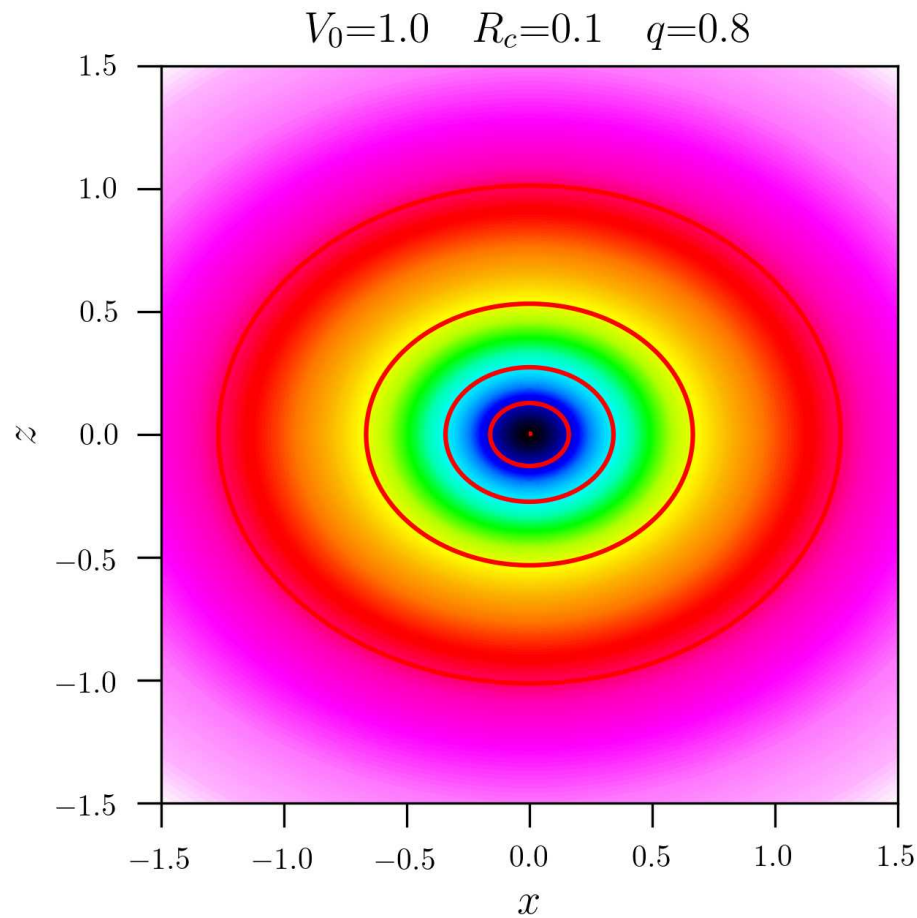


Surfaces of section

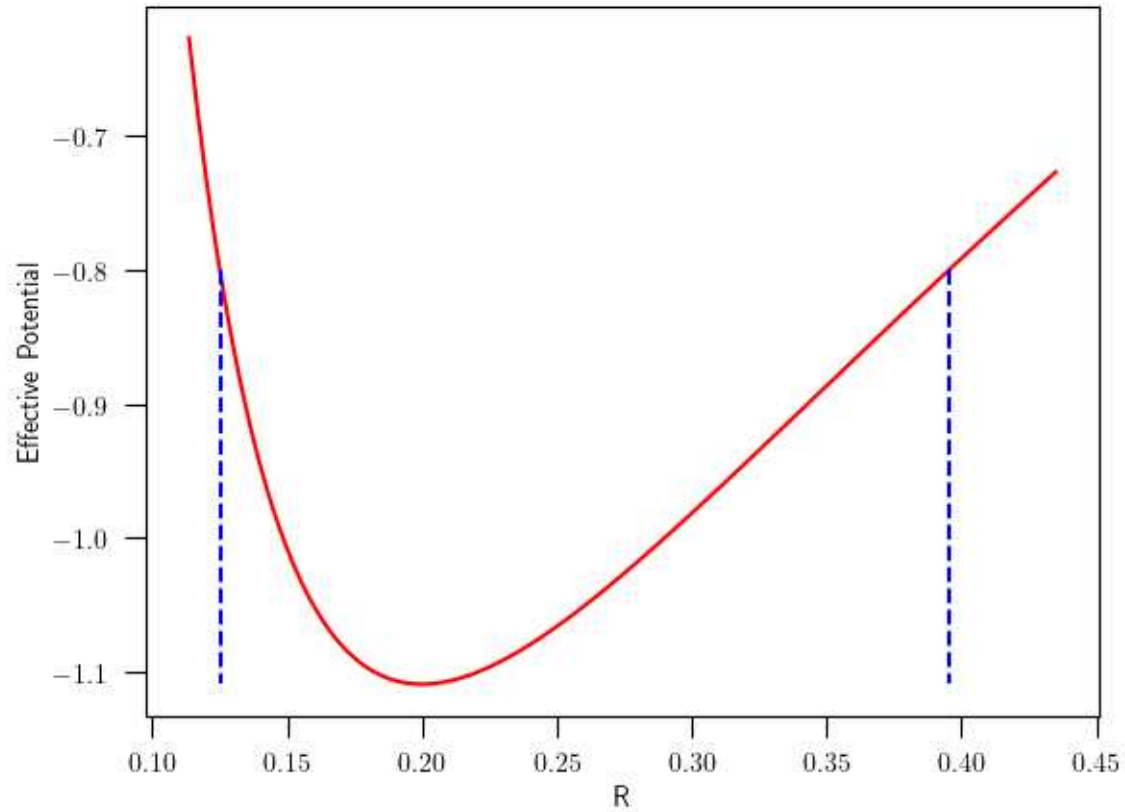
Examples

Logarithmic potential

$$\Phi_{\log}(R, z) = \frac{1}{2} V_0^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q^2} \right)$$

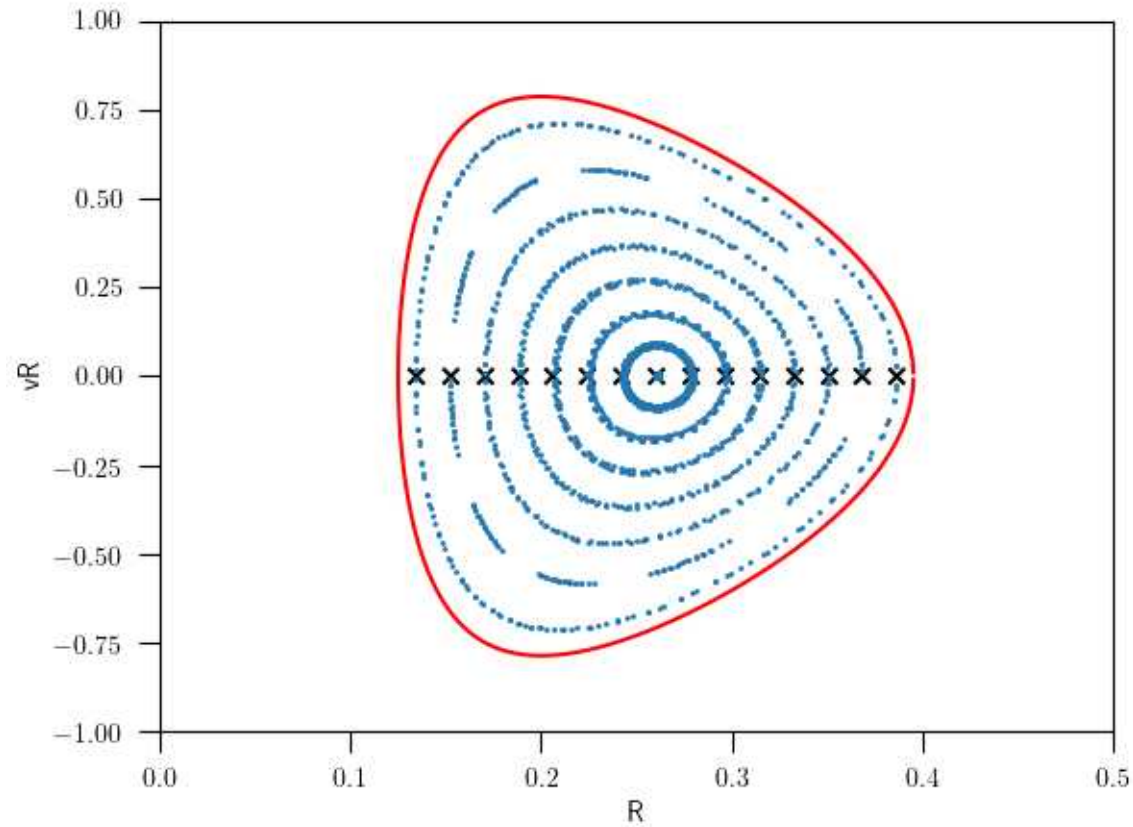


Effective Potential



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --plotpotential
```

Invariant curves : Third Integral



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --norbits 30 --nlaps 1000
```

The Third Integral I (I is in general non analytical)

