Stellar orbits

2nd part

Outlines

Orbits in axisymmetric potentials

- orbits in the equatorial plane
- orbits outside the equatorial plane
- equations of motion
- orbits in the meridian plane
- examples

Nearly circular orbits

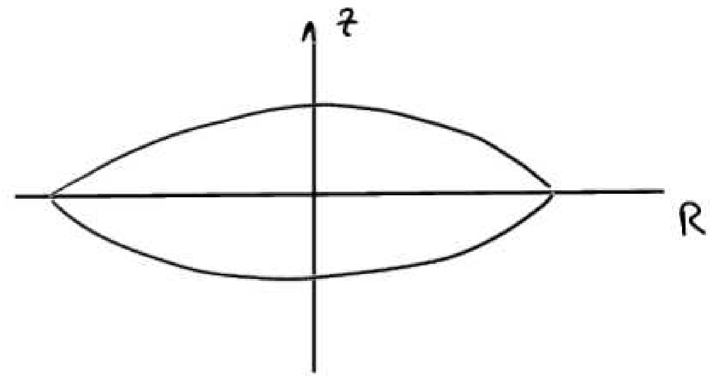
- Epicycle frequences

Axisymmetric Systems

Orbits in axisymmetric potentials

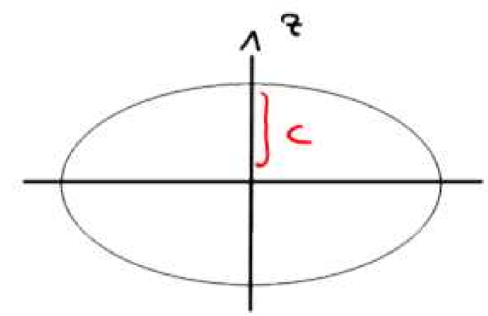
Axisymmetric potential

$$\phi(\bar{x}) = \phi(R, |z|)$$

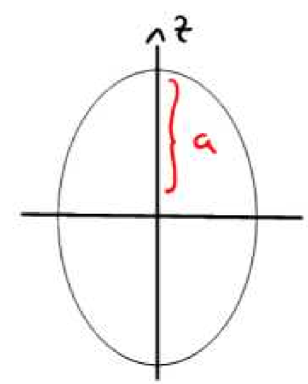


- symmetry of revolution around z
- reflection symmetry with respect to the $z = 0$ plane

Definitions



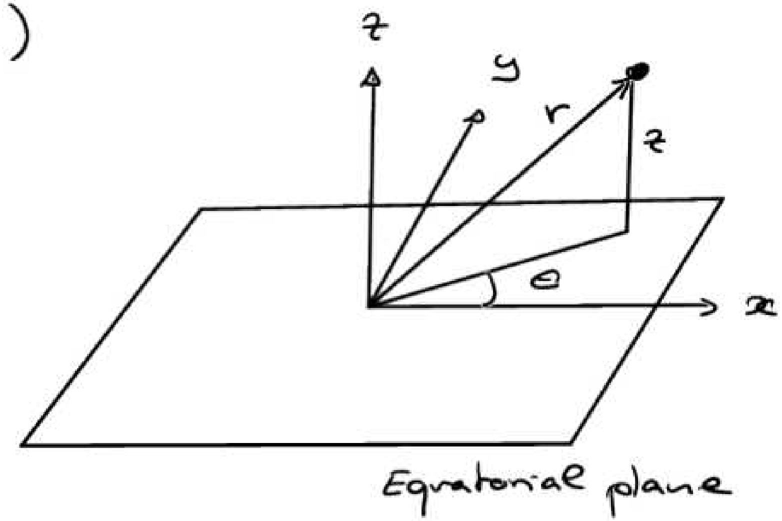
Oblate systems : c, the semi-minor axis is parallel to $\vec{z}$



Prolate systems : a, the semi-major axis is parallel to $\vec{z}$

Description of the dynamics

Cylindrical coordinates (R, θ, z)



Orbits in the equatorial plane

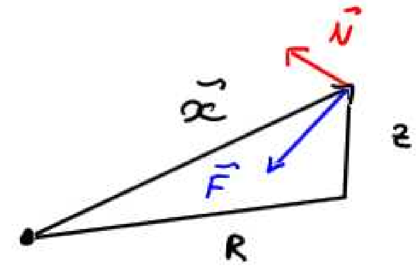
$\forall t, \; \underline{z = 0}$

$$\boxed{\phi(R, |z| = 0) \equiv \phi(R)}$$

The potential seen by the stars is similar to a spherical potential

- description of the orbits in polar coordinates r, φ
- recycle all results developped for spherical potentials

Angular momentum derivative



$$\frac{d\vec{L}}{dt} = \vec{x} \times \vec{g}(\vec{x}) = \vec{N}$$

$$\vec{x} = R\,\vec{e}_R + z\,\vec{e}_z$$

$$\vec{g}(\vec{x}) = -\vec{\nabla}\phi(x)$$

$$= -\frac{\partial \phi}{\partial R}\vec{e}_R - \frac{1}{R}\underbrace{\cancel{\frac{\partial \phi}{\partial \theta}}}_{=0}\vec{e}_\theta - \frac{\partial \phi}{\partial z}\vec{e}_z$$

$$\boxed{\frac{d\vec{L}}{dt} = \left(z\frac{\partial \phi}{\partial R} - R\frac{\partial \phi}{\partial z}\right)\vec{e}_\theta} \qquad (1)$$

$$\begin{cases} \dot{\vec{e}}_R = \dot{\theta}\,\vec{e}_\theta \\ \dot{\vec{e}}_\theta = -\dot{\theta}\,\vec{e}_R \\ \dot{\vec{e}}_z = 0 \end{cases}$$

But

$$\vec{L} = L_R\,\vec{e}_R + L_\theta\,\vec{e}_\theta + L_z\,\vec{e}_z$$

$$\frac{d\vec{L}}{dt} = \dot{L}_R\,\vec{e}_R + L_R\,\dot{\theta}\,\vec{e}_\theta + \dot{L}_\theta\,\vec{e}_\theta - L_\theta\,\dot{\theta}\,\vec{e}_R + \dot{L}_z\,\vec{e}_z$$

$$= \left(\dot{L}_R - L_\theta\,\dot{\theta}\right)\vec{e}_R + \left(\dot{L}_\theta - L_R\,\dot{\theta}\right)\vec{e}_\theta + \dot{L}_z\,\vec{e}_z$$

comparing with (1)

$$\begin{cases} \dot{L}_z = 0 & \Rightarrow \boxed{L_z = cte} \\ \dot{L}_R - L_\theta\,\dot{\theta} = 0 & \Rightarrow \boxed{L_z = cte} \end{cases}$$

EXERCICE

The z-component of the angular momentum is conserved

Orbits that moves outside the equatorial plane

Cylindrical coordinates

$$\begin{cases} x = R\cos\theta & \dot{x} = \dot{R}\cos\theta - R\sin\theta\,\dot{\theta} \\ y = R\sin\theta & \dot{y} = \dot{R}\sin\theta + R\cos\theta\,\dot{\theta} \\ z = z & \dot{z} = \dot{z} \end{cases}$$

$$\underline{\dot{x}^2 + \dot{y}^2 = \dot{R}^2 + R^2\dot{\theta}^2}$$

Lagrangian (specific) in cylindrical coordinates

$$\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \phi\left(\sqrt{x^2+y^2}\right) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) - \phi(R, z)$$

Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = 0}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases}$$

Lagrange equations

$$\begin{cases} \ddot{R} = R\dot{\theta}^2 - \frac{\partial \phi}{\partial R} & (1) \\ \frac{d}{dt}\left(R^2\dot{\theta}\right) = \cancel{\left(-\frac{\partial \phi}{\partial \theta}\right)} = 0 & (2) \\ \ddot{z} = -\frac{\partial \phi}{\partial z} & (3) \end{cases}$$

(2) $R^2\dot{\theta} = \text{cte} = L_z$

The z-component of the angurar momentum is conserved

Solution

$$\boxed{\theta(t) = L_z \int_{t_0}^{t_1} \frac{1}{R^2(t)}\, dt}$$

(1) + (3) two <u>coupled</u> through $\phi(R, z)$ equations for R and z

Hamiltonian / Energy

$$\boxed{H(\vec{q},\vec{p},t) := \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}(\vec{q},\dot{\vec{q}},t)}$$

$$\vec{q} = \begin{cases} R \\ \theta \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{\theta} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{R}} = \dot{R} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = R^2\dot{\theta} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{z}} = \dot{z} \end{cases}$$

$$\boxed{p_\theta = R^2\dot{\theta} = L_z}$$

$$\boxed{H(R,\dot{R},\theta,\dot{\theta},z,\dot{z}) = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right) + \phi(R,z) = E}$$

E (Energy) is conserved

as $\mathcal{L}$ is time independant

ϕ

Effective potential

with $L_z = R^2\dot{\theta}$

$L_z^2 = R^4\dot{\theta}^2$

Definition

$$\phi_{eff}(R,z) = \phi(R,z) + \frac{L_z^2}{2R^2}$$

$$\begin{cases} \dfrac{\partial \phi_{eff}}{\partial R} = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} \equiv \dfrac{\partial \phi}{\partial R} - R\dot{\theta}^2 \\[2ex] \dfrac{\partial \phi_{eff}}{\partial z} = \dfrac{\partial \phi}{\partial z} \end{cases}$$

The equations of motion 1 + 3 becomes

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial \phi_{eff}}{\partial z}(R,z) \end{cases}}$$

The 3D motion of a star in an axisymmetric potential is reduced to a 2D motion in the meridian plane (R,z)

phase space 6D $\to$ 4D

Hamiltonian in the meridian plane

Those equations of motion may be derived from the lagrangian

$$\mathcal{L}(R, \dot{R}, z, \dot{z}) = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 - \phi_{\text{eff}}(R, z)$$

The corresponding Hamiltonian writes $(p_R = \dot{R}\ ,\ p_z = \dot{z})$

$$H(R, \dot{R}, z, \dot{z}) = \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi_{\text{eff}}(R, z)$$

$$= \frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right) + \phi(R, z) + \frac{L_z^2}{2R^2}$$

$$= \underbrace{\frac{1}{2}\left(\dot{R}^2 + \dot{z}^2\right)}_{\text{kinetic energy in the orbital plane}} + \phi(R, z) + \frac{1}{2}R^2\dot{\theta}^2 = \underbrace{E}_{\text{orbit's total energy}}$$

E is conserved as ϕ_{eff} is time independent

Illustration in the $z = 0$ plane

for $R \to \infty$

$$\phi_{eff} = \phi + \underbrace{\frac{L_z^2}{2}\frac{1}{R^2}}_{\to 0} \sim \phi$$

for $R \to 0$

$$\phi_{eff} = \underbrace{\phi}_{\text{bounded}} + \underbrace{\frac{L_z^2}{2}\frac{1}{R^2}}_{\text{diverges}} \sim \frac{1}{R^2}$$

