

Potential Theory I

Outlines

Potential Theory : general results

- Gravitational field force, gravitational potential
- Gauss Law
- Poisson Equation
- Total potential energy

Spherical systems:

- Newton's Theorems
- Circular speed, circular velocity, circular frequency, escape speed, potential energy

Examples of spherical models:

- “Potential based” models
- “Density based” models

Potential theory : general results

Goal : compute the gravitational potential / forces
due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

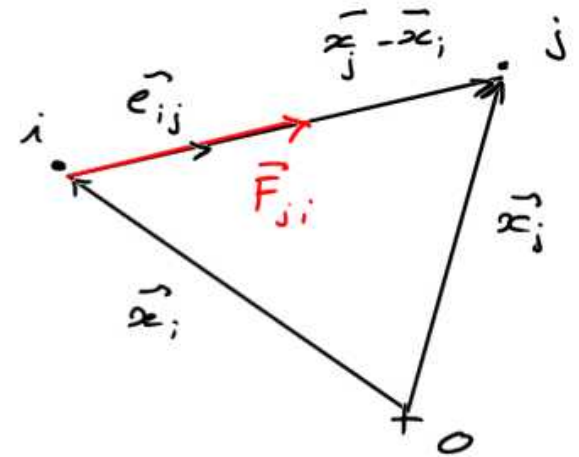
As the relaxation time of such system is very
large (\gg the age of the Universe) we can describe
the system with a smooth analytical potential / density.

Newton Law

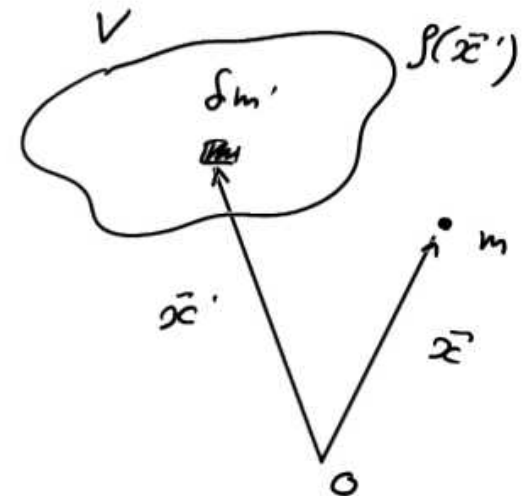
$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

Force on a particle of mass m in \vec{x}
due to a distribution of mass $\rho(\vec{x})$

$$\begin{aligned} \delta \vec{F}(\vec{x}) &= \frac{G m \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \\ &= \frac{G m \rho(\vec{x}') d^3 \vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \end{aligned}$$



$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_i$$



So, the total force writes :

$$\vec{F}(\vec{x}) = \int_V \frac{G m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'$$

$$= m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'}_{\vec{g}(\vec{x})}$$

$\vec{g}(\vec{x})$: gravitational field

$$[\vec{g}] = \frac{\text{cm}}{\text{s}^2} = \frac{\text{erg}}{\text{g}} \frac{1}{\text{cm}}$$

Gravitational Potential

It is easy to see that the function

$$\delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|} \quad \text{is such that}$$

$$\vec{\nabla} \delta V(\vec{x}) = - \frac{G m \delta m}{|\vec{x}' - \vec{x}|^2} \frac{(\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|} = - \delta \vec{F}(\vec{x})$$

so, by defining

$$V(\vec{x}) = - G \int_V \frac{m \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3\vec{x}'$$

we ensure that

$$\vec{\nabla} V(\vec{x}) = - \vec{F}(\vec{x})$$

We define the specific potential $\phi(\bar{x}) = \frac{V(\bar{x})}{m}$

which writes

$$\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} d^3\bar{x}'$$

$$[\phi] = \frac{\text{erg}}{g} \\ \equiv \text{specific energy}$$

The gravitational field writes :

$$\vec{g}(\bar{x}) = -\vec{\nabla} \phi(\bar{x})$$

Notes

- The gravity is a conservative force
 - $\phi(\vec{x})$: scalar field
 $\vec{g}(\vec{x})$: vector field
- } contain the same information
- we will always use "specific" quantities

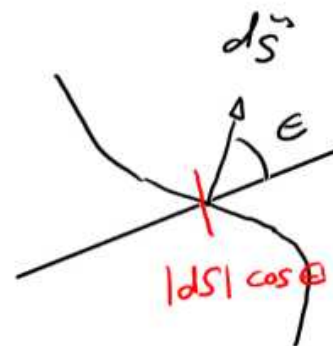
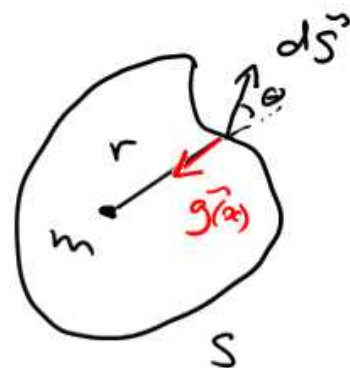
$$V(\vec{x}) \rightarrow \phi(\vec{x})$$

$$K = \frac{1}{2} m \vec{v}^2 \rightarrow \frac{1}{2} \vec{v}^2$$

$$\frac{1}{2} v^2 + \phi(\vec{x}) = \text{specific energy (conserved quantity)}$$