Spherical systems : $|\vec{L}| \equiv L$ is conserved

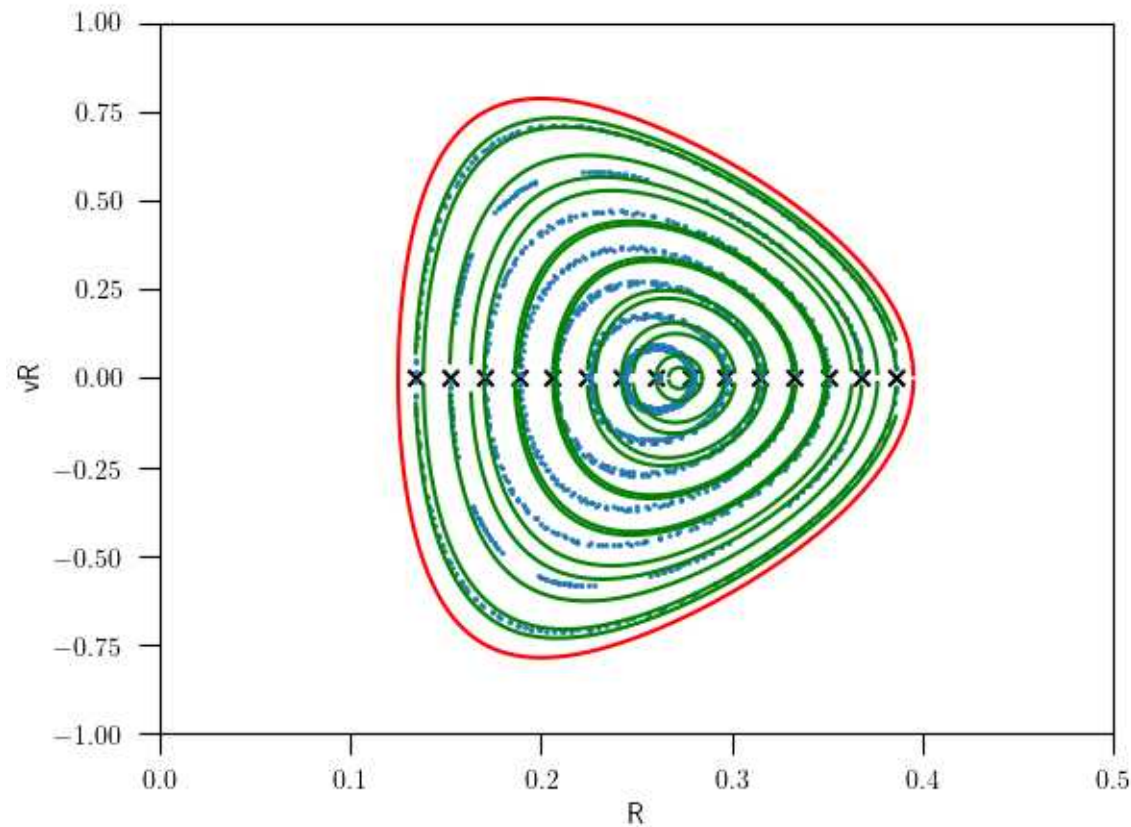
Nearly spherical potential : L is nearly an integral $\approx I$?

What is the curve in the Poincaré map that satisfies $L = \text{cte}$?

EXERCICE

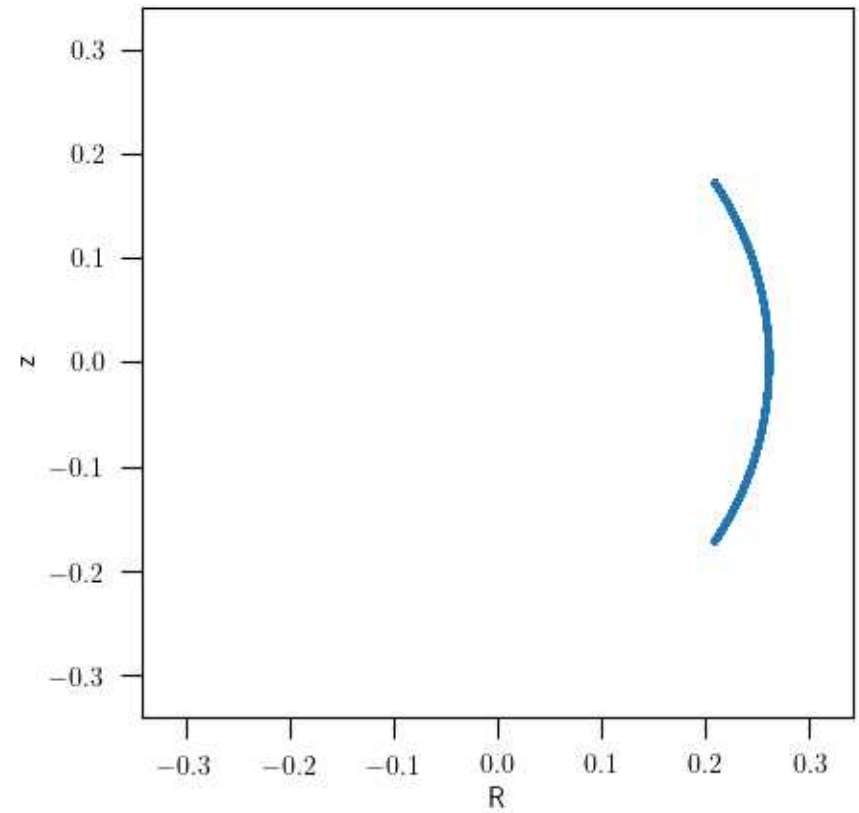
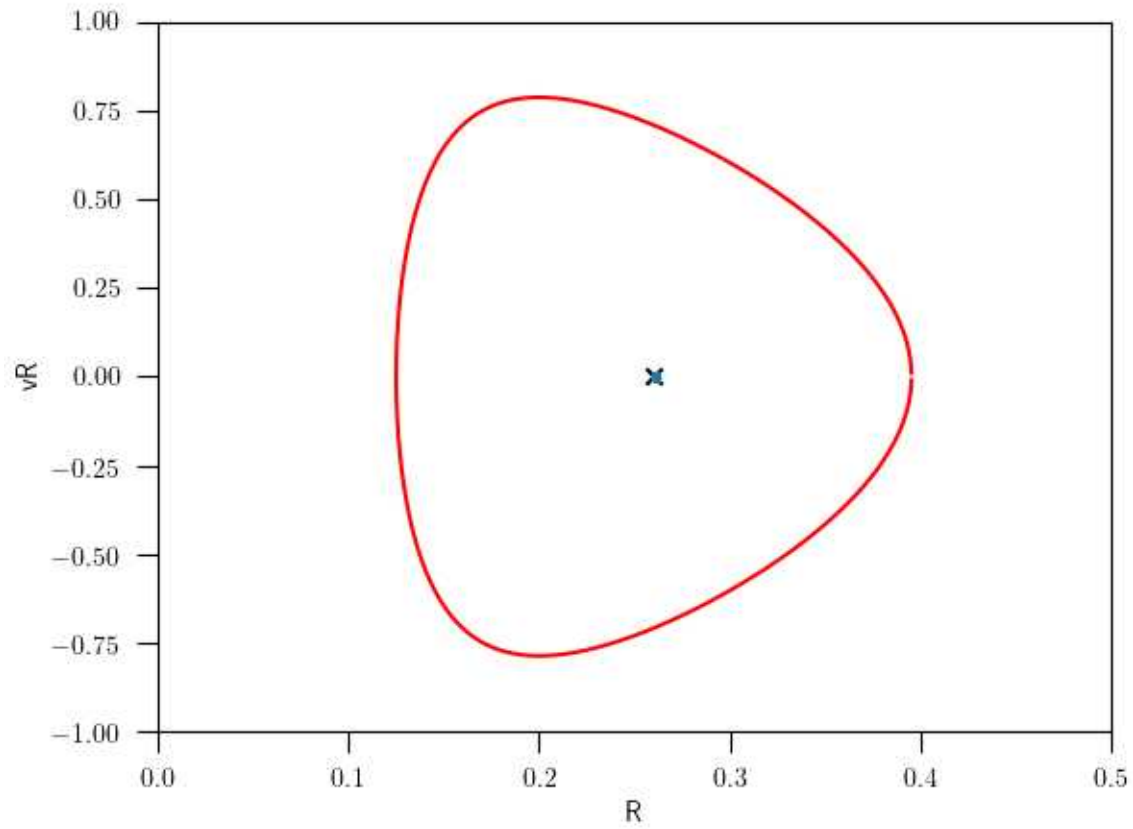
Invariant curves : Third Integral

