Astrophysics III, Dr. Yves Revaz — EPFL

4th year physics — Exercises week 2

29.09.2021 — Autumn semester 2021

Astrophysics III : Stellar and galactic dynamics

Solutions

Problem 1 :

The disk surface density can be expressed as a function of the total mass and radius of the disk :

$$\Sigma = \frac{f M_{\rm tot}}{\pi R^2}. \quad (1)$$

Hence :

$$R = \sqrt{\frac{f \cdot M_{\rm tot}}{\pi \Sigma}}. \quad (2)$$

With $f = 1/45$, $M_{\rm tot} = 2 \cdot 10^{12}\,\mathrm{M}_\odot$, we thus have : $R = 16.8\,\mathrm{kpc}$.

The mean density is :

$$< \rho >= \frac{f \cdot M_{\rm tot}}{\pi R^2 \cdot 500} \sim 0.1\ \mathrm{M}_\odot\,\mathrm{pc}^{-3} \quad (3)$$

with a thickness of $500\,\mathrm{pc}$, $R = 16800\,\mathrm{pc}$ and $f = 1/45$.

The period of a circular orbit is :

$$T = \frac{2\pi R}{\sqrt{GM(R)/R}}, \quad (4)$$

So :

$$M = \frac{4\pi^2 R^3}{GT^2}. \quad (5)$$

With $G = 6.67 \cdot 10^{-11}\,\mathrm{m}^3/\mathrm{kg}/\mathrm{s}^2$, $R = 8\,\mathrm{kpc} = 2.46 \cdot 10^{20}\,\mathrm{m}$ and $T = 220\,\mathrm{Myr} = 6.9 \cdot 10^{15}\,\mathrm{s}$, we find approximately $10^{11}\,\mathrm{M}_\odot$.

Problem 2 :

For a galaxy cluster, the cluster typical radius is 1 Mpc and a typical galaxy size is 10 kpc :

$$\frac{\text{Volume (N galaxies)}}{\text{Volume (galaxy cluster)}} \simeq \frac{10^3 \cdot (10\ \mathrm{kpc})^3}{(1000\ \mathrm{kpc})^3} = 10^{-3}$$

For a galaxy :

$$\frac{\text{Volume (N stars)}}{\text{Volume (galaxy)}} \simeq \frac{10^{11} \cdot (10^6\ \mathrm{km})^3}{(10^4\ \mathrm{pc} \cdot 3.09 \cdot 10^{13}\ \mathrm{km/pc})^3} \simeq 4 \times 10^{-24}$$

We thus see that those dynamical systems are largely made of void.

Problem 3 :

In a galaxy cluster, the typical speed of galaxies is $v = 10^3$ km/s$\cdot(3.09\cdot10^{16}$ km/kpc$)^{-1} = 3.09\cdot10^{-14}$ kpc/s, hence :

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy cluster)}} \simeq \frac{\pi(10\text{ kpc})^2\cdot v\cdot t}{\frac{4\pi}{3}(1000\text{kpc})^3} \simeq 7.7\cdot10^{-4}$$

Within a galaxy, stars typically have $v = 200$ km/s, hence :

$$\frac{\text{Volume (tube)}}{\text{Volume (galaxy)}} \simeq \frac{\pi(10^6\text{ km})^2\cdot v\cdot t}{\frac{4\pi}{3}(10^4\text{ pc}\cdot3\cdot10^{13}\text{ km/pc})^3} \simeq 1.6\cdot10^{-21}$$

Given the tiny portion of the volume crossed by the components, we conclude that the probability of a collision between two components of a given dynamical system is extremely low.

Problem 4 :

We consider that the gravitational influence of an object is significant when the mutual potential energy is of the same order than the kinematic energy of the relative motion.

$$E_{\text{cin}} \simeq E_{\text{grav}} \quad\Leftrightarrow\quad \frac{1}{2}mv^2 \simeq \frac{\text{G}m^2}{R_G} \quad\Leftrightarrow\quad R_G \simeq \frac{2\text{G}m}{v^2}$$

We observe that this result gives the value of $b_{90} = 2Gm/v^2$ seen in the course, so that $R_G = b_{90}$.

