Astrophysics IV, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 3

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Astrophysics IV: Stellar and galactic dynamics

Solutions

Problem 1:

From Poisson's equation in spherical coordinates we get:

$$\nabla^2\Phi = 4\pi G\rho$$

$\nabla^2\Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

one then obtains

$$\nabla^2\Phi = \frac{3GMb^2}{(r^2+b^2)^{5/2}}$$

and finally:

$$\rho = \frac{3M}{4\pi b^3}\left(1+\frac{r^2}{b^2}\right)^{-5/2}$$

Problem 2:

a) Point mass:

$$V_{\mathrm{c}}^2(r) = \frac{GM}{r}$$

b) Homogeneous sphere of radius a:

$$V_{\mathrm{c}}^2(r) = \begin{cases} \frac{GMr^2}{a^3} & \text{if } r < a \\ \frac{GM}{r} & \text{if } r \geq a \end{cases}$$

c) Plummer-Schuster potential:

$$V_{\mathrm{c}}^2(r) = \frac{GMr^2}{(r^2+a^2)^{3/2}}$$

d) Miyamoto-Nagai potential:

$$V_{\mathrm{c}}^2(R) = \frac{GMR^2}{\left[R^2+(a+b)^2\right]^{3/2}}$$

Problem 3:

We are still in the plane $z = 0$ (where the rotation curves are defined.) With the parametrization:

$$h_R = a + b$$
$$h_z = b$$

the circular velocity of the Miyamotio-Nagai potential can be written:

$$V_c^2(R) = \frac{GMR^2}{(R^2 + h_R^2)^{3/2}}$$

which is obviously independent of the scale height h_z. This parametrization is more telling than the a, b one: it shows how a Miyamoto-Nagai system has a circular velocity independent of the flattening of the potential. The two extremes are:

- spherical symmetry: $a = 0 \implies h_R = h_z = b$,
- thin disk: $b = 0 \implies h_R = a,\ h_z = 0$.

The rotation in the plane $z = 0$ is the same for these two extreme cases since V_c^2 is independent of h_z.

Problem 4:

From Poisson's equation in spherical coordinates we get:

$$\nabla^2\Phi = 4\pi G\rho$$

$\nabla^2\Phi$ written in spherical coordinates, and considering a spherical potential we get:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)$$

a lot of straight-forward algebra follows, but finally we get

$$\rho = \frac{v_s^2}{4\pi G r_s^2}\frac{1}{(r/r_s)(1 + r/r_s)^2}$$

The circular velocity also follows simply:

$$\begin{aligned} v_c^2 &= r\,\frac{\partial\Phi}{\partial r} = r\,v_s^2\left[-\frac{1}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)} + \frac{\ln\left(1+\frac{r}{r_s}\right)}{r_s\left(\frac{r}{r_s}\right)^2}\right] = v_s^2\left[\frac{\ln\left(1+\frac{r}{r_s}\right)}{\frac{r}{r_s}} - \frac{1}{\left(1+\frac{r}{r_s}\right)}\right] \\ &= v_s^2\left[\frac{\left(1+\frac{r}{r_s}\right)\ln\left(1+\frac{r}{r_s}\right) - \frac{r}{r_s}}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)}\right] = v_s^2\,\frac{(r_s+r)\;r_s\,\ln\left(1+\frac{r}{r_s}\right) - r\,r_s}{r\;(r_s+r)} \end{aligned}$$

Problem 5:

As per problem 4, the isochrone ρ is straightforward to derive, taking the form:

$$\rho = M \left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3} \right] \quad \text{with} \quad a \equiv \sqrt{b^2 + r^2}$$

The circular velocity is

$$v_c^2 = \frac{GMr^2}{(b+a)^2 a}$$

Problem 6:

Let's define a unit surface on the disk, corresponding to a mass Σ, which is then the surface density. Defining a slab enclosing the unit surface and making its thickness tend to a vanishing value ($\varepsilon \to 0$, see Fig. 1), the surface integral reduces to twice the gradient of the potential:

$$4\pi\, G\, \Sigma = \int \mathrm{d}^2\mathbf{S}\, \nabla\Phi_\mathrm{K} = 2\frac{\partial \Phi_\mathrm{K}}{\partial z}$$