green : contours of constant total angular momentum



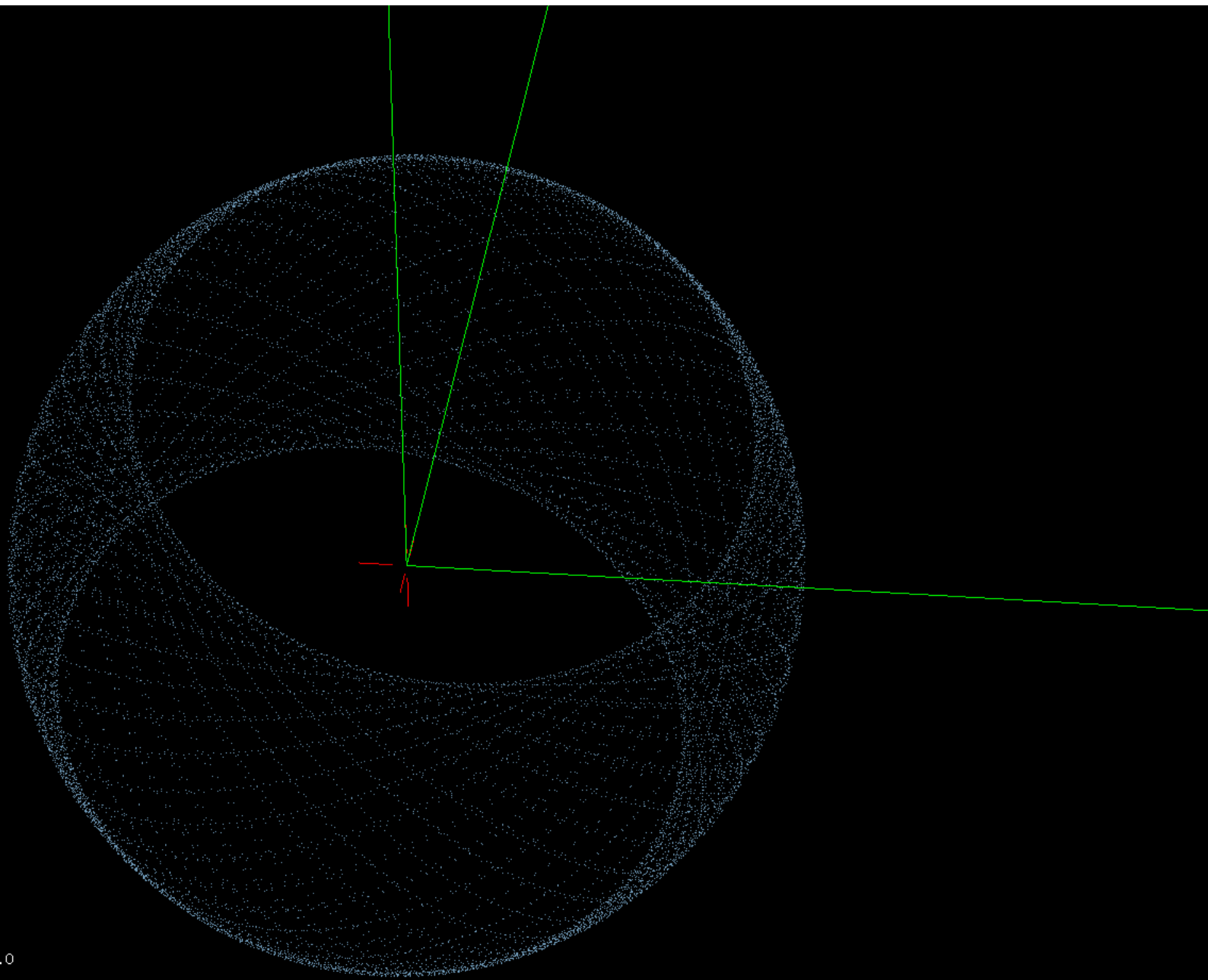
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --norbits 15 --nlaps 100 --add_IL
```

shell orbit



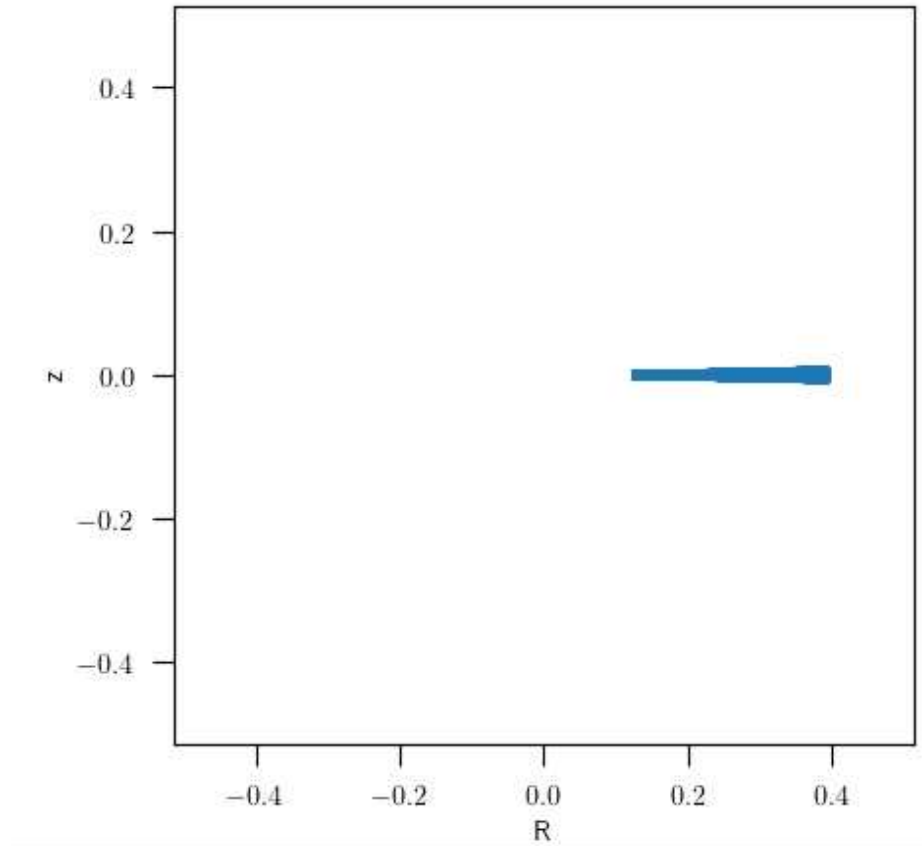
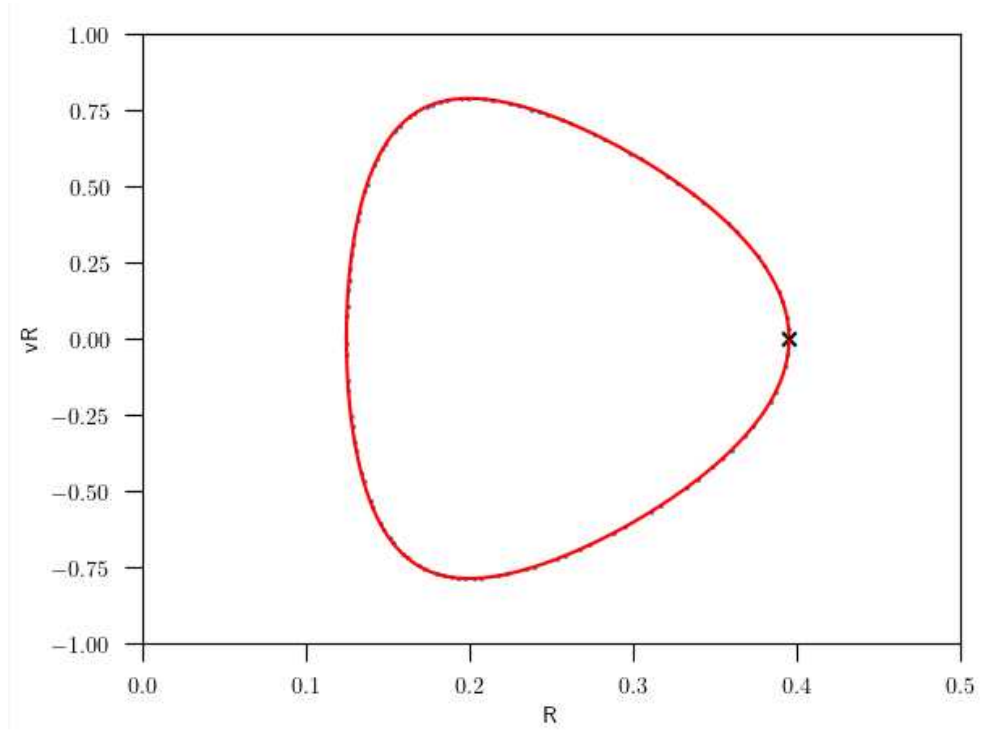
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --nlaps 100 --R 0.2612
```

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.1 10.8
Near/Far factor : 0.100 10.000



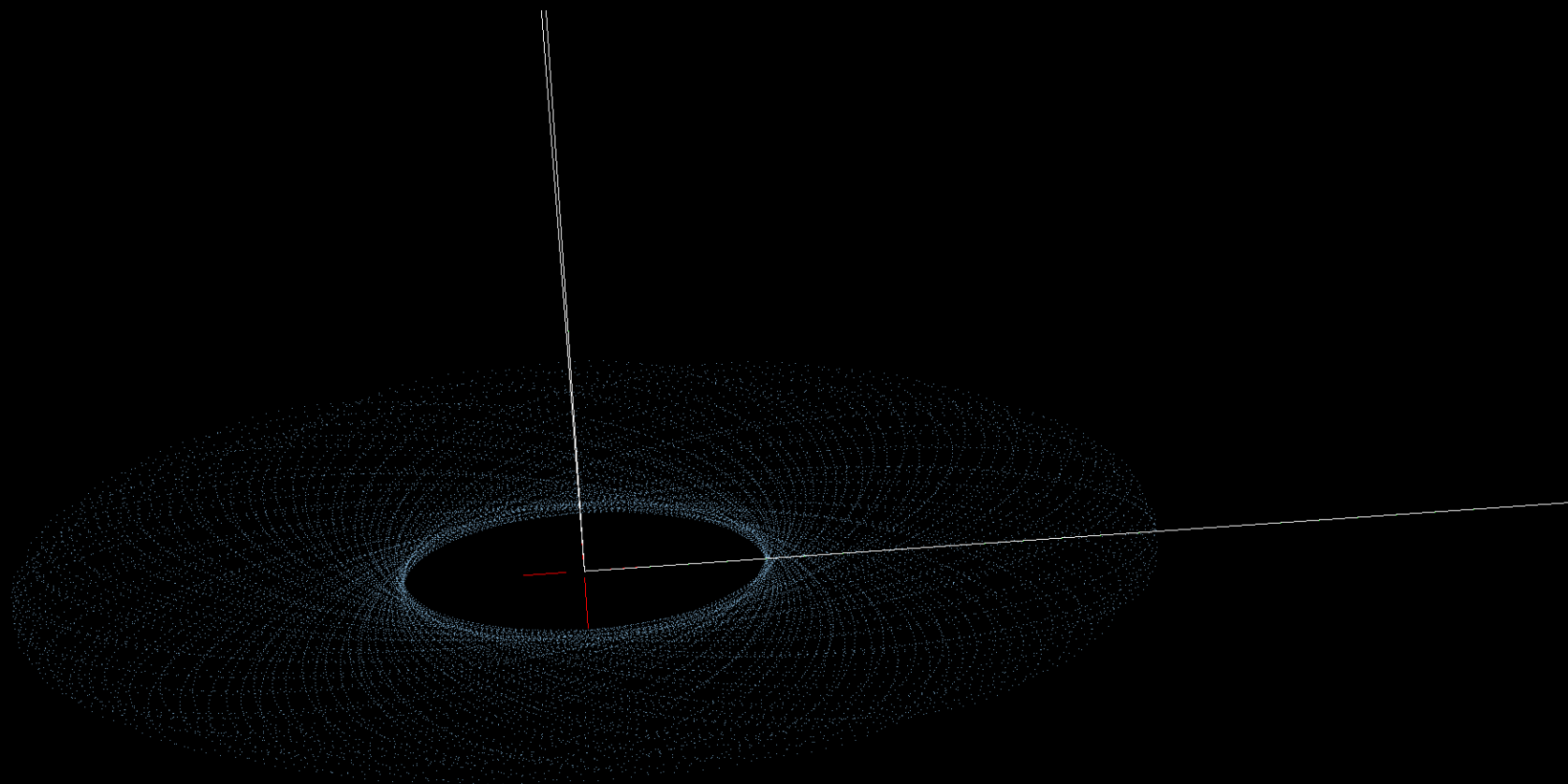
Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= 183 y= 0
Dist to IntP : d= 1.077
Observer pos : x= -0.1 y= -0.6 z= 0.9
IntP pos : x= 0.0 y= 0.0 z= -0.0