For a galaxy moving within a galaxy cluster :

$$R_G \simeq \frac{2\cdot6.67\cdot10^{-11}\text{m}^3/\text{kg/s}^2\cdot10^{11}\cdot2\cdot10^{30}\text{ kg}}{(10^6\text{ m/s})^2} \simeq 10^{19}\text{ m} \simeq 1\text{ kpc}$$

For a star moving within a galaxy :

$$R_G \simeq \frac{2\cdot6.67\cdot10^{-11}\text{m}^3/\text{kg/s}^2\cdot2\cdot10^{30}\text{ kg}}{(2\cdot10^5\text{ m/s})^2} \simeq 0.7\cdot10^{10}\text{ m} \simeq 0.05\text{ A.U.}$$

Problem 5 :

The solution to this is given by the fraction of galaxies which have their symmetry axis (normal to the disk) which covers the ten degrees of the spherical cap. We integrate one half of the hemisphere :

$$\begin{aligned}
P(i < i_0) &= \frac{\int_0^{2\pi}\text{d}\phi\int_0^{i_0}\text{d}\theta\sin\theta}{\int_0^{2\pi}\text{d}\phi\int_0^{\pi/2}\text{d}\theta\sin\theta} \\
&= \frac{2\pi\int_0^{i_0}\text{d}\theta\sin\theta}{2\pi} \\
&= \int_0^{i_0}\text{d}\theta\sin\theta \\
&= 1-\cos i_0
\end{aligned}$$

For $i_0 = 10°$, we then get

$$P(i < 10°) = 1 - \cos 10° = \quad 0.015$$

For the fraction seen edge on, we have $P(i > 80°) = 1 - P(i < 80°) = 0.17$

Problem 6 :

The relaxation time is given by

$$t_{relax} = \frac{0.1\,N}{\ln N}\, t_{cross} = \frac{0.1\,N}{\ln N}\,\frac{R}{v}$$

Therefore we have

1. For the open cluster :

$$t_{relax} = \frac{0.1 \cdot 300}{\ln 300}\,\frac{2 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{0.5\ \mathrm{km\,s^{-1}}} \simeq 6.5 \cdot 10^{14}\ \mathrm{s} \simeq 2 \cdot 10^{7}\ \mathrm{yrs}$$

   The youngest open clusters are not relaxed yet, but the oldest ones are fully relaxed.

2. For the globular cluster :

$$t_{relax} = \frac{0.1 \cdot 2 \cdot 10^{5}}{\ln(2 \cdot 10^{5})}\,\frac{3 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{6\ \mathrm{km\,s^{-1}}} \simeq 2.5 \cdot 10^{16}\ \mathrm{s} \simeq 8 \cdot 10^{8}\ \mathrm{yrs}$$

   The globular clusters are fully relaxed.

3. For a dwarf spheroidal galaxy :

$$t_{relax} = \frac{0.1 \cdot 10^{7}}{\ln(10^{7})}\,\frac{500 \cdot 3.09 \cdot 10^{13}\ \mathrm{km}}{10\ \mathrm{km\,s^{-1}}} \simeq 1 \cdot 10^{20}\ \mathrm{s} \simeq 3 \cdot 10^{12}\ \mathrm{yrs}$$

   Dwarf spheroidal galaxies are far from being relaxed.

Problem 7 :

The relaxation time is given by a ratio between the velocity (which is a factor of the total mass of the system) and the average change in velocity per unit time. If the number of particles increases, the average velocity of a particle increases as $\propto \sqrt{N}$, while the average change in velocity per unit time only increases by $\propto \log N$. Therefore, if other factors are held constant, adding more members (and therefore mass) will result in an increase in the relaxation time of a system.