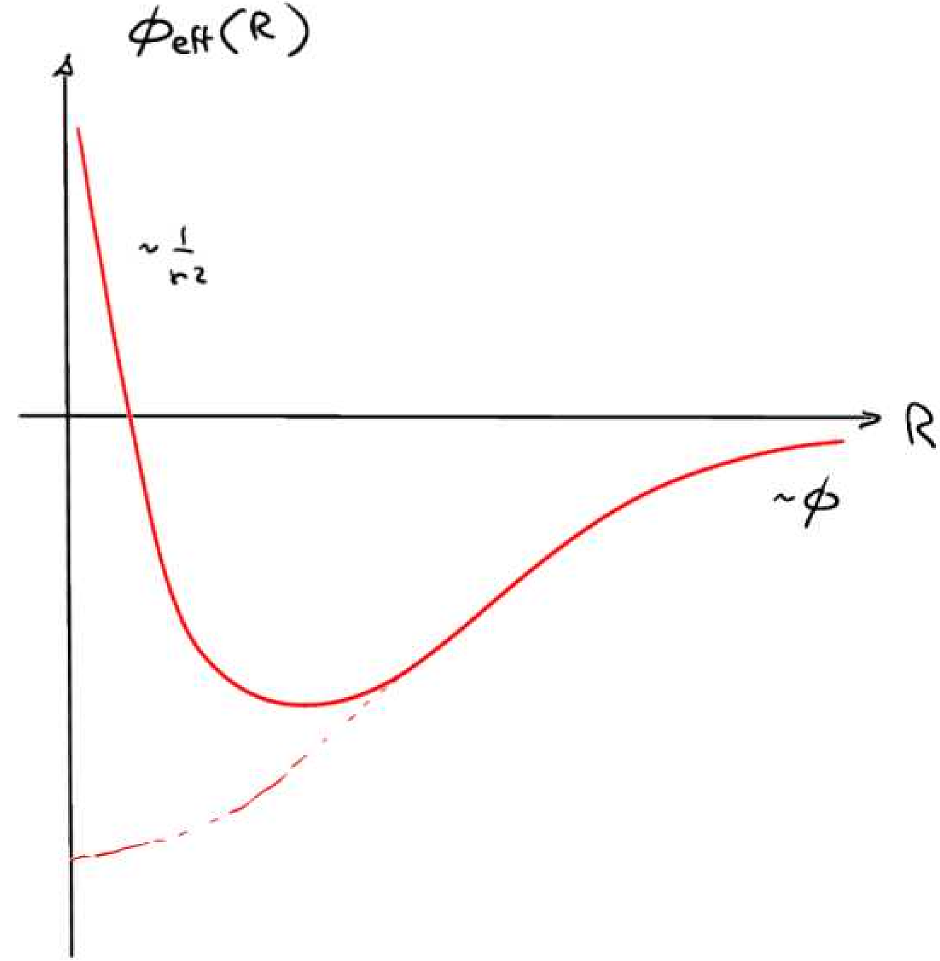


Illustration in the $z = 0$ plane

$$E = \frac{1}{2}\dot{R}^2 + \phi_{eff}(R)$$

4 cases

① $E > \phi_{eff}(\infty)$ except at $E = \phi_{eff}$

$\dot{R} \neq 0$ unbound orbits

② $E < \min(\phi_{eff}(R))$ $\dot{R}^2 < 0$

impossible

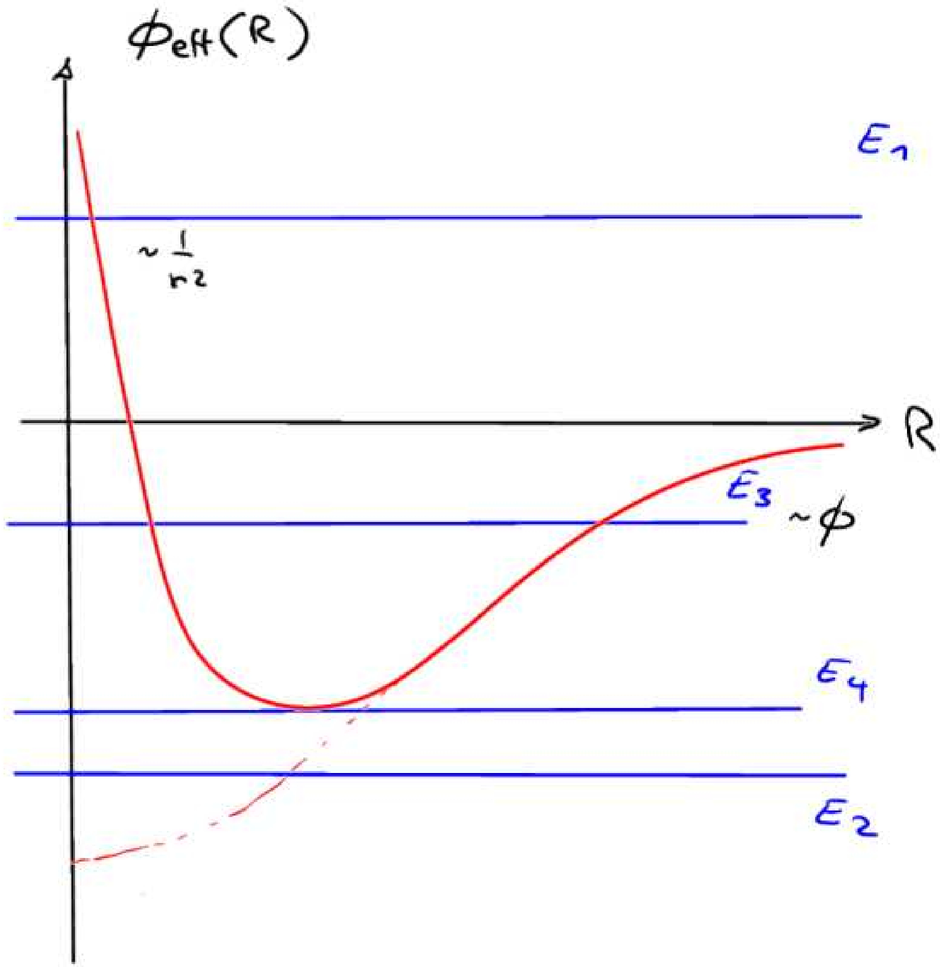
③ $\min(\phi_{eff}(R)) < E < \phi_{eff}(\infty)$

orbit bounded between

R_1 and R_2 (where $\dot{R} = 0$)

④ $E = \min(\phi_{eff}(R))$ (stationary point)

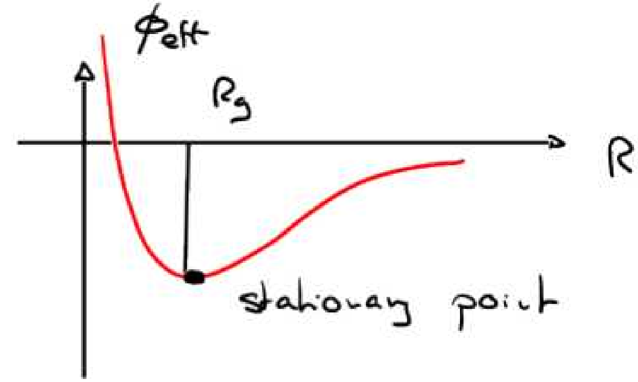
$R_1 = R_2$ (circular orbit)



Stationary point

$\dot{R} = \ddot{R} = 0 \qquad \dot{z} = \ddot{z} = 0$

from
$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \phi_{eff}}{\partial R}(R,z) \\ \ddot{z} = -\dfrac{\partial \phi_{eff}}{\partial z}(R,z) \end{cases}}$$



$$\begin{cases} \dfrac{\partial \phi_{eff}}{\partial R} = 0 = \dfrac{\partial \phi}{\partial R} - \dfrac{L_z^2}{R^3} = 0 \\ \dfrac{\partial \phi_{eff}}{\partial z} = 0 = \dfrac{\partial \phi}{\partial z} = 0 \end{cases}$$

$\longrightarrow$ by symmetry where $z = 0$

$V_\theta = R\dot{\theta}$

$$\boxed{R_g \text{ such that } \left.\frac{\partial \phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2 = \frac{V_\theta^2(R_g)}{R_g} = \frac{V_c^2(R_g)}{R_g}}$$

$V_c^2 = R \left.\dfrac{\partial \phi}{\partial R}\right)_{R,0}$

R_g : guiding center

> The stationary point in R_g in the meridional plane corresponds to a circular orbit

Circular orbits

angular speed $\quad \dot{\theta} = \dfrac{L_z}{R_g^2}$

angular momentum $\quad L_z$

energy $\quad \phi_{eff} + \dfrac{L_z^2}{2R_g}$

Note

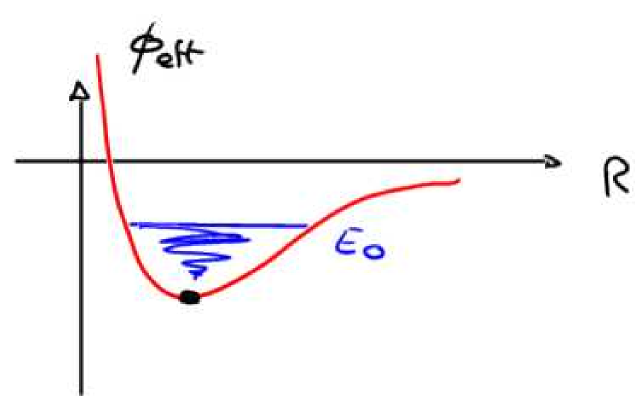
> For a given angular momentum L_z, the circular orbit is the one that minimize the energy.

① $E_0 = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\dot{z}^2 + \phi + \dfrac{L_z^2}{2R}$