The Gauss's Law

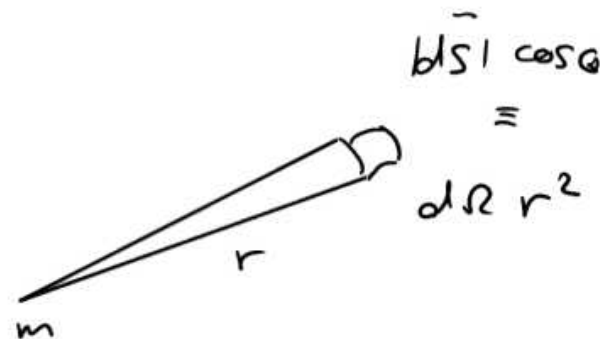
- Consider :
- a single point mass m
 - a surface S around this point
 - a point \vec{x} on the surface at a distance r
 - $\vec{g}(\vec{x})$ the gravitational field
 - $d\vec{S}$, the normal at the surface
 - θ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$



$$\vec{g}(\vec{x}) \cdot d\vec{S} = -|\vec{g}(\vec{x})| \cdot |d\vec{S}| \cos \theta$$

But $|d\vec{S}| \cos \theta = r^2 d\Omega$

$$|\vec{g}(\vec{x})| = \frac{Gm}{r^2}$$



$$\vec{g}(\vec{x}) \cdot d\vec{S} = -Gm d\Omega$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m \\ 0 \end{cases}$$

if m inside S

instead

For multiple masses m_i :

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x}) d\vec{x} = -4\pi G M$$

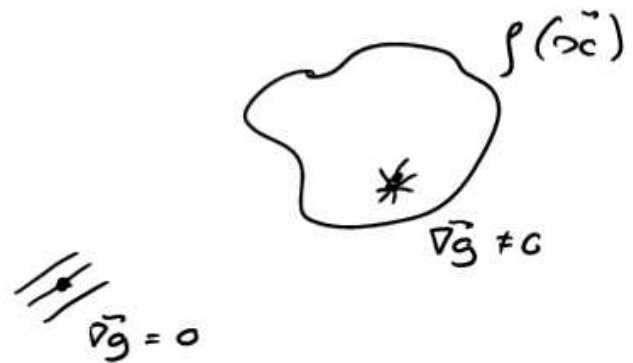
Gauss's Law

Divergence of the specific force (A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) d^3\vec{x} \stackrel{\text{div. theorem}}{=} \int_S \vec{g}(\vec{x}) d\vec{S}$$

$$\stackrel{\text{Gauss's Law}}{=} -4\pi G \int_V \rho(\vec{x}) d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})}$$



The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$

$$\vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$$

$$\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})$$

Poisson Equation

Note : To ensure a unique solution, boundary conditions are necessary (2nd order diff. eqn.)

ex : $\phi(\infty) = 0$

$$\vec{\nabla} \phi(\infty) = \vec{g}(\infty) = 0$$

Divergence of the specific force (B) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

$$\begin{aligned} \cdot \vec{\nabla}_x \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) &= \frac{d}{dx_1} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3} \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) \\ &= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5} \\ &= \underline{\underline{0}} \quad \text{if} \quad \vec{x}' \neq \vec{x} \end{aligned}$$

$$\vec{\nabla}_{\vec{x}} \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3 \vec{x}'$$

$$= G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{\vec{x}} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3 \vec{x}'$$



variable exchange

$$\vec{\nabla}_{\vec{x}} f(\vec{x} - \vec{x}') = - \vec{\nabla}_{\vec{x}'} f(\vec{x} - \vec{x}')$$

$$= - G \rho(\vec{x}) \int_{|\vec{x}' - \vec{x}| \leq h} \vec{\nabla}_{\vec{x}'} \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) d^3 \vec{x}'$$

divergence theorem

$$= - G \rho(\vec{x}) \underbrace{\int_{|\vec{x}' - \vec{x}| = h} \frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} d^2 \vec{S}'}_{4\pi h^2 \cdot \frac{1}{r^2} \Big|_{h=r}} = - 4\pi$$

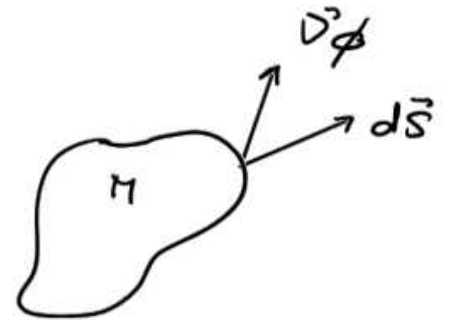
$$r = |\vec{x}' - \vec{x}| = h$$

$$\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} = \frac{1}{r^2}$$

$$\vec{\nabla}_{\vec{x}} \cdot \vec{g}(\vec{x}) = - 4\pi G \rho(\vec{x})$$

Gauss theorem (B) integrate the Poisson equation over a volume V that contains a mass M

$$\int_V \vec{\nabla}^2 \phi(\vec{x}) d^3\vec{x} = \underbrace{\int_V 4\pi G \rho(\vec{x}) d^3\vec{x}}$$