Large radius



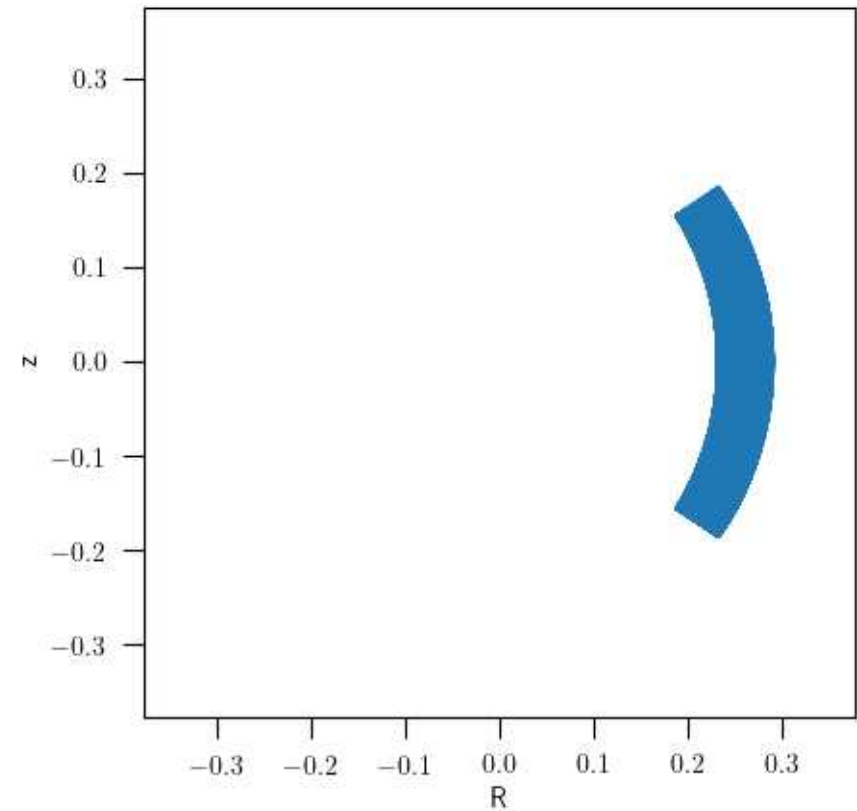
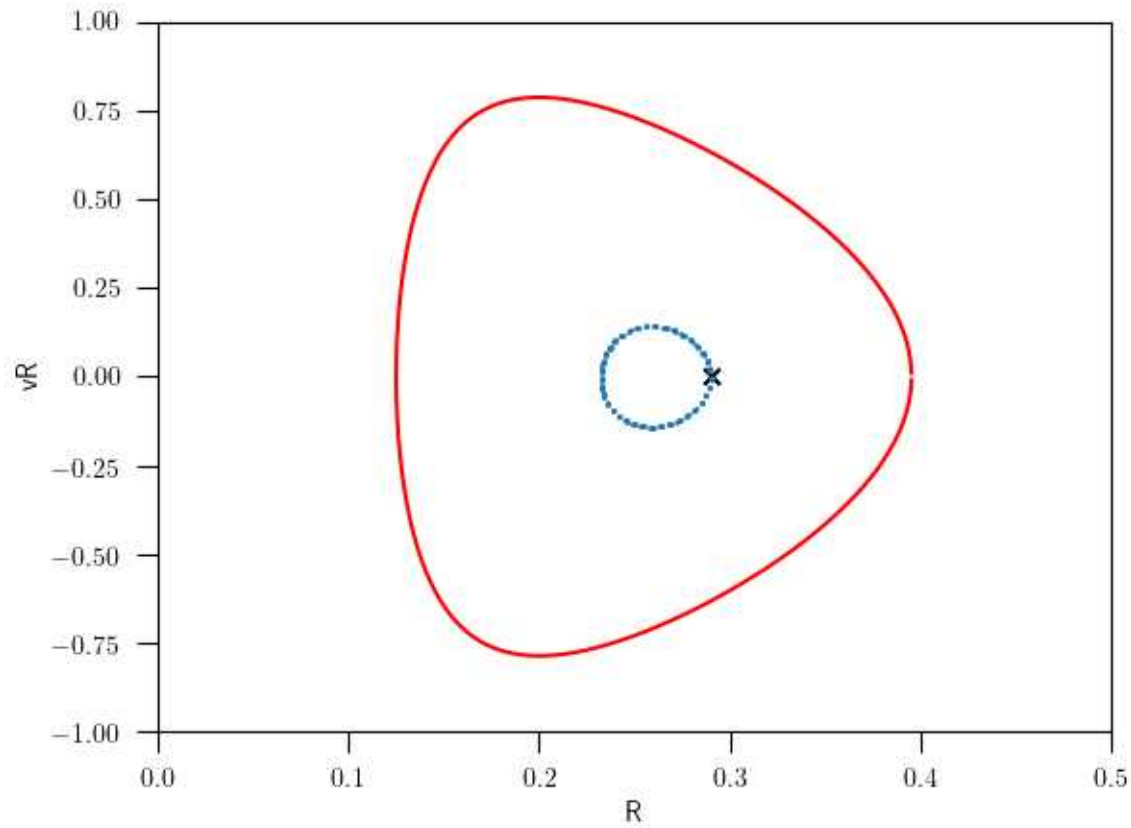
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --norbits 15 --nlaps 100 --R 0.3953
```

ect : Observer_0
Mode : 0
e : 0
e : 0
 : 35.0
lanes : 0.1 14.1
actor : 0.100 10.000



tion : x= 0.0 y= 0.0 z= 0.0
creen : x= 126 y= 197
tP : d= 1.406
os : x= -0.0 y= -1.3 z= 0.5
os : x= -0.0 y= 0.0 z= -0.0

Smaller radius



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --norbits 15 --nlaps 100 --R 0.29
```