② Dissipate energy $\quad L_z = cte$

$\dot{z} \searrow \quad \dot{R} \searrow$

③ circular orbit

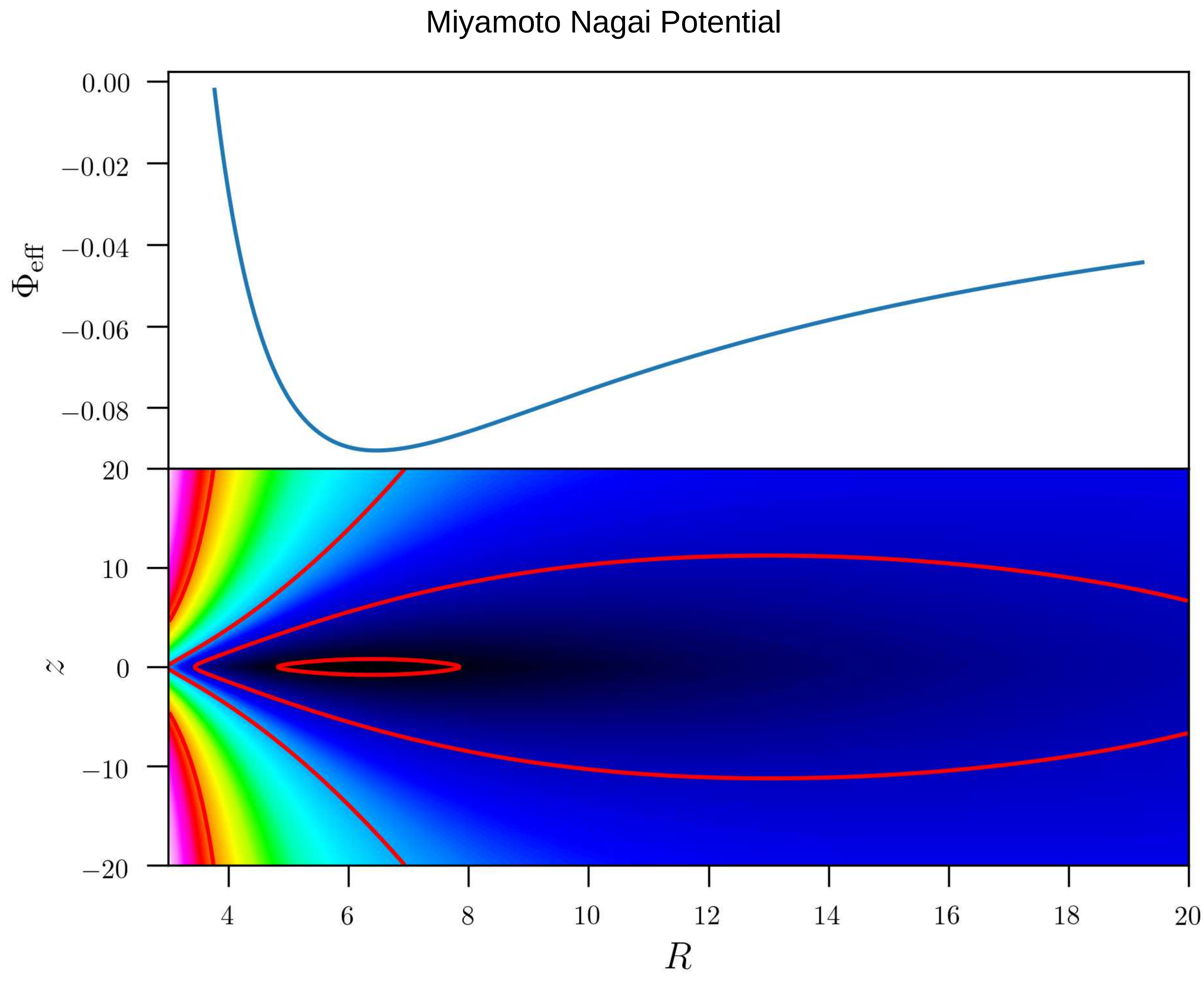


Examples

① Miyamoto - Nagai potential

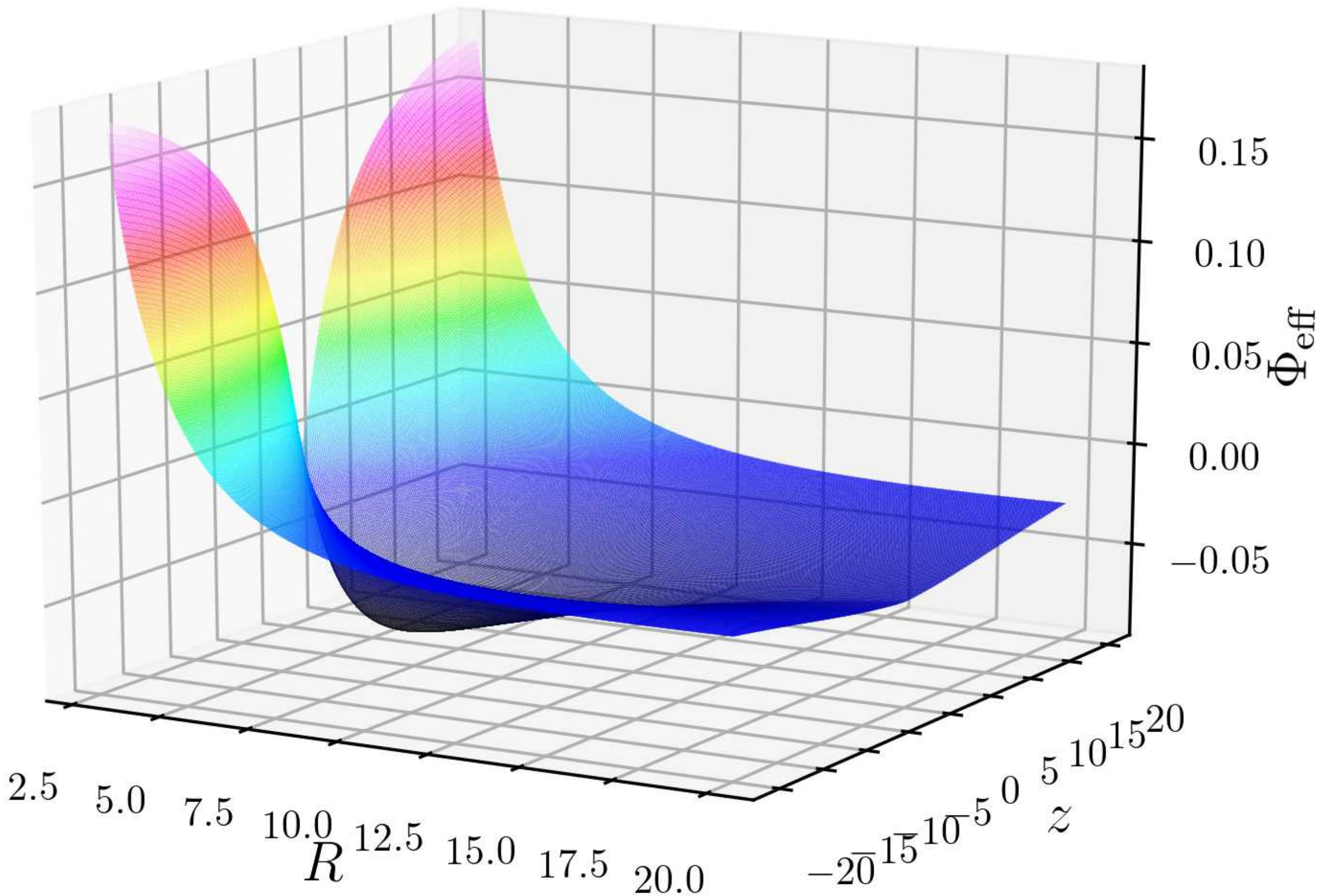
$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

$$\phi_{eff}(R, z=0) = -\frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$



Miyamoto Nagai Potential
0.00
−0.02
−0.04
−0.06
−0.08
$\Phi_{\rm eff}$
20
10
0
−10
−20
z
4
6
8
10
12
14
16
18
20
R

Miyamoto Nagai Potential



Examples

① Miyamoto - Nagai potential

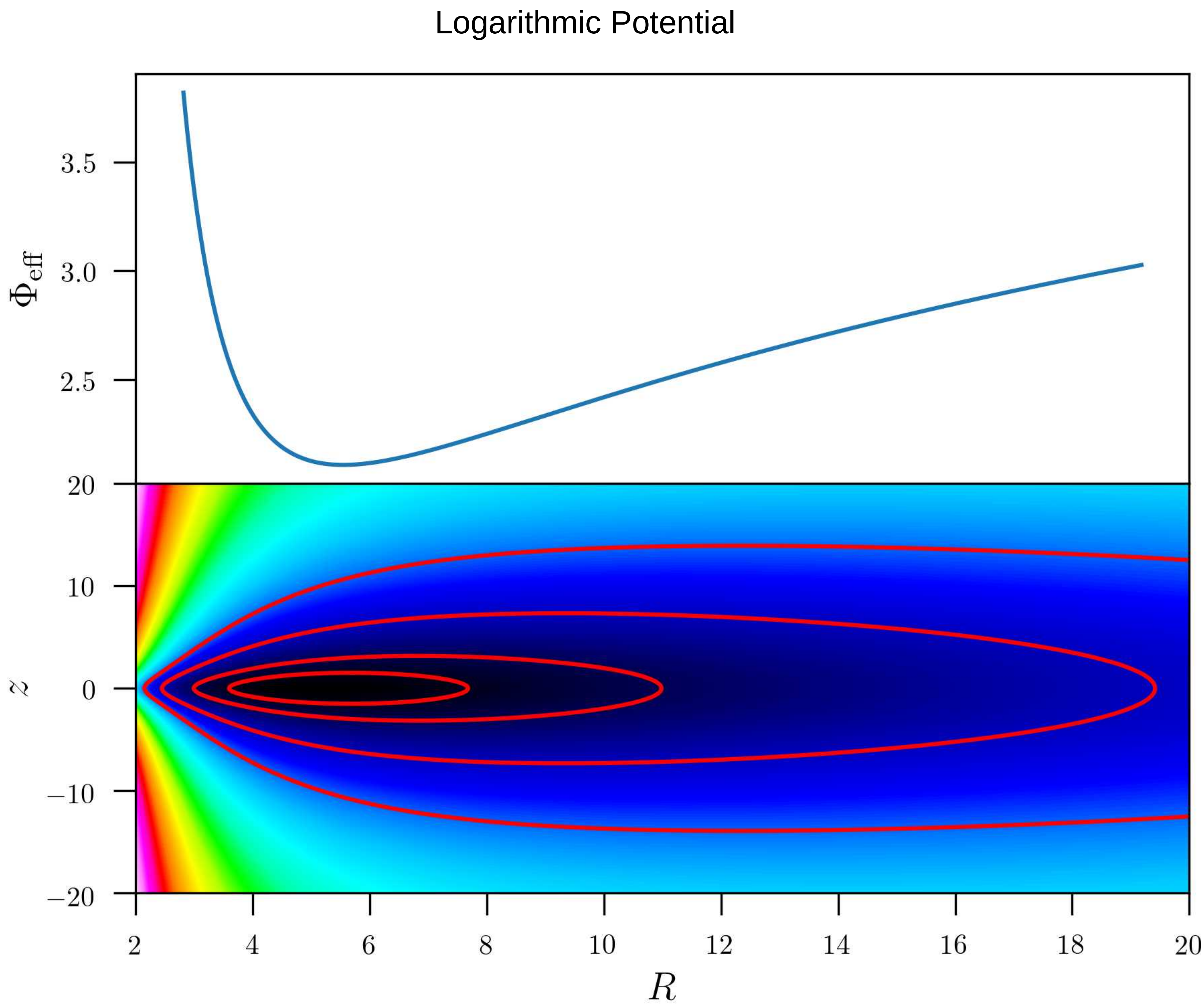
$$\phi(R, z) = -\frac{GM}{\sqrt{R^2 + \left(a + \sqrt{z^2+b^2}\right)^2}}$$

$$\phi_{eff}(R, z=0) = -\frac{GM}{\sqrt{R^2 + (a+b)^2}} + \frac{L_z^2}{2R^2}$$

② Logarithmic potential

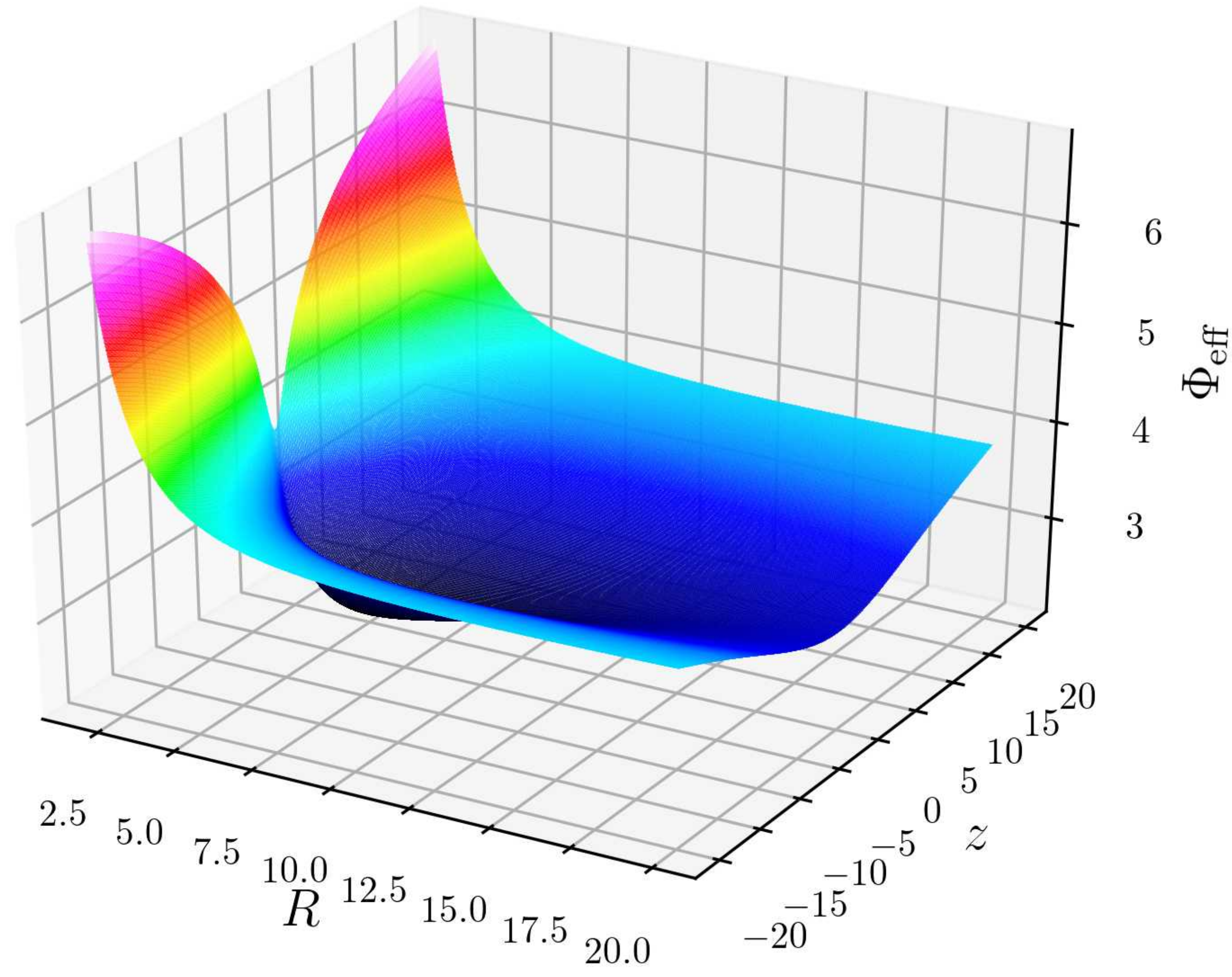
$$\phi(R, z) = \frac{1}{2} V_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$$

$$\phi_{eff}(R, z=0) = \frac{1}{2} V_0^2 \ln\left(R^2\right) + \frac{L_z^2}{2R^2}$$