div.
theorem

$$\int_S d^2\vec{s} \cdot \vec{\nabla} \phi = 4\pi G M$$

Gauss theorem

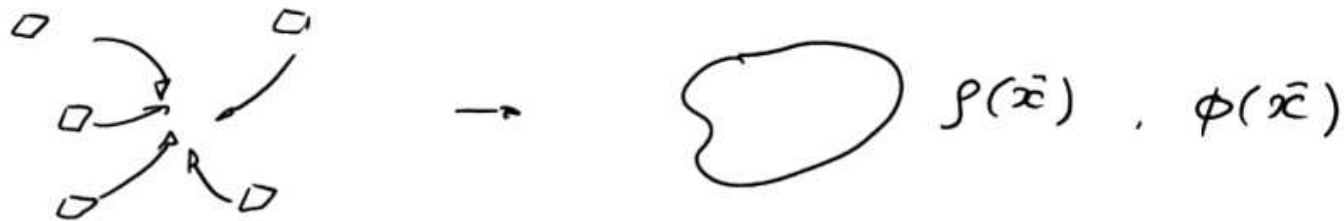
Equivalently :

$$\int_S d^2\vec{s} \cdot \vec{g}(\vec{x}) = -4\pi G M$$

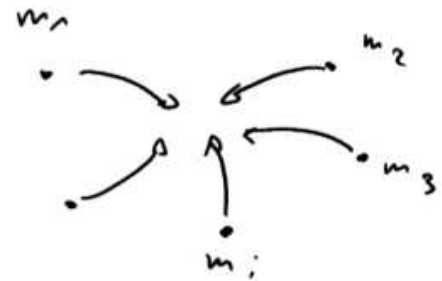
Gauss's Law

Total potential energy (1.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$



Assume a set of discrete points



- The work to bring the 1st point from ∞ to \vec{x}_1 is 0

- The work to bring the 2nd point from ∞ to \vec{x}_2 is $-\frac{Gm_1m_2}{r_{12}}$

- The work to bring the 3rd point from ∞ to \vec{x}_3 is $-\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \dots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$
$$= - \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^{i-1} \frac{G m_i m_j}{r_{ij}} = - \frac{1}{2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = - \sum_{\substack{j=1 \\ j \neq i}} \frac{G m_j}{r_{ij}}$ (potential on i)

$$W = \frac{1}{2} \sum_{i=1}^N m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^N V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3 \vec{x}$$

Total potential energy (1.1)

From
$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3\vec{x}$$

- replace $\rho(\vec{x})$ with the Poisson equation $\rho(\vec{x}) = \frac{1}{4\pi G} \nabla^2 \phi$

$$W = \frac{1}{8\pi G} \int \nabla^2 \phi \cdot \phi(\vec{x}) d^3\vec{x} = \frac{1}{8\pi G} \int \vec{\nabla} \cdot (\vec{\nabla} \phi) \cdot \phi(\vec{x}) d^3\vec{x}$$

- divergence theorem
$$\int d^3x \, g \cdot \vec{\nabla} \cdot \vec{F} = \int_S g \cdot \vec{F} d\vec{S} - \int d^3x \, \vec{F} \cdot \vec{\nabla} g$$

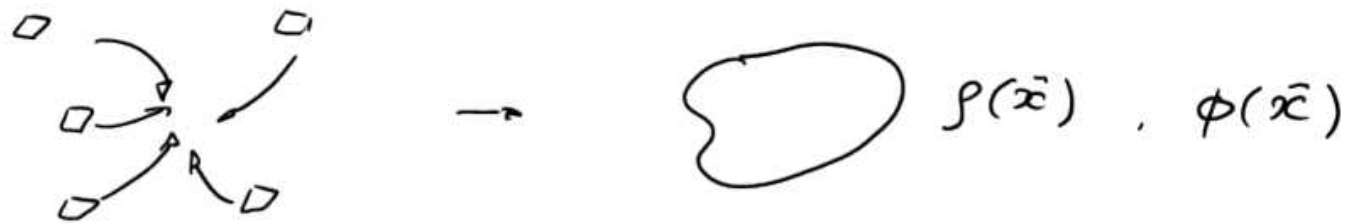
$$W = \frac{1}{8\pi G} \left[\underbrace{\int \phi \vec{\nabla} \phi d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla} \phi(\infty) = 0} - \int d^3\vec{x} \, \vec{\nabla} \phi \cdot \vec{\nabla} \phi \right]$$

$= 0$ as $\phi(\infty) = \vec{\nabla} \phi(\infty) = 0$

$$W = - \frac{1}{8\pi G} \int d^3\vec{x} \, |\vec{\nabla} \phi|^2$$

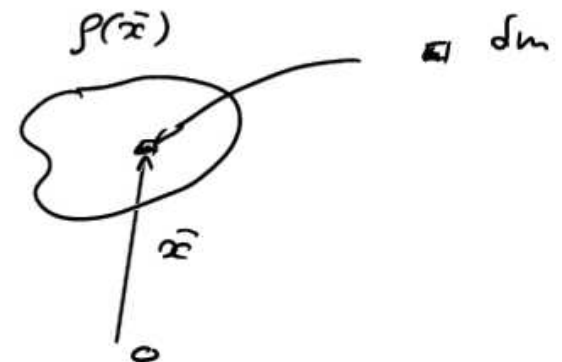
Total potential energy (2.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$



- ① Work done to assemble a piece of mass $\delta m = \delta \rho d^3 \vec{x}$ from ∞ to \vec{x} assuming an existing mass distribution $\rho(\vec{x}), \phi(\vec{x})$