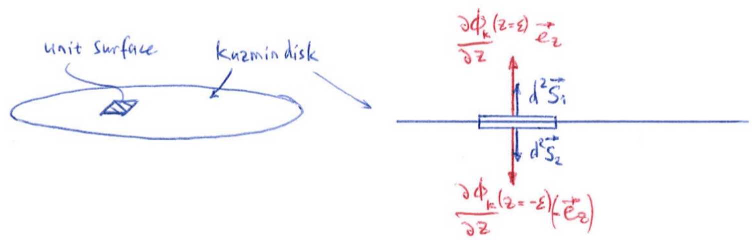

Figure 1: The Kuzmin disk with the unit surface (left) and seen edge-on (right), with the 2ε thick slab, on the surface of which the integration is made.

We have

$$\begin{aligned} \frac{\partial \Phi_\mathrm{K}}{\partial z} &= \frac{\partial}{\partial z}\left[-GM \left[R^2 + (a+|z|)^2 \right]^{-1/2} \right] \\ &= GM \left[R^2 + (a+|z|)^2 \right]^{-3/2} (a+|z|) \end{aligned}$$

With $|z| \to 0$, we then have:

$$\begin{aligned} 4\pi\, G\Sigma_\mathrm{K} &= 2\frac{\partial \Phi_\mathrm{K}}{\partial z} = 2aGM \left[R^2 + a^2 \right]^{-3/2} \\ \Rightarrow \Sigma_\mathrm{K} &= \frac{aM}{2\pi \left(R^2 + a^2 \right)^{3/2}} \end{aligned}$$

Problem 7:

The velocity curve may be obtained from the formula (see course: result from a razor-thin homeoid since we cannot use Gauss law here):

$$v_{\rm c}^2(R) = -4G \int_0^R \mathrm{d}a \frac{a}{\sqrt{R^2-a^2}} \frac{\mathrm{d}}{\mathrm{d}a} \int_a^\infty \mathrm{d}R' \frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} \tag{1}$$

Replacing $\Sigma(R')$ using the Mestel's surface density we get:

$$\begin{aligned}
\int_a^\infty \mathrm{d}R' \frac{R'\Sigma(R')}{\sqrt{R'^2-a^2}} &= \frac{v_0^2}{2\pi G} \int_a^\infty \mathrm{d}R' \frac{1}{\sqrt{R'^2-a^2}} \\
&= \frac{v_0^2}{2\pi G} \int_a^{R_{\rm max}} \mathrm{d}R' \frac{1}{\sqrt{(R'/a)^2-1}} \frac{1}{a} \\
&= \frac{v_0^2}{2\pi G} \int_a^{R_{\rm max}} \mathrm{d}R' \frac{\mathrm{d}}{\mathrm{d}R} \left(\mathrm{arccosh}(R/a)\right) \\
&= \frac{v_0^2}{2\pi G} \left[\mathrm{arccosh}(R_{\rm max}/a) - \mathrm{arccosh}(1)\right] \\
&= \frac{v_0^2}{2\pi G} \mathrm{arccosh}(R_{\rm max}/a)
\end{aligned} \tag{2}$$

The derivative with respect to a of this latter result writes:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{v_0^2}{2\pi G}\mathrm{arccosh}(R_{\rm max}/a)\right) &= \frac{v_0^2}{2\pi G}\frac{\mathrm{d}}{\mathrm{d}a}\mathrm{arccosh}(R_{\rm max}/a) \\
&= -\frac{v_0^2}{2\pi G}\frac{R_{\rm max}}{\sqrt{R_{\rm max}^2-a^2}}\frac{1}{a}
\end{aligned} \tag{3}$$

which, in the limit $R_{\rm max} \to \infty$ gives:

$$-\frac{v_0^2}{2\pi G a} \tag{4}$$

This leads to the circular velocity:

$$\begin{aligned}
v_{\rm c}^2(R) &= \frac{2v_0^2}{\pi} \int_0^R \mathrm{d}a \frac{1}{\sqrt{R^2-a^2}} \\
&= v_0^2
\end{aligned} \tag{5}$$