Logarithmic Potential
$\Phi_{\rm eff}$
3.5
3.0
2.5
20
10
0
−10
−20
z
2
4
6
8
10
12
14
16
18
20
R

Logarithmic Potential



General solutions for the equations of motion

$$\boxed{\begin{cases} \ddot{R} = -\dfrac{\partial \Phi_{eff}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial \Phi_{eff}}{\partial z}(R,z) \end{cases}}$$

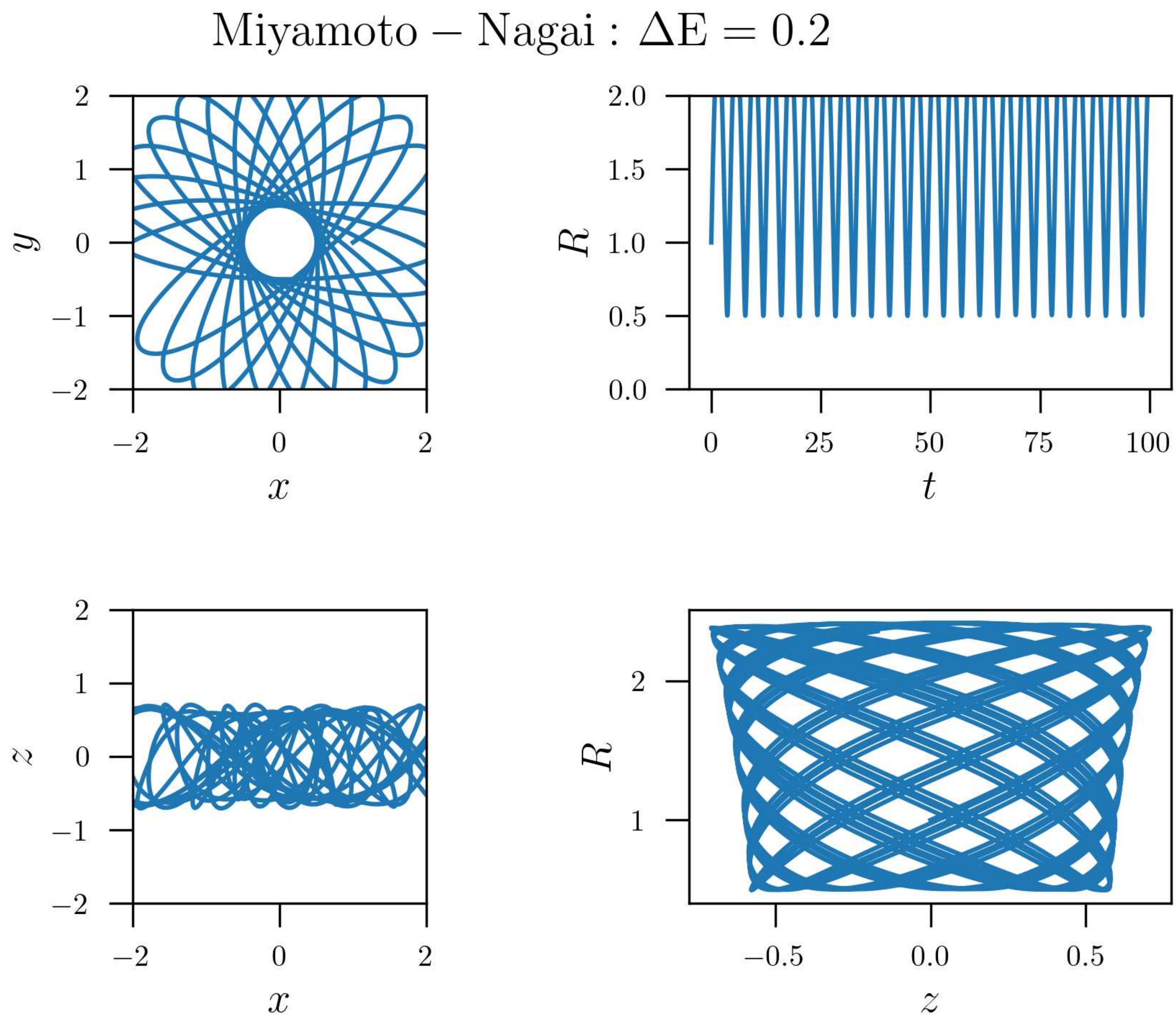
no simple solutions ☹

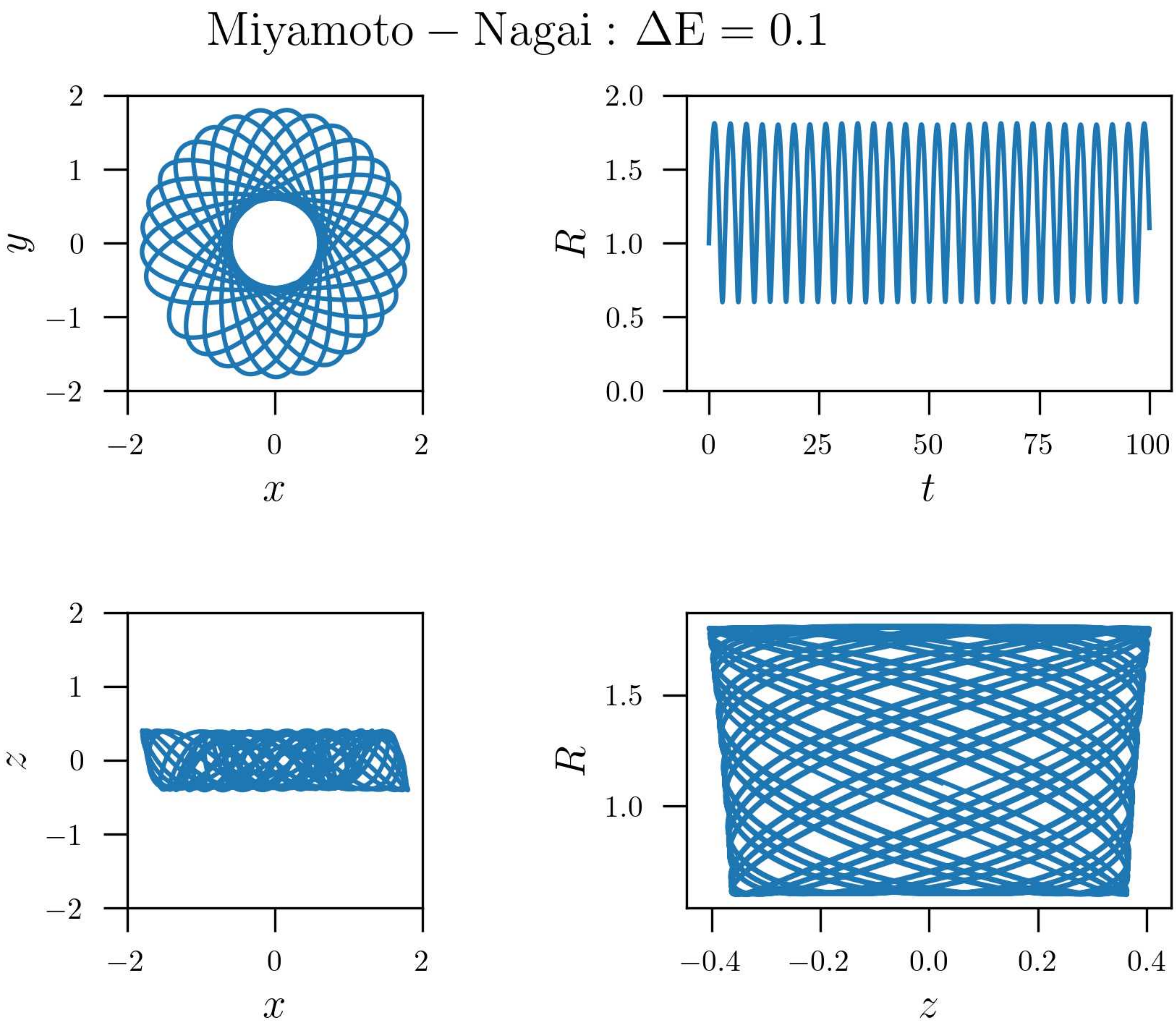
need numerical integration

Hamilton's Equations

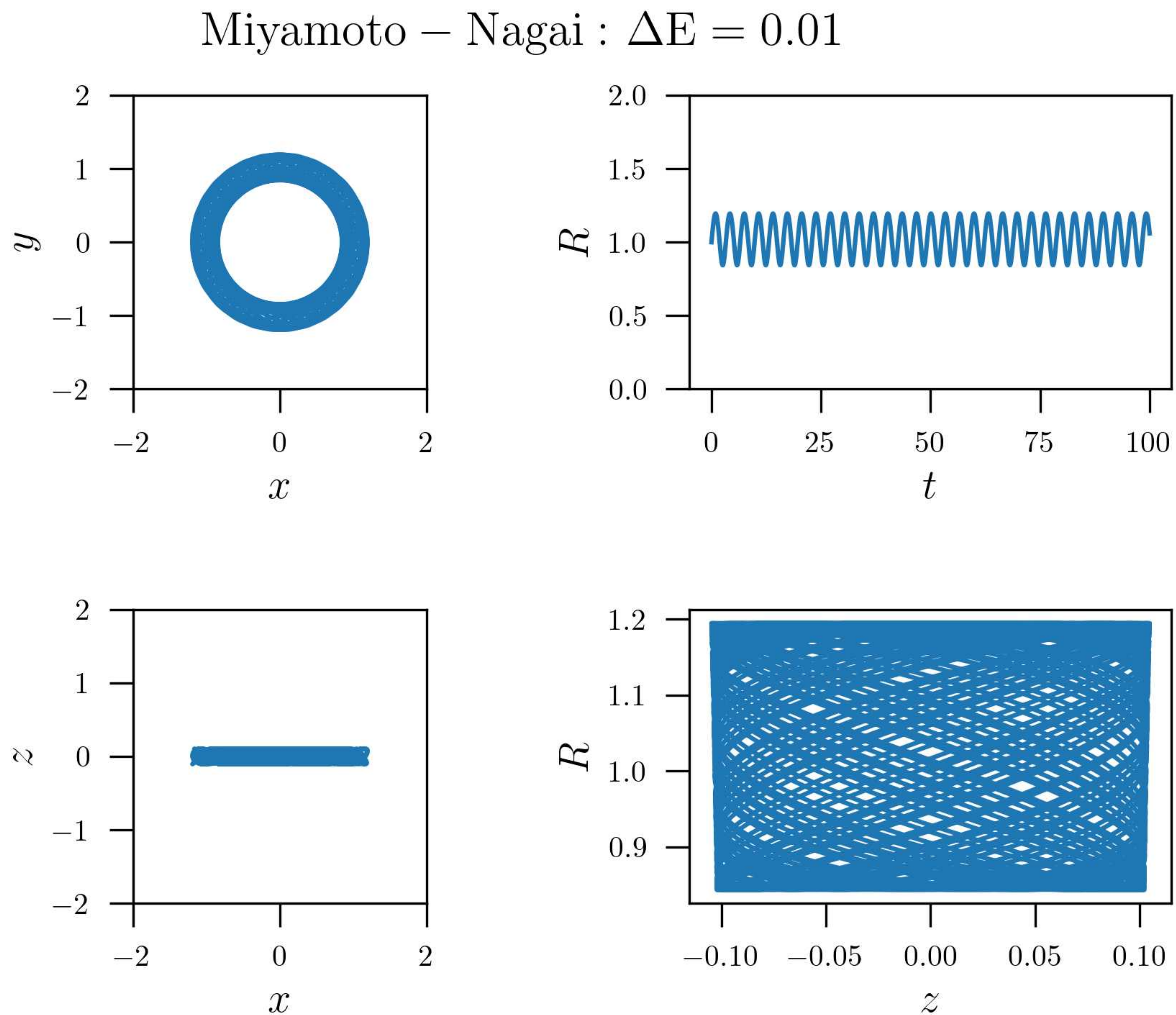
$$\vec{q} = \begin{cases} R \\ z \end{cases} \qquad \dot{\vec{q}} = \begin{cases} \dot{R} \\ \dot{z} \end{cases} \qquad \vec{p} = \begin{cases} \dot{R} \\ \dot{z} \end{cases}$$

$$\begin{cases} \dot{q}_R = p_R & \equiv \dot{R} \\[1ex] \dot{q}_z = p_z & \equiv \dot{z} \\[1ex] \dot{p}_R = -\dfrac{\partial \Phi_{eff}}{\partial q_R}(q_R, q_z) & \equiv -\dfrac{\partial \Phi_{eff}}{\partial R}(R,z) \\[2ex] \dot{p}_z = -\dfrac{\partial \Phi_{eff}}{\partial q_z}(q_R, q_z) & \equiv -\dfrac{\partial \Phi_{eff}}{\partial z}(R,z) \end{cases}$$