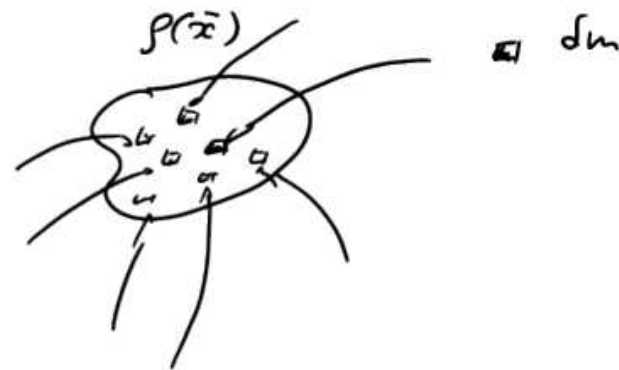
$$\begin{aligned} \delta W_{\vec{x}} &= V(\vec{x}) - \underbrace{V(\infty)}_{=0} \\ &= \delta m \phi(\vec{x}) = \delta \rho(\vec{x}) d^3 \vec{x} \phi(\vec{x}) \end{aligned}$$



To increase energy where the mass distribution by $\delta\rho$

$$\rho(\vec{x}) \rightarrow \rho(\vec{x}) + \delta\rho(\vec{x})$$

$$\delta W = \int \delta\rho(\vec{x}) d^3\vec{x} \phi(\vec{x})$$



Poisson: $\delta\rho(\vec{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(\vec{x})$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(\vec{x}) \phi(\vec{x}) d^3\vec{x}$$

divergence theorem

$$\int_V d^3x \nabla \cdot \vec{F} = \int_S \vec{F} \cdot d\vec{s} - \int_V d^3x \vec{F} \cdot \vec{\nabla} g$$

$$= \frac{1}{4\pi G} \underbrace{\int_{S \rightarrow \infty} \phi(\vec{x}) \vec{\nabla} \delta\phi(\vec{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} (\delta\phi(\vec{x})) d^3\vec{x}$$

as $\phi(\infty) = 0$

$$\vec{\nabla} \delta\phi(\infty) = \delta g(\infty) = 0$$

$$\delta W = - \frac{1}{4\pi G} \int \bar{\nabla} \phi(\vec{x}) \cdot \bar{\nabla}(\delta \phi(\vec{x})) d^3 \vec{x}$$

with

$$\frac{1}{2} \delta |\bar{\nabla} \phi(\vec{x})|^2 = \delta \bar{\nabla} \phi(\vec{x}) \cdot \bar{\nabla} \phi(\vec{x}) = \bar{\nabla}(\delta \phi(\vec{x})) \cdot \bar{\nabla} \phi(\vec{x})$$

$$\delta W = - \frac{1}{8\pi G} \int \delta |\bar{\nabla} \phi|^2 d^3 x = - \frac{1}{8\pi G} \delta \int |\bar{\nabla} \phi|^2 d^3 x$$

② Contribution of all δW to W

$$W = - \frac{1}{8\pi G} \int |\bar{\nabla} \phi|^2 d^3 x$$

Total potential energy (2.2)

$$\text{From } W = - \frac{1}{8\pi G} \int |\vec{\nabla} \phi|^2 d^3x = - \frac{1}{8\pi G} \int \vec{\nabla} \phi \cdot \vec{\nabla} \phi d^3x$$

• divergence theorem $\int d^3x \vec{F} \cdot \vec{\nabla} g = \int_S g \cdot \vec{F} d\vec{S} - \int d^3x g \vec{\nabla} \cdot \vec{F}$

$$W = - \frac{1}{8\pi G} \left[\underbrace{\int_S \phi \vec{\nabla} \phi d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla} \phi(\infty) = 0} - \int d^3x \phi \underbrace{\vec{\nabla}(\vec{\nabla} \phi)}_{4\pi G \rho \text{ (Poisson)}} \right]$$

$$= \frac{1}{8\pi G} 4\pi G \int d^3x \phi(\vec{x}) \rho(\vec{x})$$

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3x$$

Total potential energy : Summary

$$W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) d^3\vec{x}$$

$$W = - \frac{1}{8\pi G} \int |\vec{\nabla} \phi|^2 d^3x$$

Other useful expression

$$W = - \int \rho(\vec{x}) \vec{x} \cdot \vec{\nabla} \phi(\vec{x}) d^3\vec{x}$$

Relation between the potential energy and the Poisson equation

What is the relation that must hold between the density $\rho(\vec{x})$ and potential $\phi(\vec{x})$ in order to minimize the potential energy of a system ?



Answer : the Poisson equation $\nabla^2 \phi = 4\pi G \rho$

Potential Theory

Spherical Systems

$$\rho(\vec{x}) = \rho(r)$$

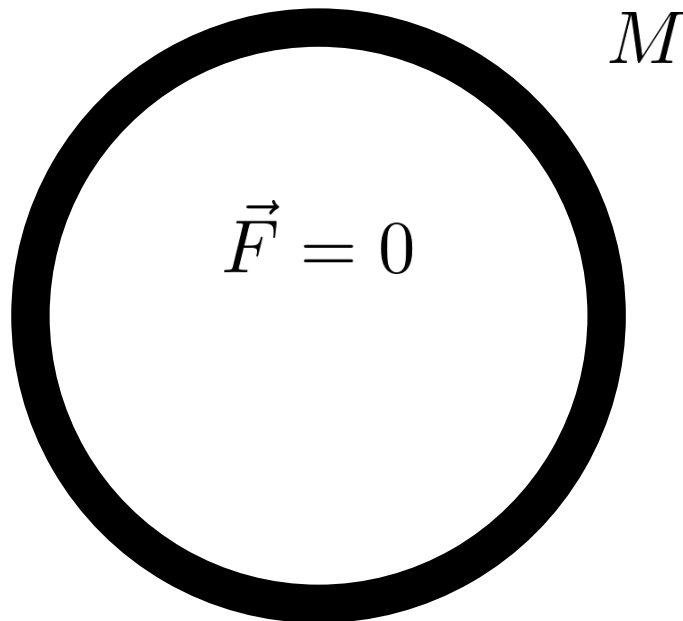
$$r = \sqrt{x^2 + y^2 + z^2}$$

Newton's Theorems

Newton (1642-1727)