orbit.dat

Active object : Observer_0

Projection Mode : 0

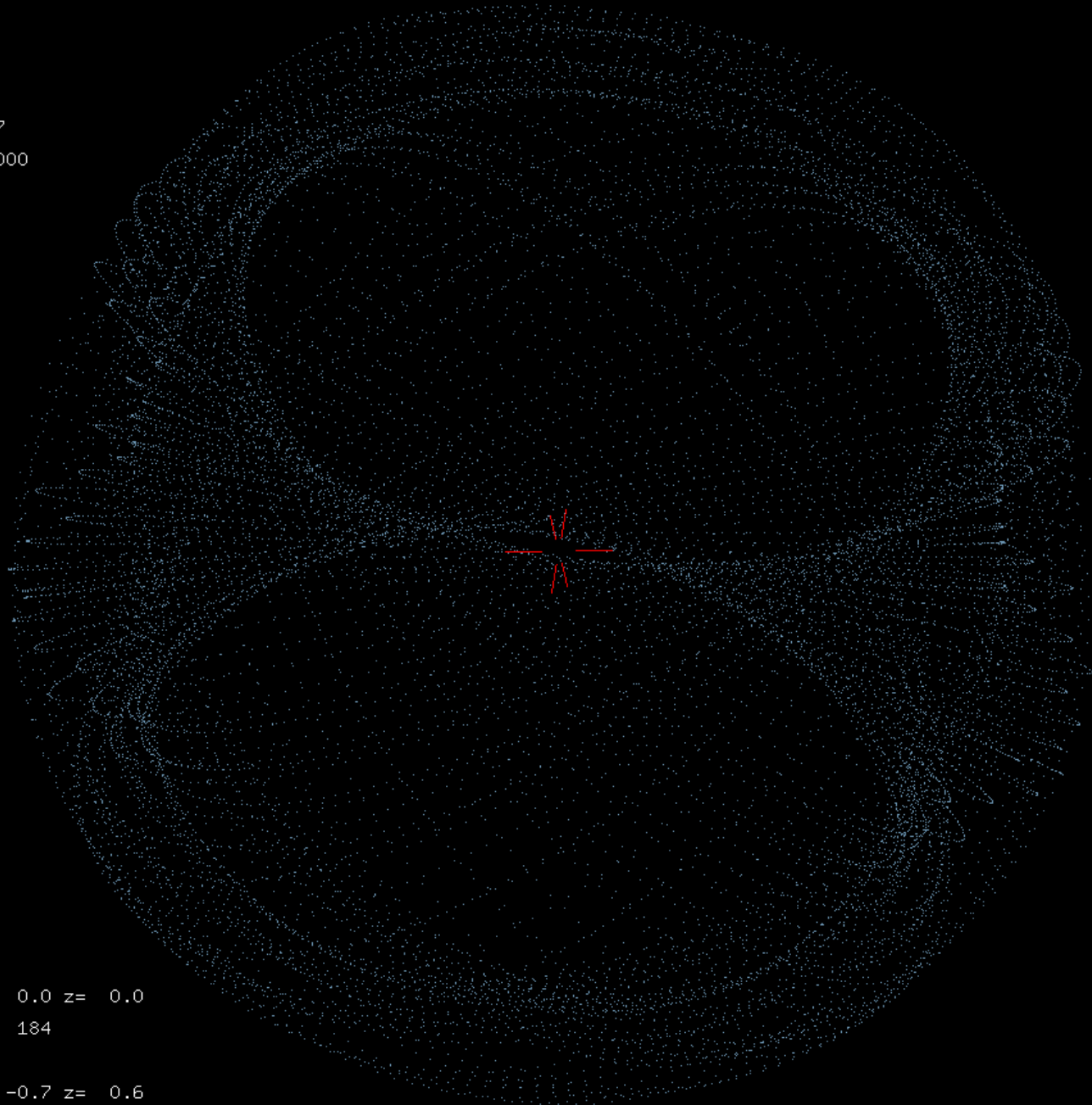
Stereo Mode : 0

Motion Mode : 0

Fov : 35.0

Near/Far planes : 0.1 9.7

Near/Far factor : 0.100 10.000



Mouse Position : x= 0.0 y= 0.0 z= 0.0

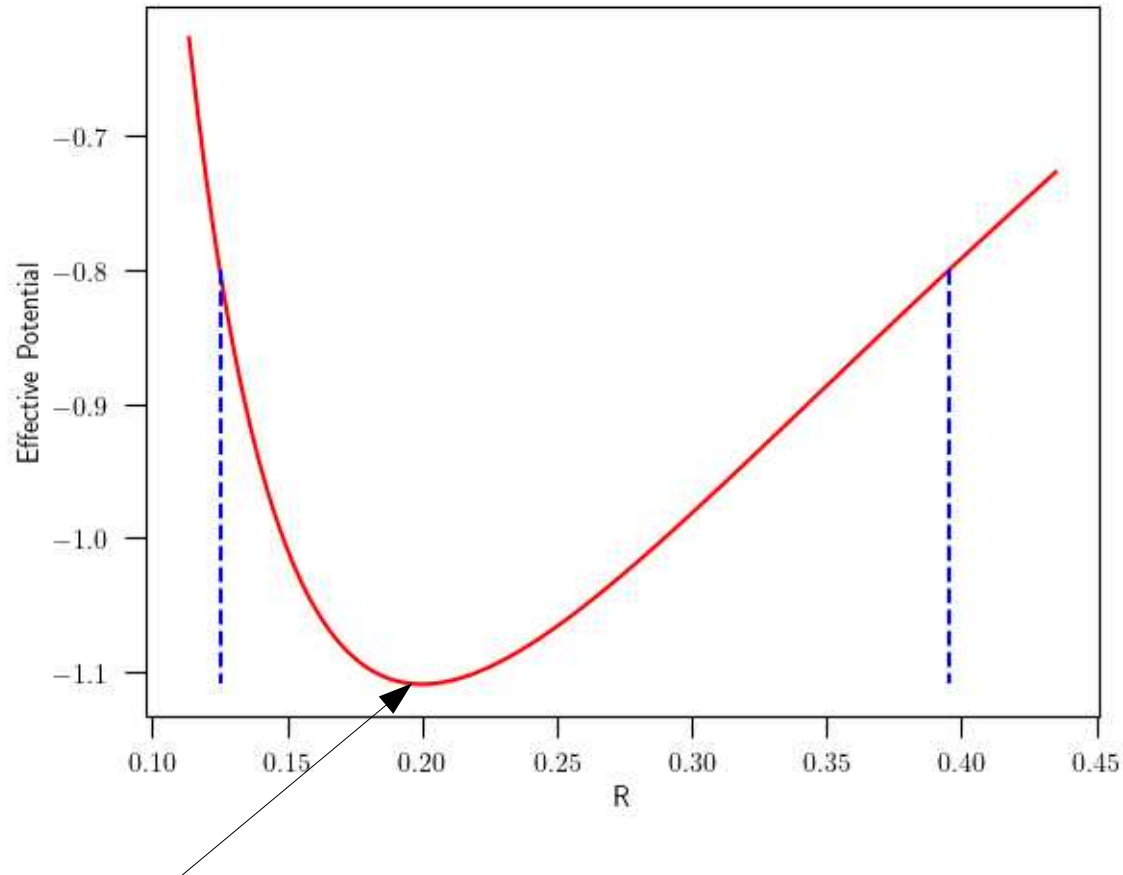
Mouse On screen : x= 424 y= 184

Dist to IntP : d= 0.975

Observer pos : x= -0.2 y= -0.7 z= 0.6

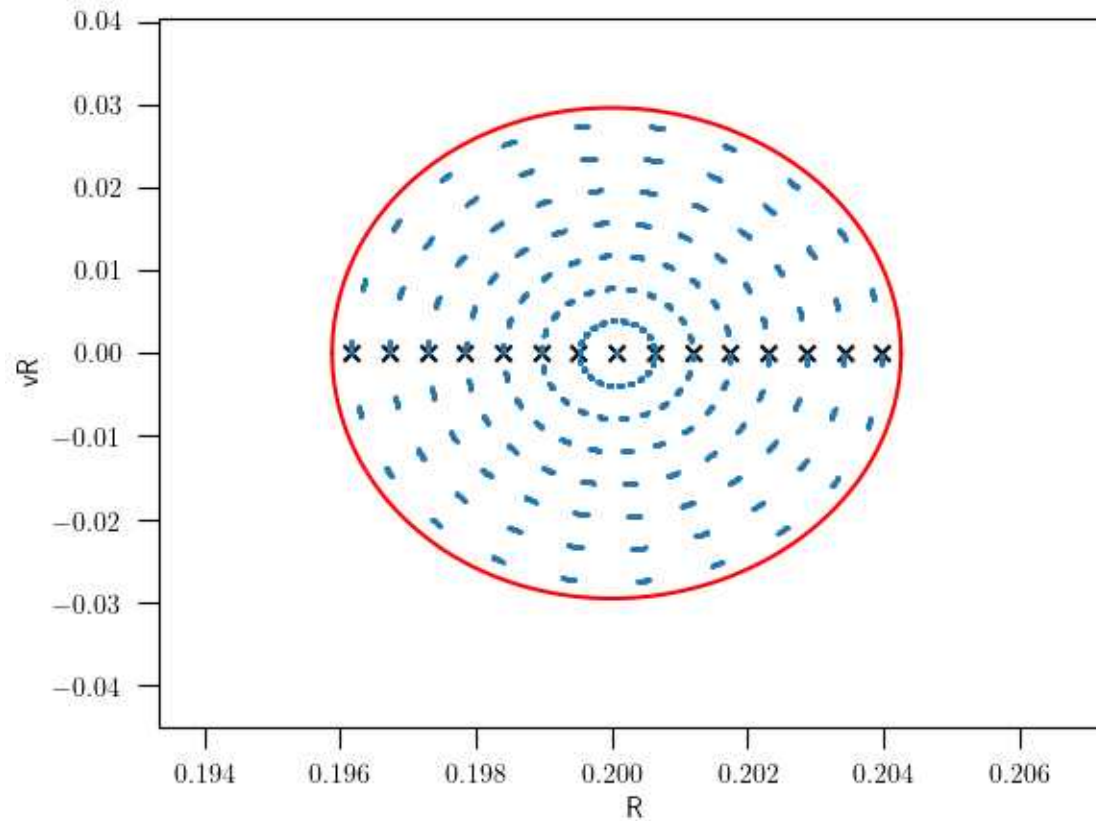
IntP pos : x= 0.0 y= 0.0 z= 0.0

Exploring orbits at lower energy



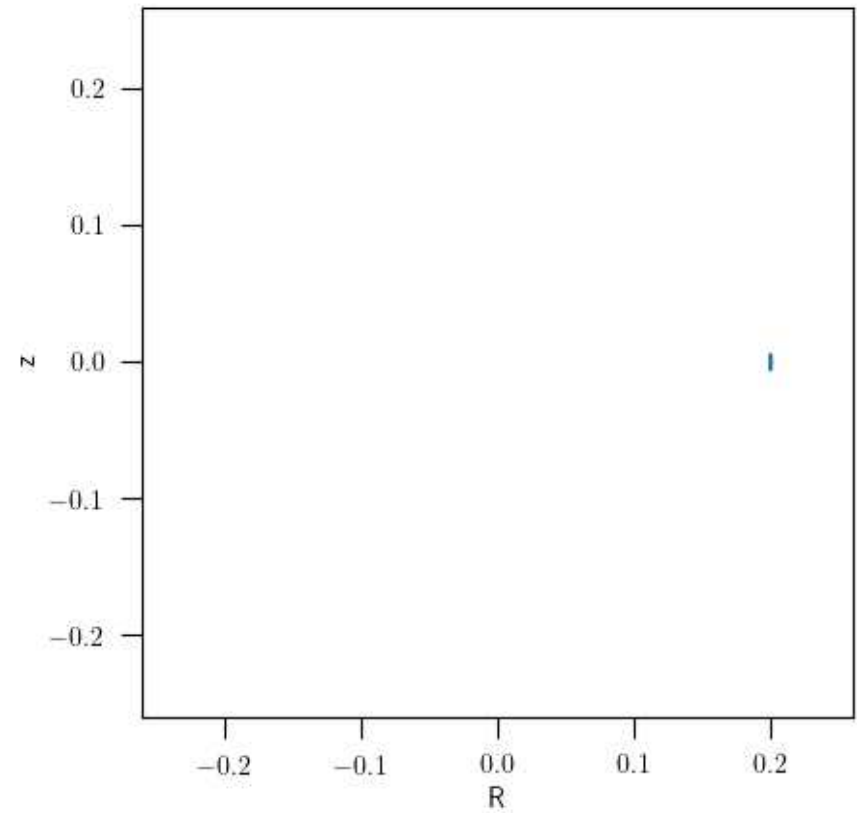
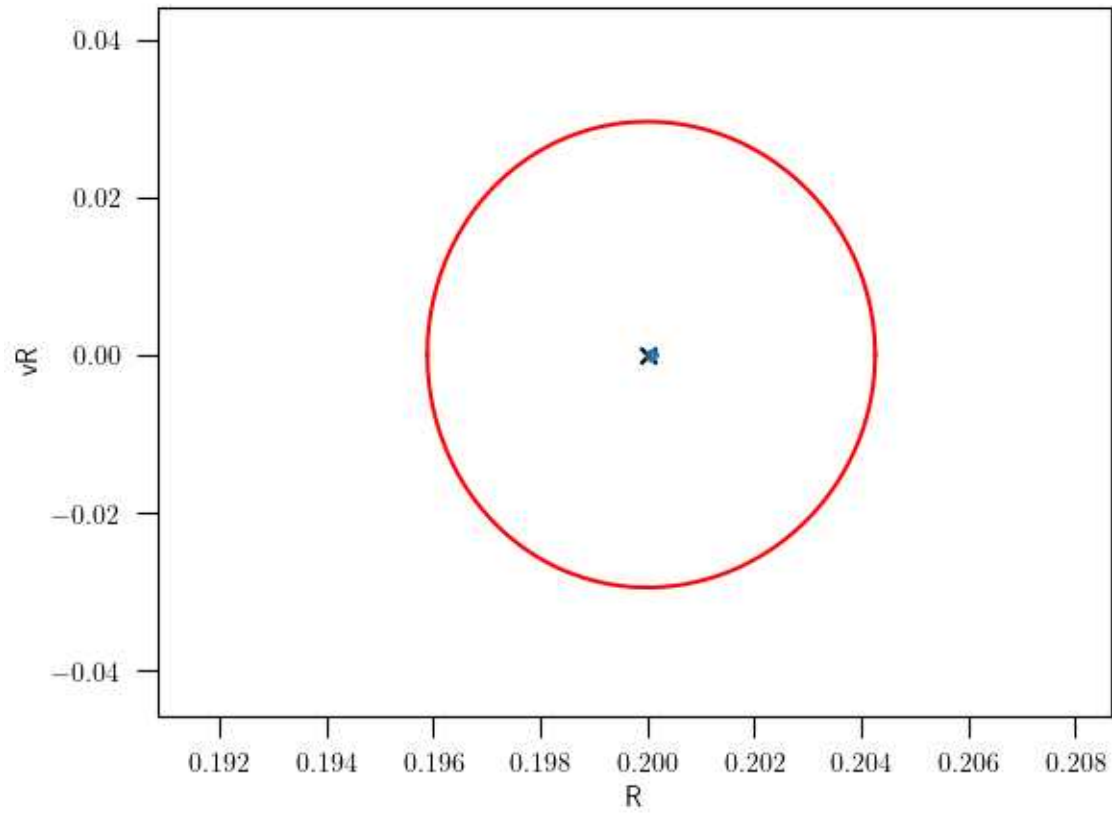
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --plotpotential
```

Orbits near the circular orbit energy



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --norbits 15 --nlaps 100
```