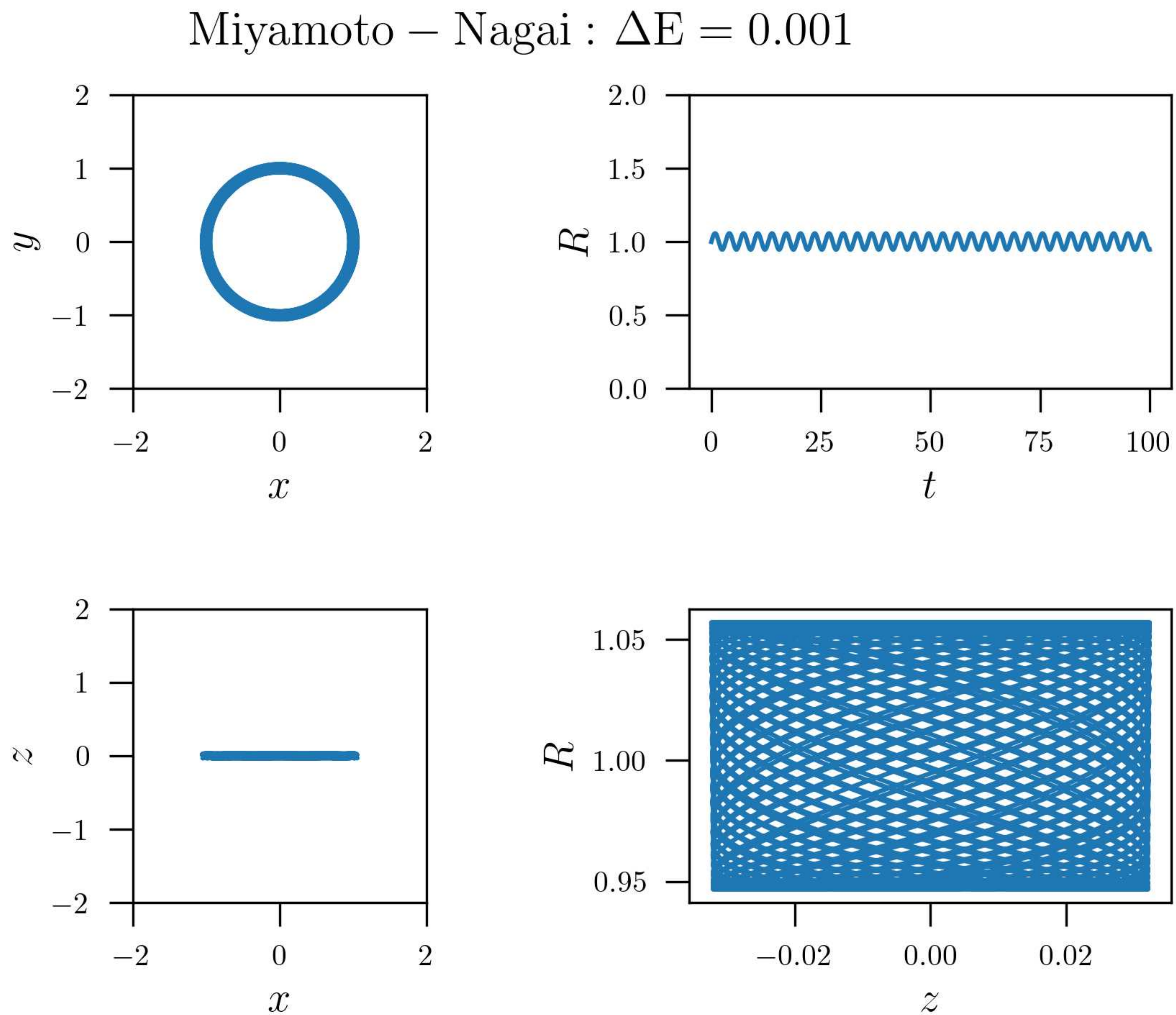

Miyamoto – Nagai : ΔE = 0.2
x
y
t
R
x
z
z
R



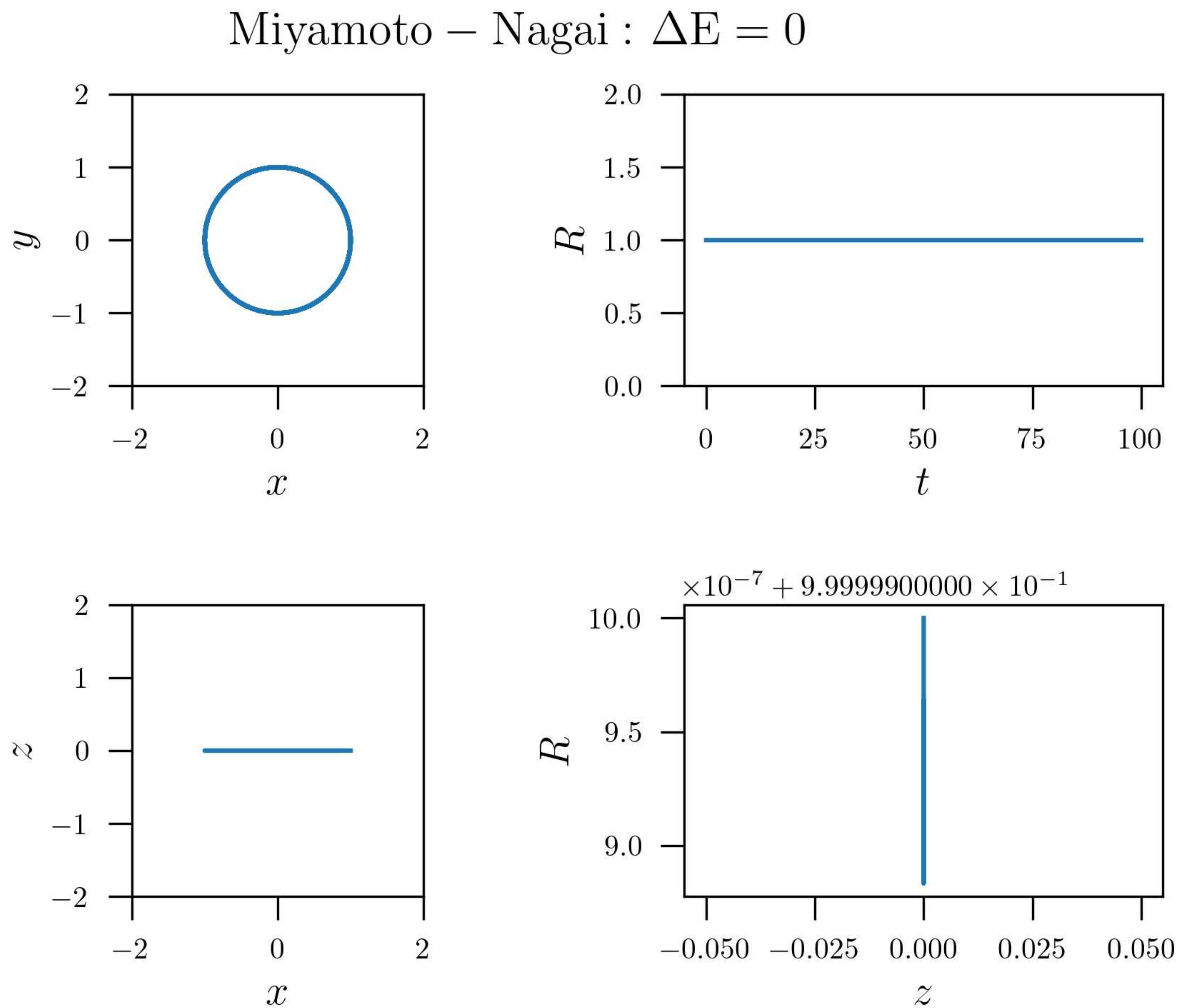
Miyamoto − Nagai : ΔE = 0.1
y
x
R
t
z
x
R
z



Miyamoto – Nagai : ΔE = 0.01
y
x
R
t
z
x
R
z
2
1
0
−1
−2
−2
0
2
2.0
1.5
1.0
0.5
0.0
0
25
50
75
100
1.2
1.1
1.0
0.9
−0.10
−0.05
0.00
0.05
0.10



Miyamoto − Nagai : ΔE = 0.001
y
x
R
t
z
x
R
z

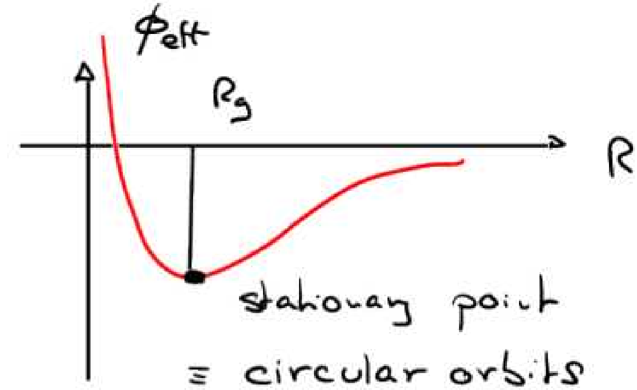


$\mathrm{Miyamoto-Nagai: \Delta E = 0}$
x
y
t
R
z
$\times 10^{-7} + 9.9999900000 \times 10^{-1}$

Nearly circular orbits

Nearly circular orbits

From the previous study of orbits in axisymmetric potentials



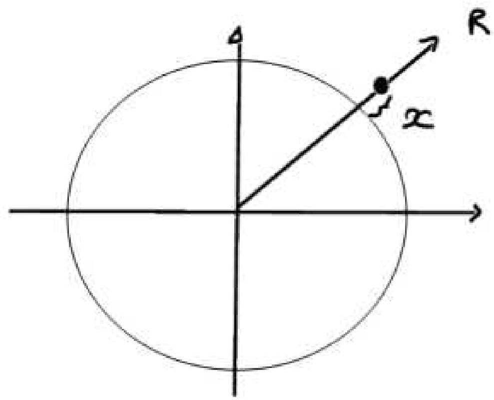
Goal Study orbits in the neighbourhood of circular orbits

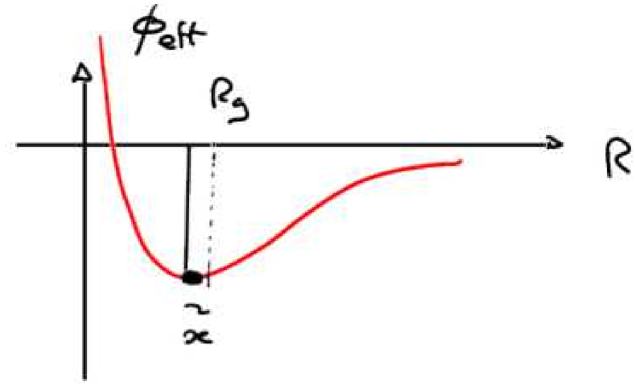
Justifications In a disk galaxy, many stars are found in nearly circular orbits

Recall R_g : the guiding center

$$\boxed{R_g \text{ such that } \left.\frac{\partial \phi}{\partial R}\right)_{R_g,0} = \frac{L_z^2}{R_g^3} = R_g \dot{\theta}^2}$$