First theorem:

A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

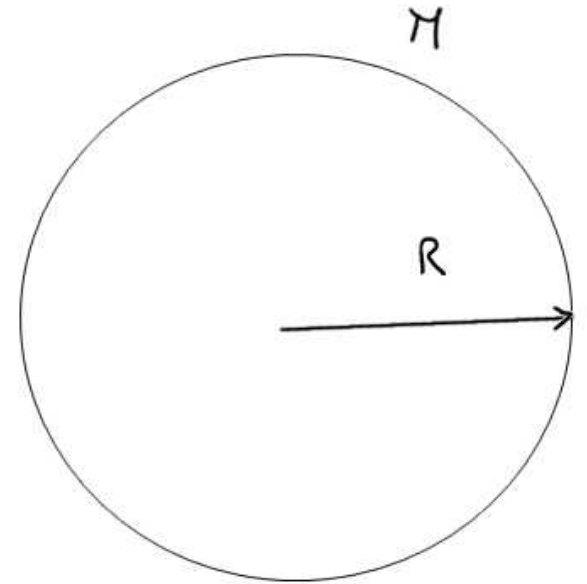


Spherical infinitely thin shell

Radius : R

Mass : M

Density : $\rho(r) = \frac{M}{4\pi r^2} \delta(R-r)$



indeed :

$$M := 4\pi \int_0^{\infty} dr r^2 \rho(r)$$
$$= 4\pi \int_0^{\infty} dr r^2 \frac{M}{4\pi r^2} \delta(R-r) = M$$

First Newton Theorem

A body that is inside a spherical shell of matter experiences no net gravitational force from that shell

a shell of constant density $\rho(\vec{a}) = \rho$

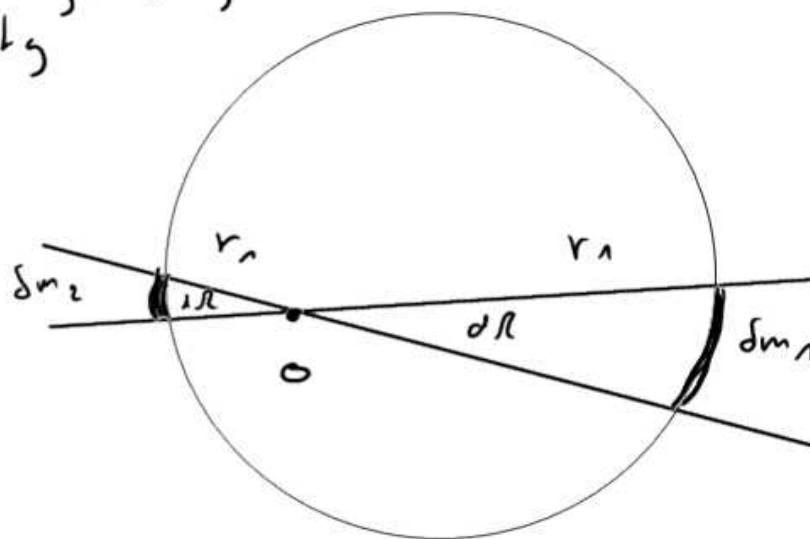
$$\begin{cases} \delta m_1 = \rho(r_1) \cdot r_1^2 d\Omega dr \\ \delta m_2 = \rho(r_2) \cdot r_2^2 d\Omega dr \end{cases}$$

thus :

$$\frac{\delta m_1}{\delta m_2} = \frac{r_1^2}{r_2^2}$$

and

$$\frac{\delta m_1}{r_1^2} = \frac{\delta m_2}{r_2^2}$$



consequently : $\delta \vec{F}_1 = -\delta \vec{F}_2$
by integrating over the entire shell (dR)
all forces cancel out ! \neq

Corollary

The gravitational potential $\phi(\vec{x})$ is constant inside the sphere.

$$\text{As } \vec{\nabla}_* \phi(\vec{x}) = \vec{g} = 0 \quad \phi(\vec{x}) = \text{const} \quad \#$$

What is the value of $\phi(\vec{x})$?

$$\phi(\vec{x}) = - \int_V \frac{G \rho(\vec{x}')}{|\vec{x}' - \vec{x}|} d^3 \vec{x}'$$

Spherical coordinates

$$d^3 \vec{x} = r^2 dr d\Omega = 4\pi r^2 dr$$

At the center $\vec{x} = 0$

$$\phi(0) = - 4\pi G \int_0^\infty \frac{\rho(r')}{r'} r'^2 dr' = - 4\pi G \int_0^\infty \rho(r') r' dr'$$