Circular orbit



```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.109 --vR 0 --R 0.2 --nlaps 10
```

orbit.dat

Active object : Observer_0

Projection Mode : 0

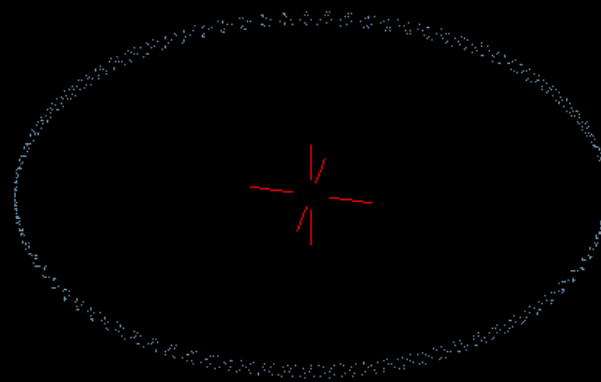
Stereo Mode : 0

Motion Mode : 0

Fov : 35.0

Near/Far planes : 0.1 14.3

Near/Far factor : 0.100 10.000



Mouse Position : x= 0.0 y= 0.0 z= 0.0

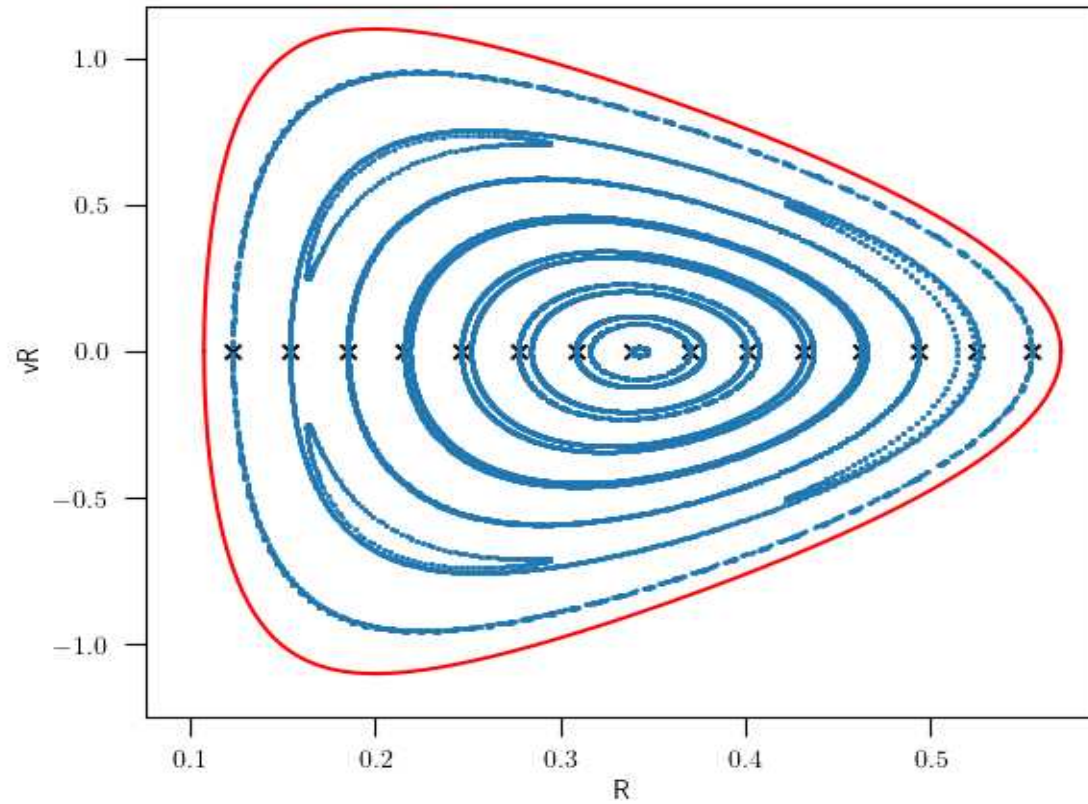
Mouse On screen : x= 247 y= -53

Dist to IntP : d= 1.431

Observer pos : x= 0.3 y= -1.1 z= 0.9

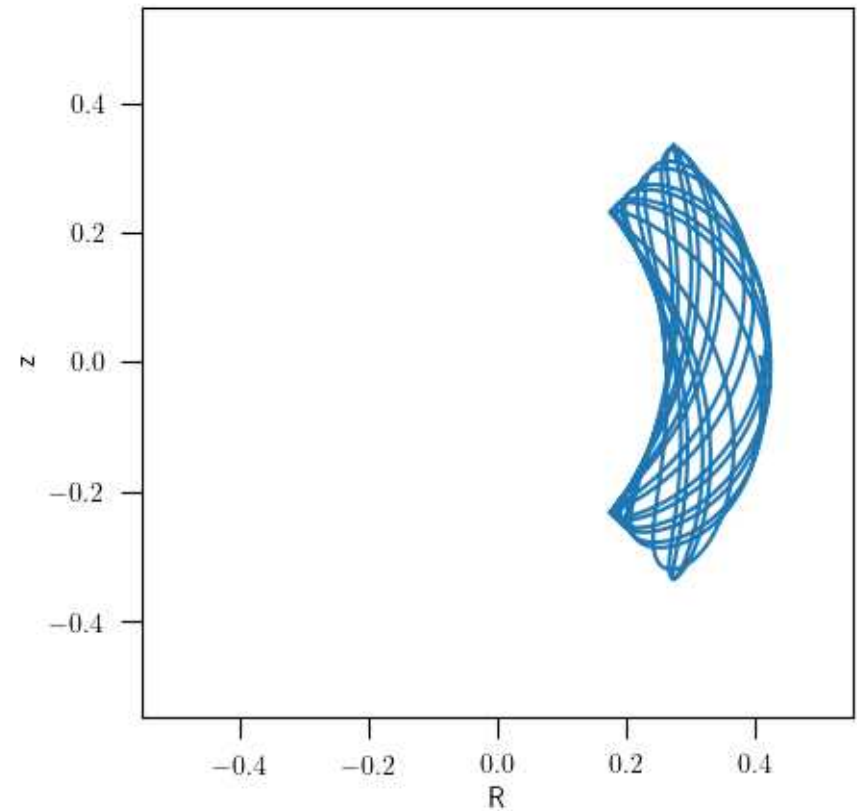
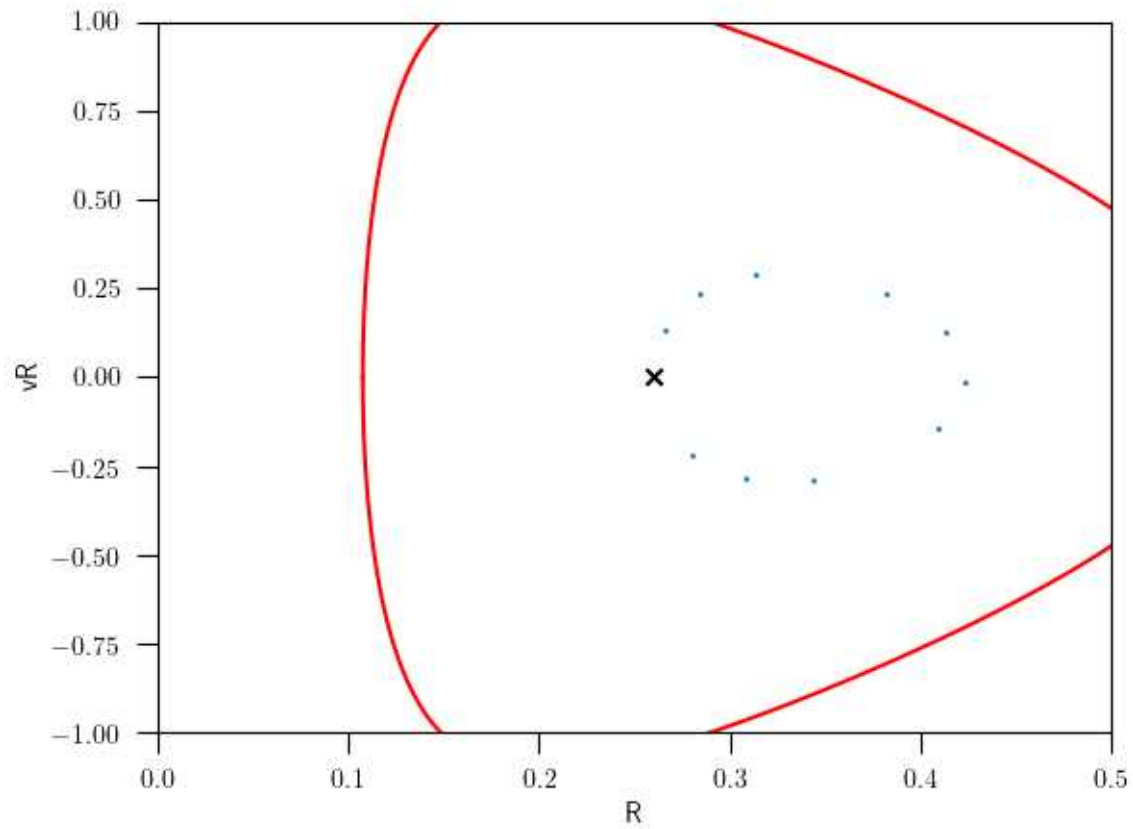
IntP pos : x= 0.0 y= 0.0 z= 0.0