We define $\quad x := R - R_g \quad$ the distance to the guiding center R_g





Taylor expansion of ϕ_{eff} around $R = R_g$, $z = 0$

$$\phi_{eff}(R,z) \cong \phi_{eff}(R_g,0) + \underbrace{\frac{\partial \phi_{eff}}{\partial R}(R_g,0)}_{=0 \ \text{min}} (R - R_g) + \underbrace{\frac{\partial \phi_{eff}}{\partial z}(R_g,0)}_{=0 \ \text{sym.}} z$$

$$+ \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial R^2}(R_g,0)(R-R_g)^2 + \frac{1}{2} \frac{\partial^2 \phi_{eff}}{\partial z^2}(R_g,0)\, z^2$$

$$+ \frac{1}{2} \underbrace{\frac{\partial^2 \phi_{eff}}{\partial z \, \partial R}}_{=0}(R_g,0)(R-R_g)\, z + O\left(\left((R-R_g)z\right)^3\right)$$

$=0 \quad \phi_{eff}(R,z)$ must be sym. with respect to $z = 0$

$$\boxed{\phi_{eff}(R,z) \simeq \phi_{eff}(R_g,0) + \frac{1}{2}\frac{\partial^2\phi_{eff}}{\partial R^2}(R_g,0)\,x^2 + \frac{1}{2}\frac{\partial^2\phi_{eff}}{\partial z^2}(R_g,0)\,z^2}$$

<u>Definition</u>

$$\begin{cases} \kappa^2(R_g) = \left(\dfrac{\partial^2\phi_{eff}}{\partial R^2}\right)_{(R_g,0)} \\[2ex] \nu^2(R_g) = \left(\dfrac{\partial^2\phi_{eff}}{\partial z^2}\right)_{(R_g,0)} \end{cases}$$

$$[\phi] = \left(\frac{m}{s}\right)^2$$

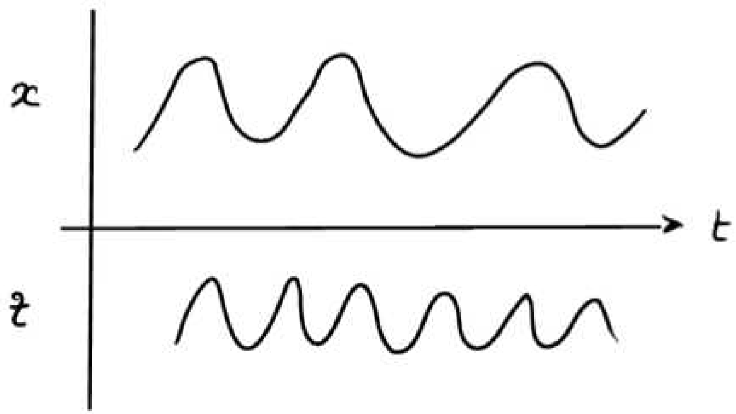
$$\left[\left(\frac{\partial^2\phi}{\partial R^2}\right)^{\frac{1}{2}}\right] \equiv \left[\left(\frac{\partial^2\phi}{\partial z^2}\right)^{\frac{1}{2}}\right] = \frac{1}{s}$$

frequency

<u>Equations of motion near R_g</u>

$$\begin{cases} \ddot{R} = -\dfrac{\partial\phi_{eff}}{\partial R}(R,z) \\[2ex] \ddot{z} = -\dfrac{\partial\phi_{eff}}{\partial z}(R,z) \end{cases} \Rightarrow \boxed{\begin{cases} \ddot{x} = -\kappa^2(R_g)\,x \\[2ex] \ddot{z} = -\nu^2(R_g)\,z \end{cases}}$$

$$\begin{cases} \ddot{x} = -\varkappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$



Two decoupled harmonic oscillators with frequencies $\varkappa$ and ν

$\boxed{\begin{array}{l} \varkappa : \text{ epicycle (radial) frequency} \\ \nu : \text{ vertical frequency} \end{array}}$

Expressions of κ and ν from the total potential

$$\kappa^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial R^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\frac{L_z^2}{R_g^4}$$

$$L_z^2 = V_c^2 R_g^2 = R_g^3 \left.\frac{\partial \phi}{\partial R}\right)_{R_g}$$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + \frac{3}{R_g}\left.\frac{\partial \phi}{\partial R}\right)_{(R_g,0)}$$

circ. frequency $\Omega^2 = \frac{1}{R_g}\left.\frac{\partial \phi}{\partial R}\right)_{R_g}$

$$= \left.\frac{\partial^2 \phi}{\partial R^2}\right)_{(R_g,0)} + 3\Omega^2$$

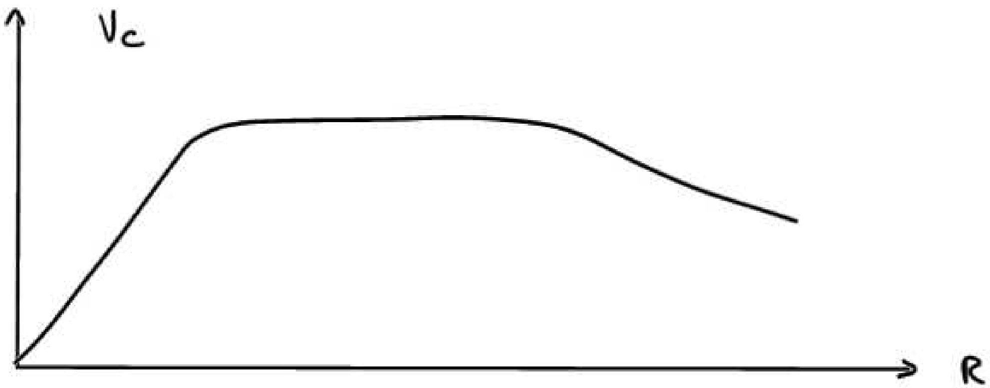
$$\Omega^2 = \frac{V_c^2}{R^2}$$

$$= \left(R\frac{\partial(\Omega^2)}{\partial R} + 4\Omega^2\right)_{(R_g,0)}$$

$$= \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\Omega^2\right)_{(R_g,0)} = \left(\frac{1}{R}\frac{\partial(V_c^2)}{\partial R} + 2\frac{V_c^2}{R^2}\right)_{(R_g,0)}$$

$$\nu^2(R_g) = \left.\frac{\partial^2 \phi_{eff}}{\partial z^2}\right)_{(R_g,0)} = \left.\frac{\partial^2 \phi}{\partial z^2}\right)_{(R_g,0)}$$

Note : $\varkappa$ depends only on V_c



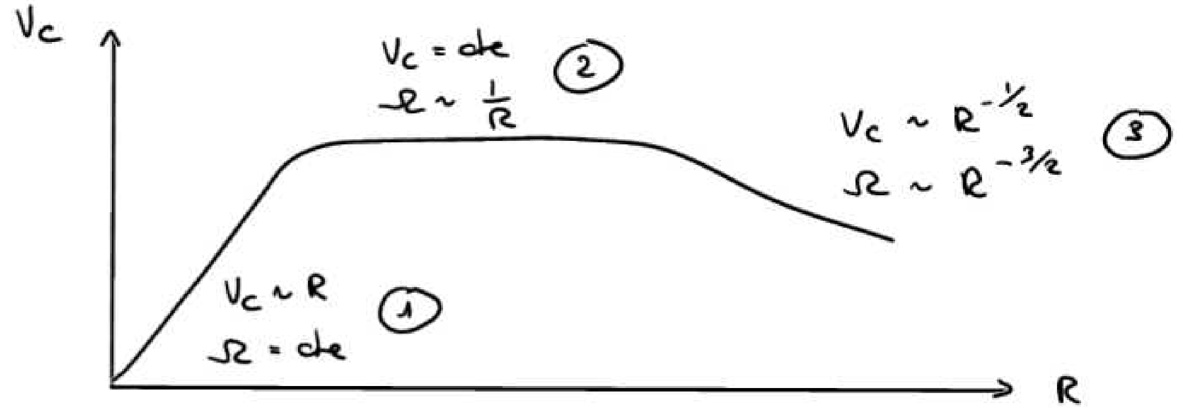
$\varkappa$ obtained by derivating V_c^2

Periods :

$$\begin{cases} \text{radial} & T_R := \dfrac{2\pi}{\varkappa} \\ \text{vertical} & T_z := \dfrac{2\pi}{\nu} \\ \text{azimuthal} & T_\theta := \dfrac{2\pi}{\Omega} \end{cases}$$

Radial dependency of κ, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- 1 <u>near the center</u>

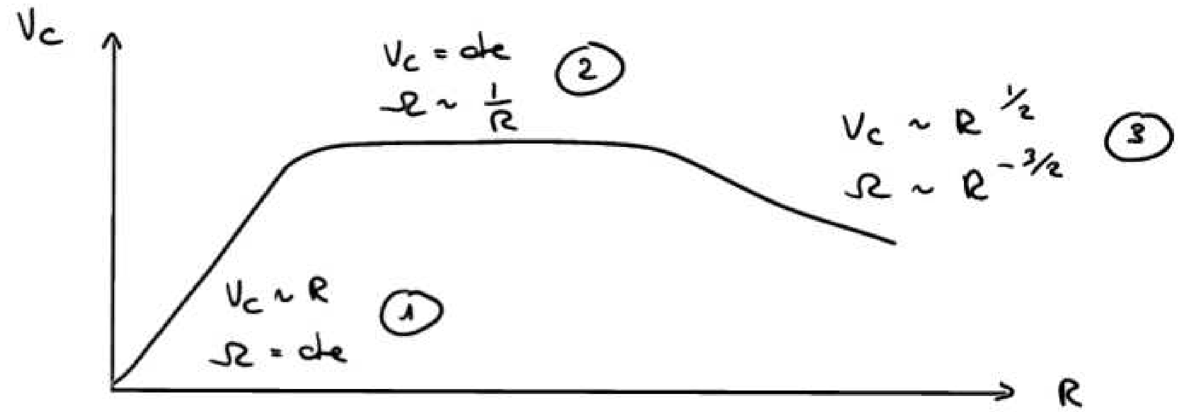
$V_c \sim R$ (rigid rotation) $\Rightarrow$ $\Omega = cte$

$$\kappa^2 = R\frac{d}{dR}(\Omega^2) + 4\Omega^2 \quad \Rightarrow \quad \kappa^2 = 4\Omega^2$$

$$\boxed{\kappa \sim 2\Omega}$$

Radial dependency of $\varkappa$, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



• 2 flat rotation part

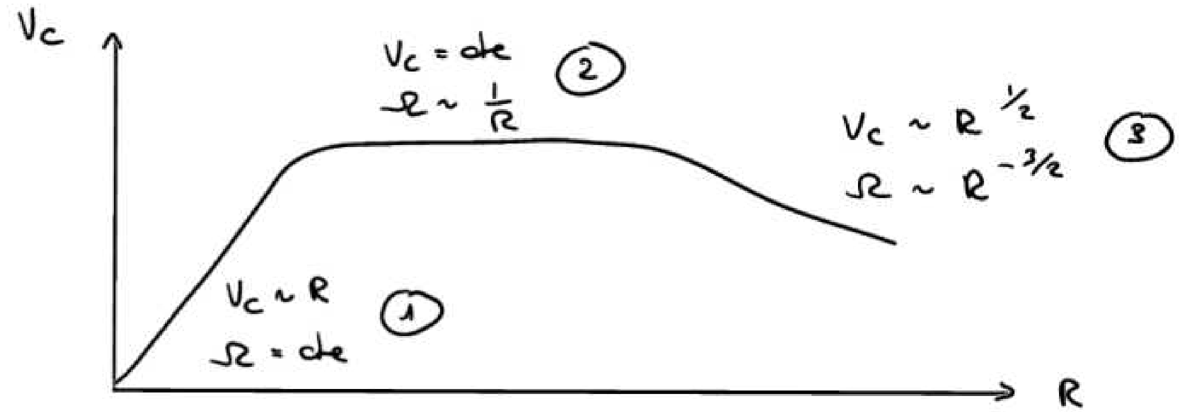
$$V_c = \text{cte} \quad \Rightarrow \quad \Omega \sim \frac{1}{R}$$

$$\varkappa^2 = \frac{1}{R}\frac{\partial}{\partial R}(V_c^2) + 2\Omega^2 \quad \Rightarrow \quad \varkappa^2 = 2\Omega^2$$

$$\boxed{\varkappa \sim \sqrt{2}\,\Omega}$$

Radial dependency of κ, ν for a typical galaxy

$$\boxed{\Omega = \frac{V_c}{R}}$$



- 3 further out

$$V_c \sim R^{-1/2} \quad \text{(Keplerian decrease)}$$

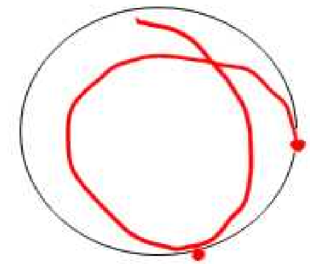
$$\Omega = \frac{V_c}{R} \sim R^{-3/2} \qquad \Omega^2 \sim R^{-3} \qquad \frac{\partial}{\partial R}(\Omega^2) = -3\,\frac{\Omega^2}{R}$$

$$\kappa^2 = \underbrace{R\,\frac{\partial}{\partial R}(\Omega^2)}_{-3\Omega^2} + 4\Omega^2$$

$$\boxed{\kappa = \Omega}$$

Thus, in general

$$\boxed{\Omega \leqslant \varkappa \leqslant 2\,\Omega}$$



Integrals of motions

$$\begin{cases} \ddot{x} = -\kappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$\Rightarrow$ Two integrals of motion (one for each oscillator)