with : $\rho(r') = \frac{M}{4\pi r'^2} \delta(R-r')$

$$\phi(r) = -GM \int_0^\infty \frac{\delta(R-r)}{r^2} r dr = -\frac{GM}{R}$$

As the potential is constant for $r < R$

$$\phi(\tilde{x}) = -\frac{GM}{R} \quad \tilde{x} \in \text{sphere}$$

Newton's Theorems

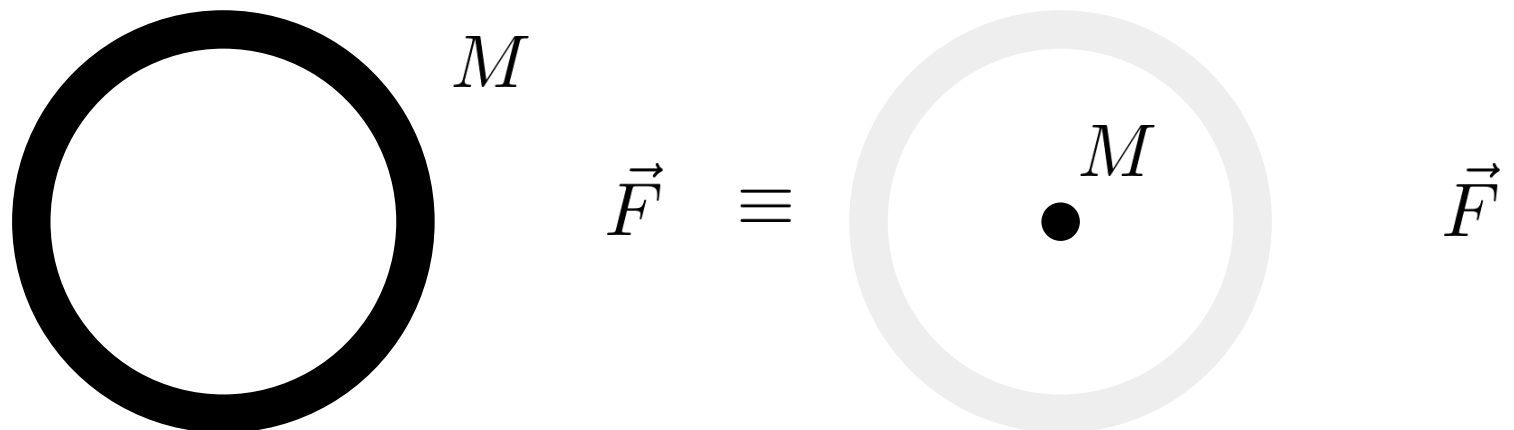
Newton (1642-1727)

First theorem:

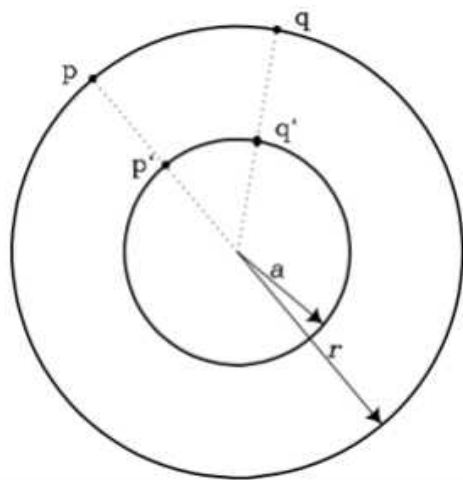
A body that is inside a spherical shell of matter experiences no net gravitational force from the shell.

Second theorem:

The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its centre.



Second Newton Theorem



The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center

Consider two shells

- 1. inner, with radius a and mass M
- 2. outer, with radius r and mass M

Compute

$$1. \quad \phi_p = \phi_i(r)$$

$$2. \quad \phi_{p'} = \phi_o(a) = - \frac{GM}{r}$$

1. contribution of shell_i , in q' with solid angle $\delta\Omega$

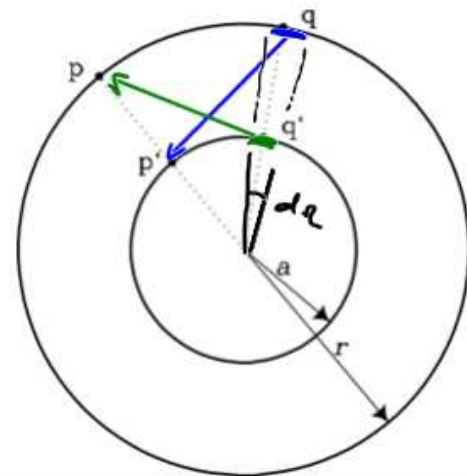
$$\bullet \quad \delta\phi_i(p) = - \frac{G\delta m_{q'}}{|p-q'|} = - \frac{GM}{|p-q'|} \frac{\delta\Omega}{4\pi}$$

2. contribution of shell_o , in q with solid angle $\delta\Omega$

$$\bullet \quad \delta\phi_o(p') = - \frac{G\delta m_q}{|p'-q|} = - \frac{GM}{|p'-q|} \frac{\delta\Omega}{4\pi} = \delta\phi_i(p)$$