At higher energy



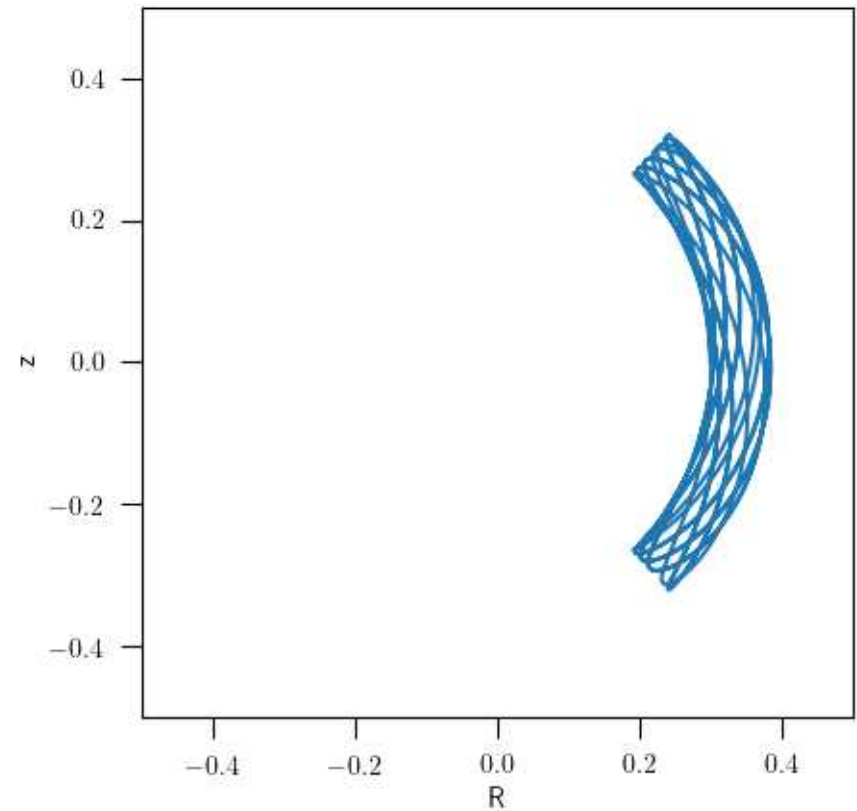
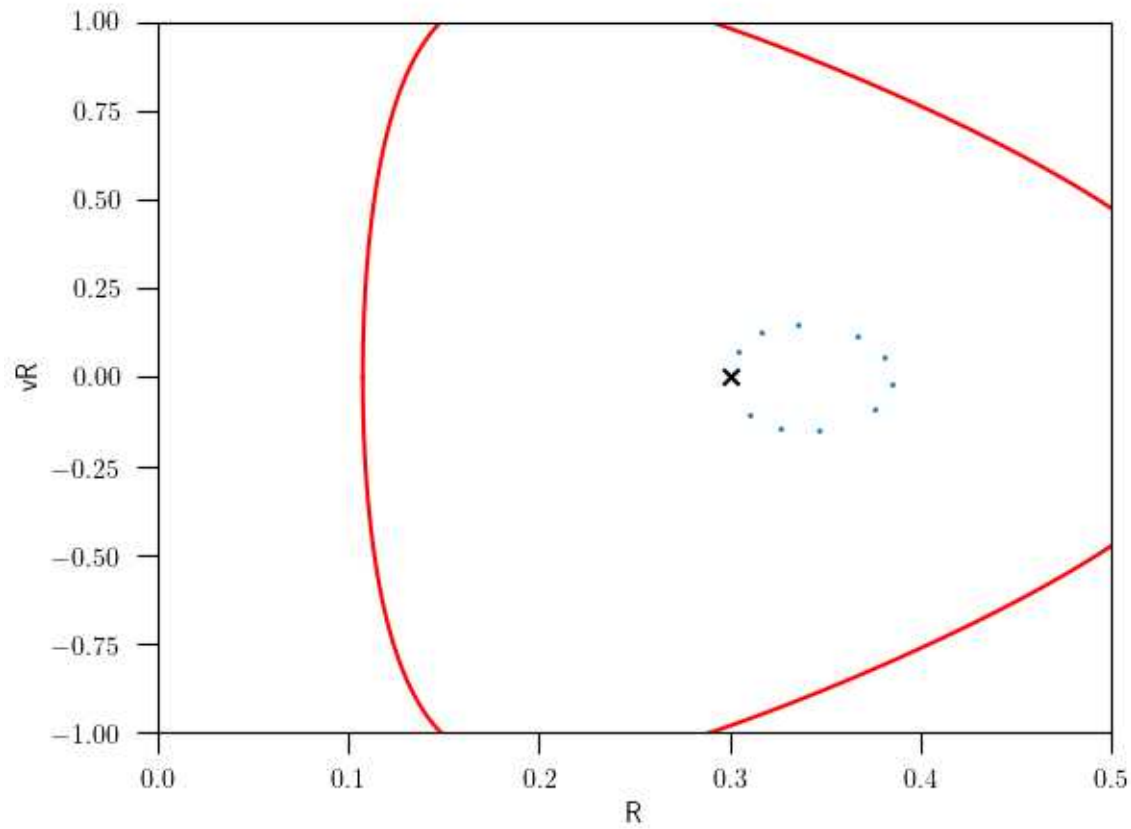
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --norbits 15 --nlaps 1000
```

At higher energy



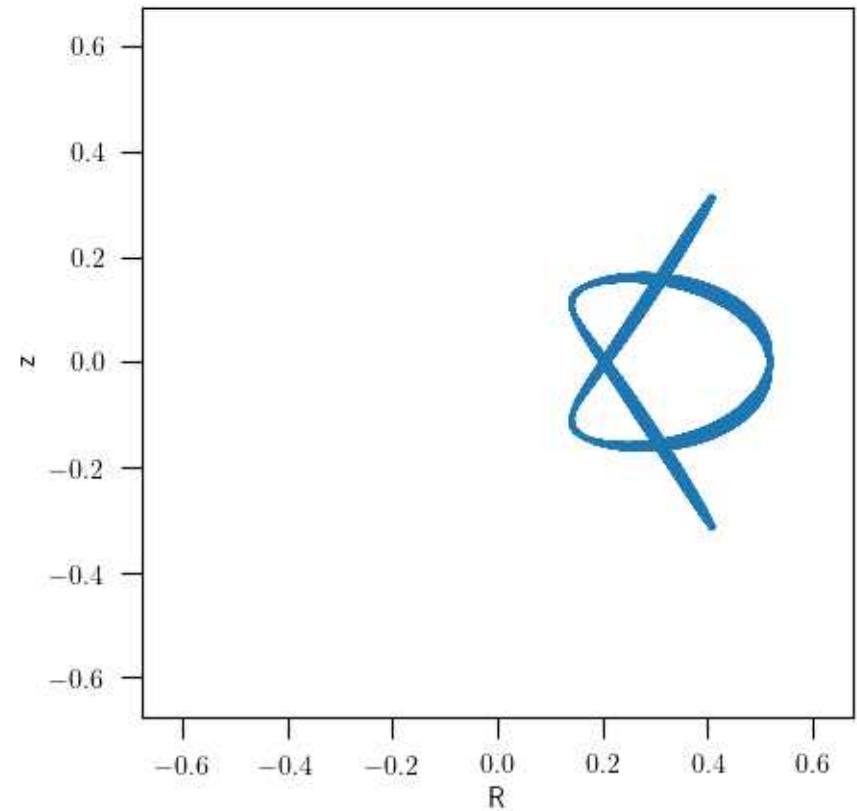
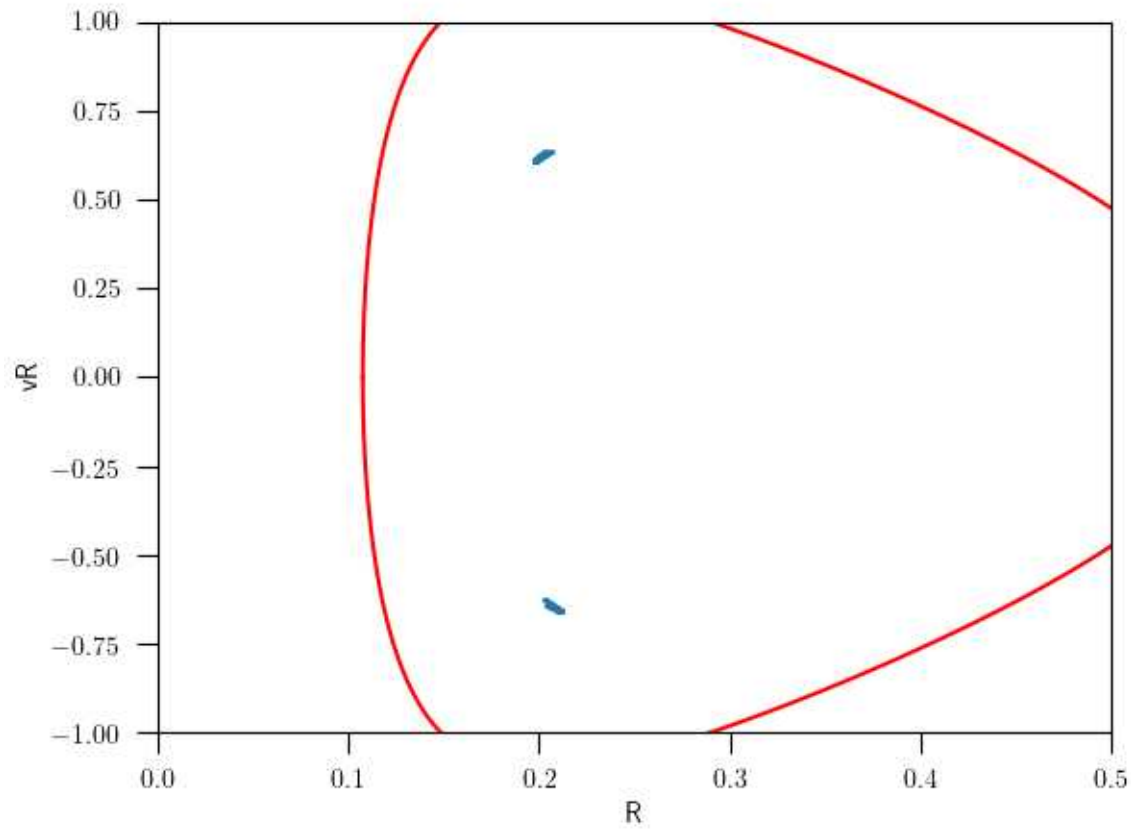
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.26 --nlaps 10
```