1) $H_R = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\kappa^2 x^2$

2) $H_z = \frac{1}{2}\dot{z}^2 + \frac{1}{2}\nu^2 z^2$

Thus, if a star oscillates near a circular orbit :

3 integrals of motions L_z, H_R, H_z

Total Hamiltonian (near a circular orbit of radius R_s)

$$\boxed{H(R,\dot R,\theta,\dot\theta,z,\dot z) = \frac{1}{2}\left(\dot R^2 + R^2\dot\theta^2 + \dot z^2\right) + \phi(R,z)}$$

$L_z = R^2\dot\theta$

$$= \frac{1}{2}\dot R^2 + \frac{1}{2}\dot z^2 + \underbrace{\phi(R,z) + \frac{L_z^2}{2R^2}}_{\phi_{eff}(R,z)}$$

$$= \frac{1}{2}\dot R^2 + \frac{1}{2}\dot z^2 + \phi_{eff}(R_s,0) + \frac{1}{2}\kappa^2(R-R_s)^2 + \frac{1}{2}\nu^2 z^2$$

$$\boxed{H(R,p_R,z,p_z) = H_R(R,p_R) + H_z(z,p_z) + \phi_{eff}(R_s,0)}$$

Orbital motions

$$\begin{cases} \ddot{x} = -\kappa^2(R_g)\, x \\ \ddot{z} = -\nu^2(R_g)\, z \end{cases}$$

$+ \quad R^2\dot{\theta} = L_z$

Solutions

① motion in z

$$\boxed{z(t) = Z \cos(\nu t + \xi)}$$

② motion in x

$$\boxed{x(t) = X \cos(\kappa t + \alpha)}$$

Note valid only for small oscillations

as long as $\nu^2 = \frac{\partial^2 \phi}{\partial z^2} \simeq cte$

ie $\rho_{disk} \simeq cte \quad \left(\nu^2 = \frac{\partial^2 \phi}{\partial z^2} = 4\pi G \rho \right)$

$\rightarrow z <$ disk scale lenght

$\sim 300\,pc$

③ motion in Θ

$$\boxed{L_z = R^2 \dot{\Theta}}$$

$$\Theta(t) = L_z \int_{t_0}^{t} dt' \frac{1}{R^2(t')} = L_z \int_{t_0}^{t} dt' \frac{1}{\left(R_g + x(t')\right)^2}$$

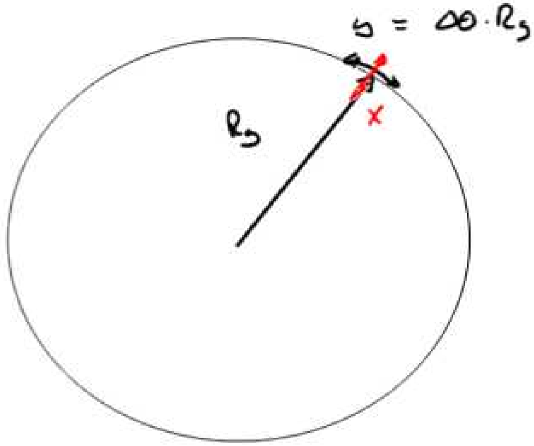
$$\simeq \frac{L_z}{R_g^2} \int_{t_0}^{t} dt' \frac{1}{\left(\frac{x}{R_g} + 1\right)^2} \overset{\text{Taylor}}{\simeq} \Omega_g \int_{t_0}^{t} dt' \left(1 - \frac{2x(t)}{R_g}\right)$$

$$\Omega_g = \frac{L_z}{R_g^2}$$

introducing $x(t) = X \cos(\varkappa t + \alpha)$

$$\boxed{\Theta(t) = \underbrace{\Omega_g \cdot t}_{\text{motion of the guiding center along the circular orbit}} - \underbrace{\frac{2\Omega_g}{\varkappa}\frac{X}{R_g} \sin(\varkappa t + \alpha)}_{\text{oscillations}} + \Theta_0}$$

New carthesian system x, y, z with an origin that follows the guiding center



$$\begin{cases} R(t) = R_g \\ \theta(t) = \Omega_g t + \theta_0 \end{cases}$$

Then, from

$$\theta(t) = \Omega_g \cdot t \underbrace{- \frac{2\Omega_g}{\varkappa} \frac{X}{R_g} \sin(\varkappa t + \alpha)}_{\Delta\theta = \frac{y}{R_g}} + \theta_0$$

$$y = - \frac{2\Omega_g}{\varkappa} X \sin(\varkappa t + \alpha)$$

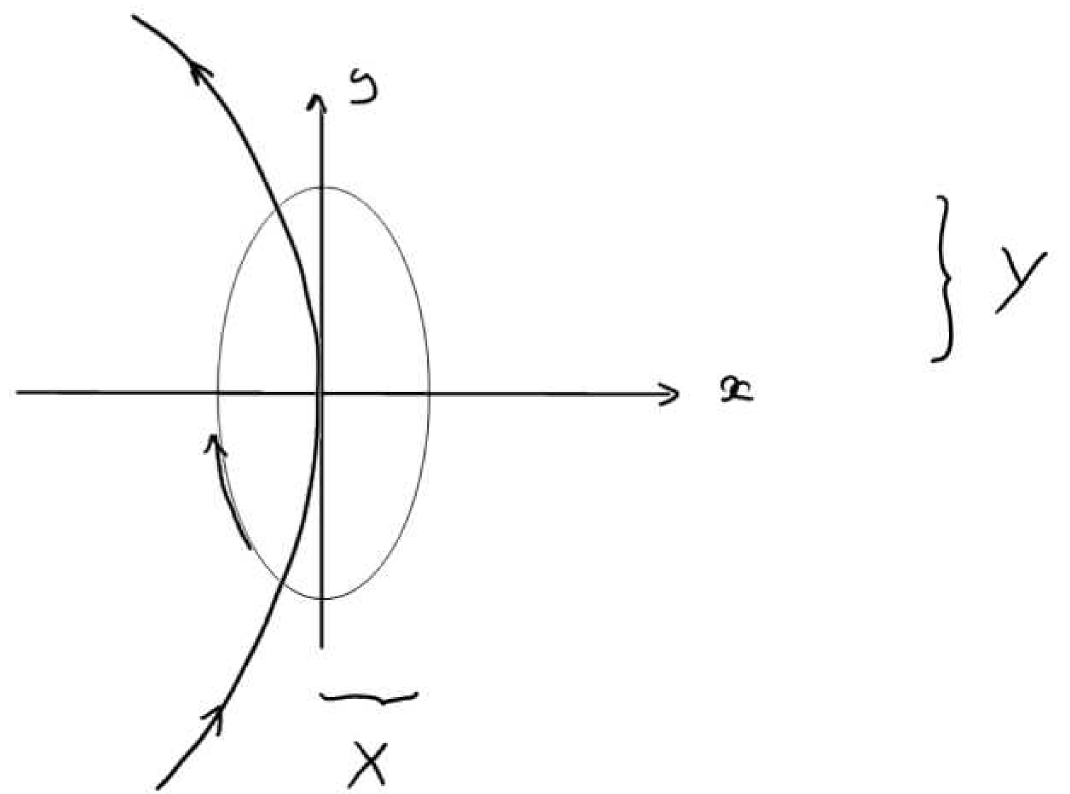
$$\boxed{y(t) = - Y \sin(\varkappa t + \alpha)} \qquad Y := \frac{2\Omega_g}{\varkappa} X$$

Complete solution

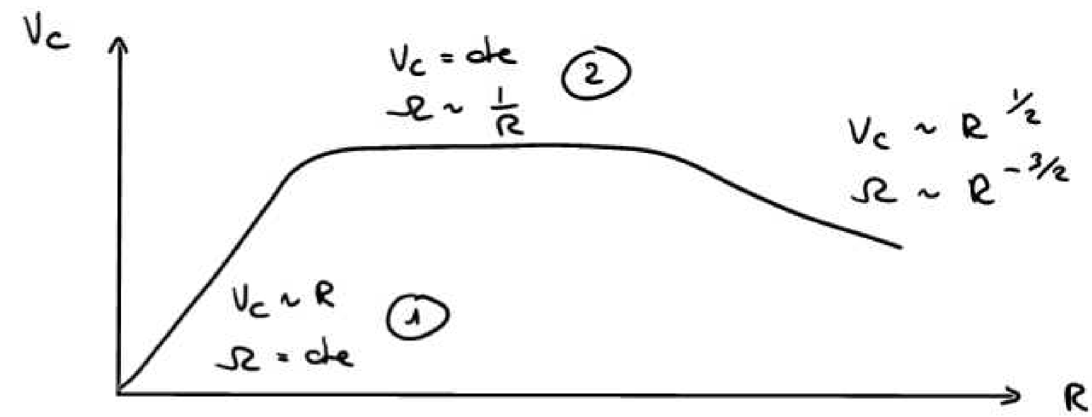
$$\begin{cases} x(t) = X\cos(\varkappa t + \alpha) \\ y(t) = -Y\sin(\varkappa t + \alpha) \\ z(t) = Z\cos(\nu t + \xi) \end{cases}$$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ ellipse

$$Y = \frac{2\Omega_y}{\varkappa} X$$



Radial dependency for a typical galaxy



1 near the center

$\varkappa = 2\Omega$ $\qquad \frac{X}{Y} = 1$ $\qquad$ circle

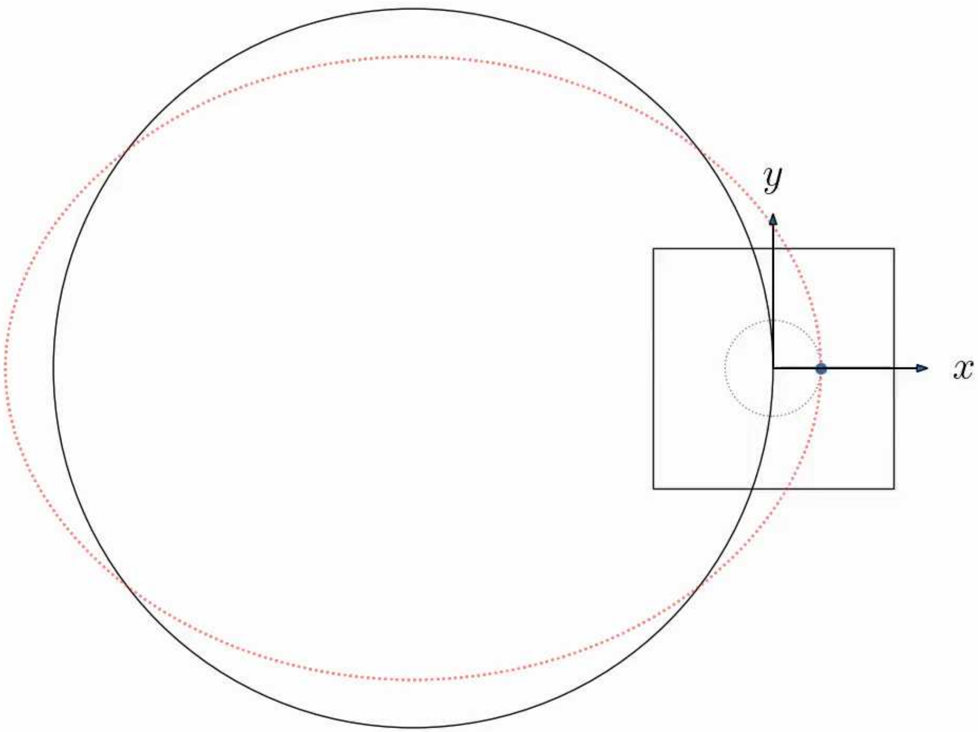
2 flat rotation part

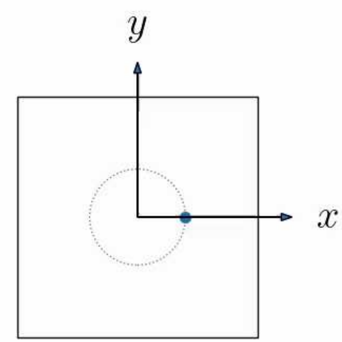
$\varkappa = \sqrt{2}\,\Omega$ $\qquad \frac{X}{Y} = \frac{\sqrt{2}\,\Omega}{2\Omega}$ $\qquad X < Y$