Summing over all q' \equiv Summing over all q

$$\phi_i(p) = \phi_o(p') = - \frac{GM}{r} \quad \#$$

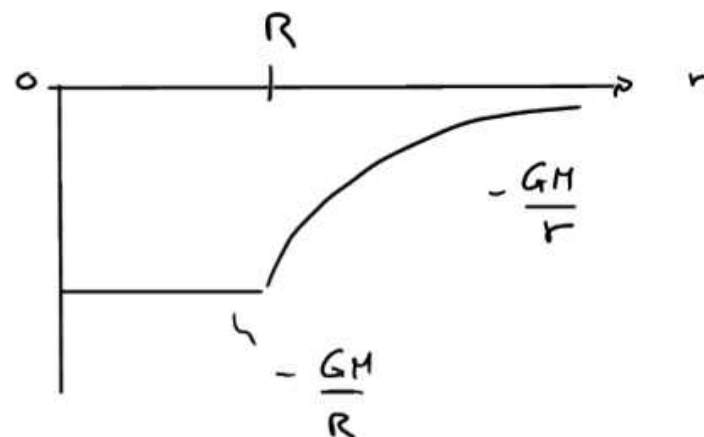


mass inside the solid angle

$$\delta m = \frac{M \delta\Omega}{4\pi}$$

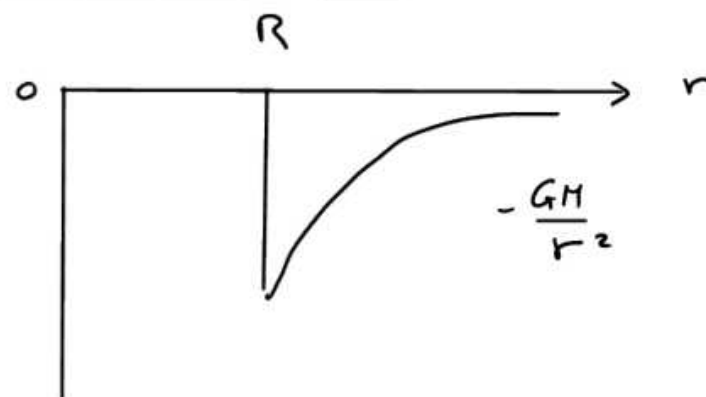
Total potential of a shell of mass M , radius R

$$\phi(r) = \begin{cases} -\frac{GM}{R} & r < R \\ -\frac{GM}{r} & r \geq R \end{cases}$$



Total gravitational field of a shell of mass M , radius R

$$\vec{g}(r) = \begin{cases} 0 & r < R \\ -\frac{GM}{r^2} \vec{e}_r & r \geq R \end{cases}$$



Potential Theory

Spherical Systems general distribution of mass

$$\rho(\vec{x}) = \rho(r)$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

Potential and gravitational field of any density $\rho(r)$

Build any density by summing shells of
of size R , mass M_R and density $\rho_R(r)$

$$\rho(r) = \sum_R \delta \rho_R(r) = \int dR \frac{\partial \rho_R(r)}{\partial R}$$

$$= \int_0^\infty dR \underbrace{\frac{\partial M_R}{\partial R}}_{\text{mass per unit length}} \frac{1}{4\pi r^2} \delta(R-r) = \frac{\partial M_r}{\partial r} \frac{1}{4\pi r^2}$$

$$\partial M_R = 4\pi R^2 \rho(R) dR$$

Each shell contributing to the total density has thus a potential

$$\delta\phi_R(r) = \begin{cases} \frac{G \ 4\pi R^2 \ \rho(R) \ dR}{R} & r < R \\ \frac{G \ 4\pi R^2 \ \rho(R) \ dR}{r} & r \geq R \end{cases}$$

$$\delta\phi_R(r) = \begin{cases} -4\pi G R \ \rho(R) \ dR & r < R \\ -\frac{4\pi G R^2 \ \rho(R) \ dR}{r} & r \geq R \end{cases}$$

Total Potential

$$\begin{aligned}\phi(r) &= \int_0^{\infty} \delta\phi_R(r) \\&= \int_0^r \underbrace{\delta\phi_R(r)}_{\substack{\text{inner shells} \\ r \geq R}} + \int_r^{\infty} \underbrace{\delta\phi_R(r)}_{\substack{\text{outer shells} \\ r < R}} \\&= -4\pi G \underbrace{\int_0^r \frac{R^2 \rho(R)}{r} dR}_{\frac{GM(r)}{r}} - 4\pi G \int_r^{\infty} R \rho(R) dR\end{aligned}$$

$$\phi(r) = - \frac{GM(r)}{r} - 4\pi G \int_r^\infty dR R \rho(R)$$

contribution
of the mass
inside r

contribution
of the mass
outside r

Gravitational field of a spherical model $\rho(r)$

From the potential $\phi(r)$ $\vec{g}(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$

$$\begin{aligned}g(r) &= -\frac{\partial\phi}{\partial r} = -\frac{\partial}{\partial r} \left[-\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r') r' dr' \right] \\&= -\frac{GM(r)}{r^2} + \frac{G}{r} 4\pi \frac{\partial}{\partial r} \int_0^r dr' r'^2 \rho(r') + 4\pi G \frac{\partial}{\partial r} \int_r^\infty dr' r' \rho(r') \\&= -\frac{GM(r)}{r^2} + \underbrace{\frac{G}{r} 4\pi r^2 \rho(r) - 4\pi G r \rho(r)}_{=0}\end{aligned}$$

$$g(r) = -\frac{GM(r)}{r^2}$$

contribution
of the mass
inside r

Gravitational field of a spherical model

$g(r)$

Sum of shells

$$g(r) = \int_0^{\infty} \delta g_{r'}(r)$$

$\delta g_{r'}(r)$ = force due to the shell of radius r'

EXERCICE

Summary : for any spherical mass distribution $\rho(r)$

$$g(r) = - \frac{GM(r)}{r^2}$$

$$M(r) = 4\pi \int_0^{\infty} \rho(r') r'^2 dr'$$

$$\phi(r) = - \frac{GM(r)}{r} - 4\pi G \int_r^{\infty} \rho(r') r' dr'$$

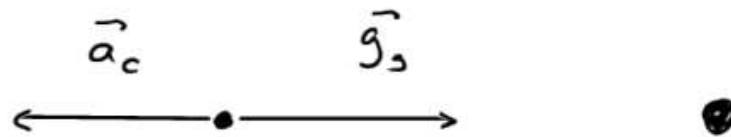
Note $g(r) = - \frac{\partial \phi}{\partial r}$

as expected from

$$\vec{g}(\vec{x}) = -\vec{\nabla} \phi(\vec{x})$$

Spherical systems : circular speed, circular velocity

Speed of a test particle in a circular orbit in the potential $\phi(r)$ at a radius r :