At higher energy



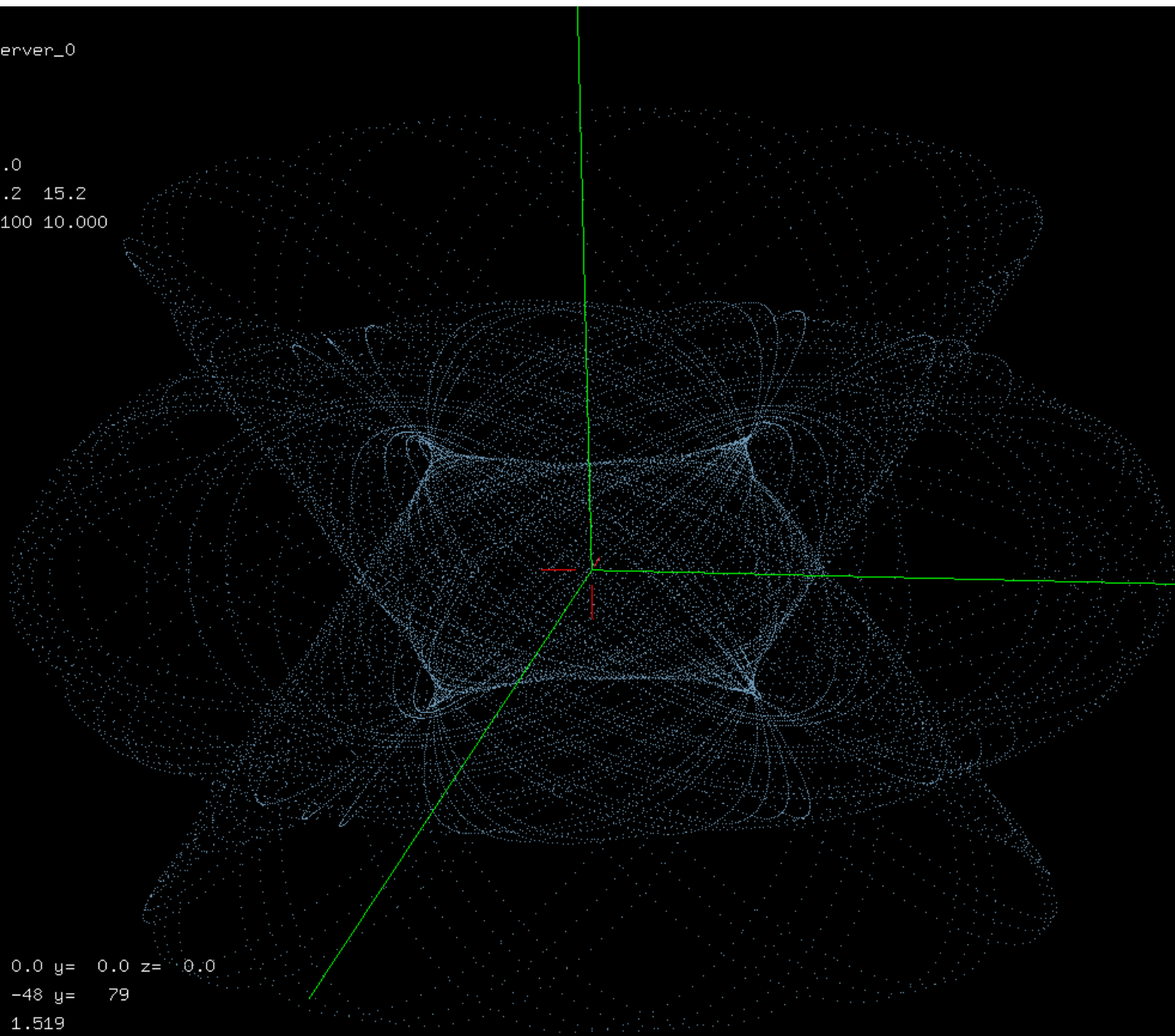
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.30 --nlaps 10
```

Bifurcation (resonance) : new orbit family



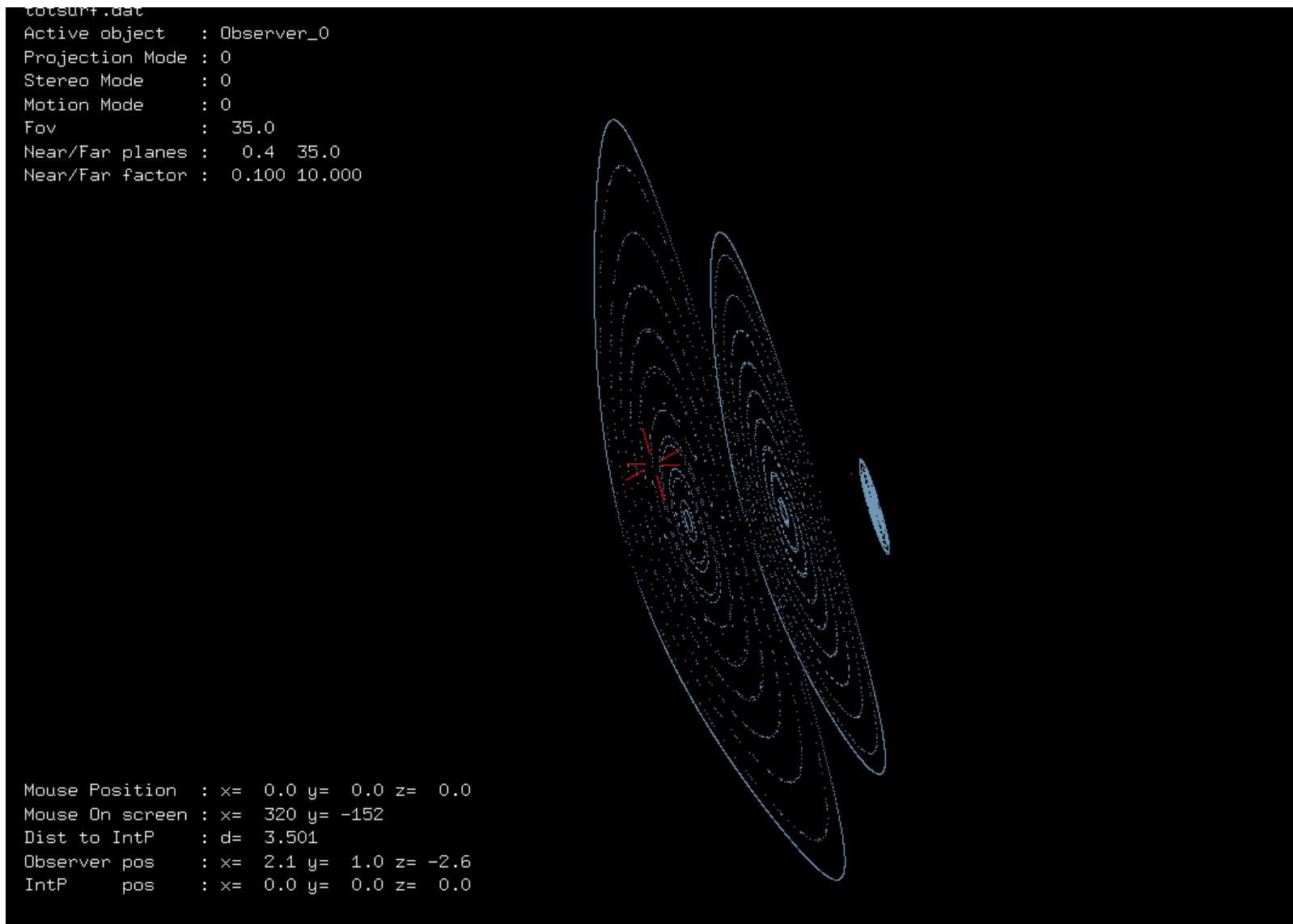
```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --R 0.52 --nlaps 100
```

orbit.dat
Active object : Observer_0
Projection Mode : 0
Stereo Mode : 0
Motion Mode : 0
Fov : 35.0
Near/Far planes : 0.2 15.2
Near/Far factor : 0.100 10.000



Mouse Position : x= 0.0 y= 0.0 z= 0.0
Mouse On screen : x= -48 y= 79
Dist to IntP : d= 1.519
Observer pos : x= 1.4 y= 0.3 z= 0.4
IntP pos : x= 0.0 y= -0.0 z= -0.0

Slices of different energies



```
rm surf-*.dat
```

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -1.1 --vR 0 --norbits 50
```

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.8 --vR 0 --norbits 50
```

```
./mapping-Rz.py --V0 1. --Rc 0.0 --p 0.9 --Lz 0.2 -E -0.5 --vR 0 --norbits 50
```

```
./concatenate.py surf-0*
```

```
glups --fullscreen -pglparameters totsurf.dat
```

The End