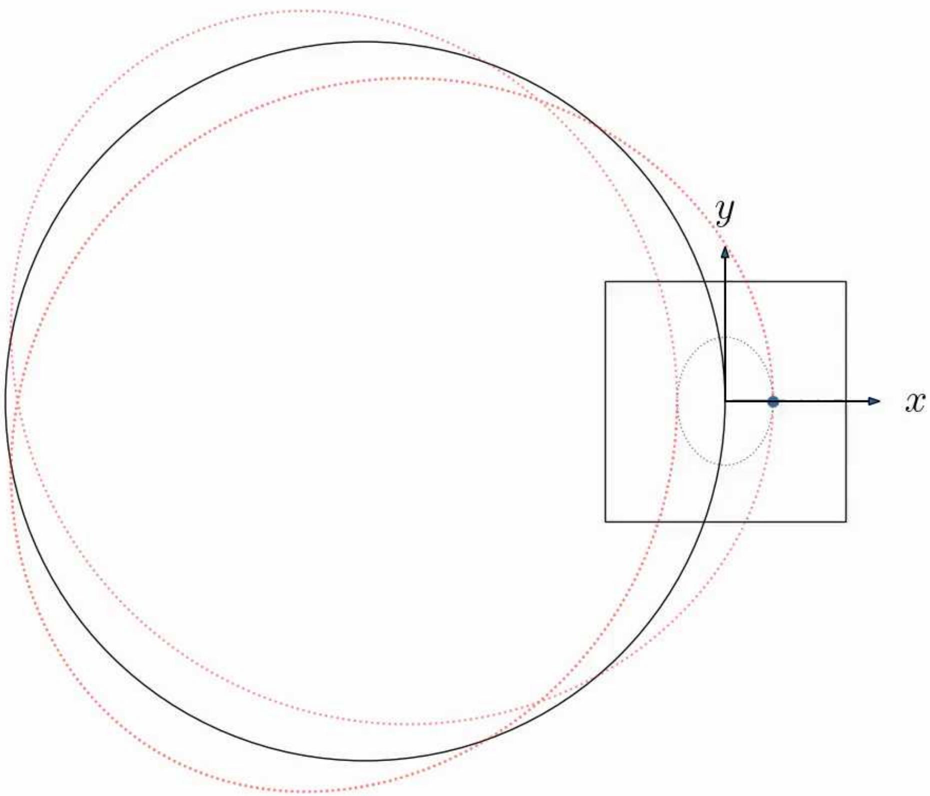
3 further out

$\varkappa = \Omega$ $\qquad \frac{X}{Y} = \frac{\Omega}{2\Omega}$ $\qquad X < Y$

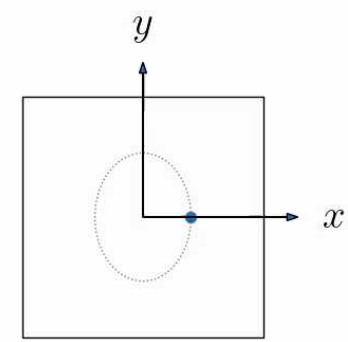
$$\kappa/\Omega = 2.0$$



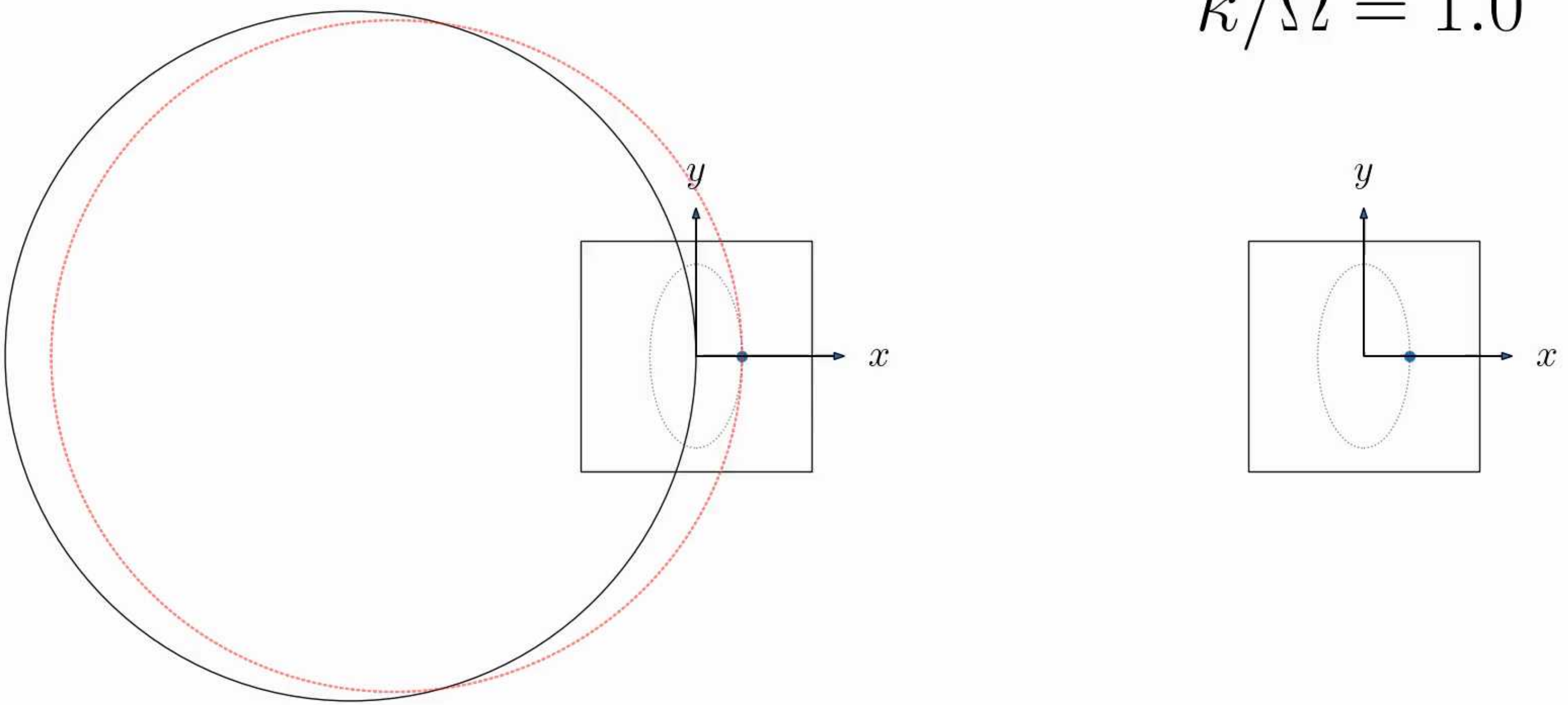




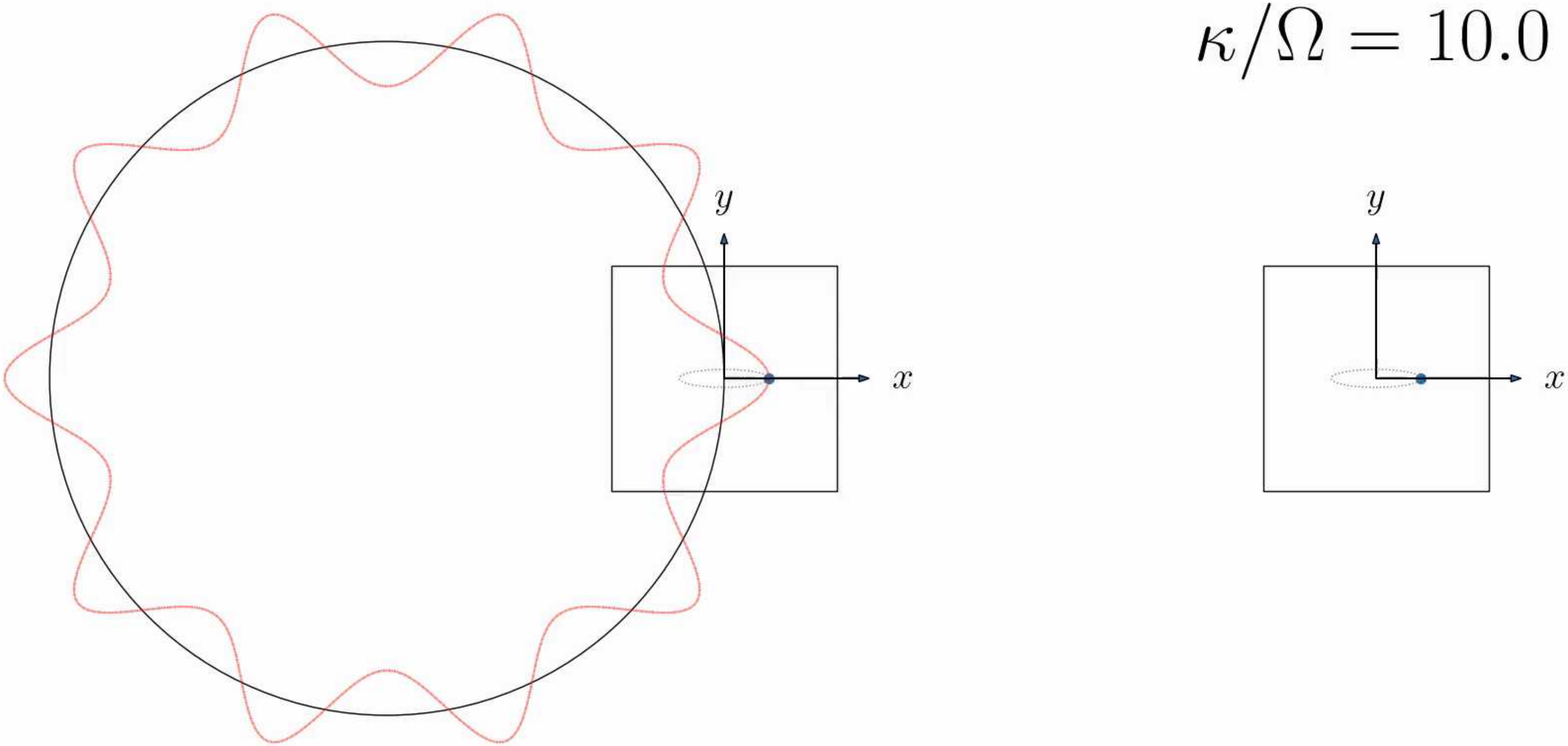
$$\kappa/\Omega = 1.5$$



$$\kappa/\Omega = 1.0$$



$$\kappa/\Omega = 10.0$$



The End