\vec{a}_c : centrifugal acceleration

$$\frac{v_c^2}{r}$$

\vec{g}_s : gravity acceleration (spec force)

$$- \frac{GM(r)}{r^2} = - \frac{\partial \phi}{\partial r}$$

$$v_c^2 = \frac{GM(r)}{r}$$

$$v_c^2 = r \frac{\partial \phi}{\partial r}$$

$$[v_c^2] : \frac{\text{erg}}{\text{s}}$$

as ϕ

\equiv specific energy

$$GM(r) = r^2 \frac{\partial \phi}{\partial r}$$

Velocity composition

Note: V_0^2 scale with the mass ($M(r)$) : it is thus the "important" quantity (spec. energy)

Multi-components system : ex: bulge + stellar halo + DM halo

$$\left\{ \begin{array}{llll} \rho_B(r) & , & M_B(r) & , & \phi_B(r) \rightarrow V_{c,B}(r) \\ \rho_H(r) & , & M_H(r) & , & \phi_H(r) \rightarrow V_{c,H}(r) \\ \rho_{DM}(r) & , & M_{DM}(r) & , & \phi_{DM}(r) \rightarrow V_{c,DM}(r) \end{array} \right.$$

$$V_{c,tot}^2 = \frac{GM_{tot}(r)}{r} = \frac{G}{r} \sum_i M(r)$$

$$V_{c,tot}^2 = \sum_i V_{c,i}^2$$

$V_c^2 \sim \text{energy} : \text{extensive quantity}$

Period of the circular orbit

$$T(r) = \frac{2\pi r}{v_c(r)} = 2\pi \sqrt{\frac{r^3}{GM(r)}} = 2\pi \sqrt{\frac{r}{\frac{\partial \phi}{\partial r}}}$$

Circular frequency (angular frequency)

$$\Omega(r) = \frac{2\pi}{T(r)} = \sqrt{\frac{GM(r)}{r^3}} = \sqrt{\frac{1}{r} \frac{\partial \phi}{\partial r}}$$

Escape speed v_e

if $\frac{1}{2}v_e^2 > \phi(r) = E > 0$

the particle may escape the system

$$v_e(r) = \sqrt{2|\phi(r)|}$$

Potential energy

from $W = - \int f(\vec{x}) \vec{x} \cdot \vec{\nabla} \phi(\vec{x}) d^3\vec{x}$

$$W = - 4\pi G \int_0^{\infty} f(r) M(r) r dr$$

Gravitational radius

radius at which $\frac{GM^2}{r} = W$

(estimation of the system size)

$$r_g = \frac{GM^2}{|W|}$$

Spherical systems : useful relations

	$\rho(r)$	$\Phi(r)$	$M(r)$	$\frac{d\Phi}{dr}$
$\rho(r)$	$\rho(r)$	$\frac{1}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right)$	$\frac{1}{4\pi r^2} \frac{dM(r)}{dr}$	$\frac{1}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right)$
$\Phi(r)$	$-\frac{GM(r)}{r} - 4\pi G \int_r^\infty dr' r' \rho(r')$	$\Phi(r)$	$-G \int_r^\infty dr' \frac{M(r')}{r'^2}$	$-\int_r^\infty dr' \frac{d\Phi}{dr}$
$M(r)$	$4\pi \int_0^r dr' r'^2 \rho(r')$	$\frac{r^2}{G} \frac{d\Phi}{dr}$	$M(r)$	$\frac{r^2}{G} \frac{d\Phi}{dr}$
$\frac{d\Phi}{dr}$	$\frac{4\pi G}{r^2} \int_0^r dr' r'^2 \rho(r')$	$\frac{d\Phi}{dr}$	$\frac{GM(r)}{r^2}$	$\frac{d\Phi}{dr}$

Poisson in spherical coordinates

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r)$$

Mass inside a radius r

$$M(r) = 4\pi \int_0^r dr' r'^2 \rho(r')$$

Potential in spherical coordinates

$$\Phi(r) = -\frac{GM(r)}{r} - 4\pi G \int_r^\infty \rho(r') r' dr'$$

Gradient of the potential in spherical coordinates

$$\frac{d\Phi(r)}{dr} = \frac{GM(r)}{r^2}$$

Examples of Spherical models

**“Potential based”
models**

Point mass

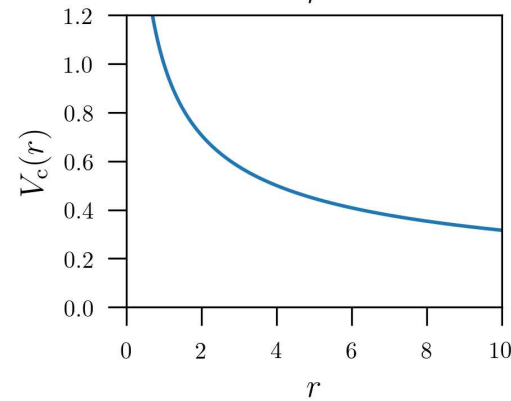
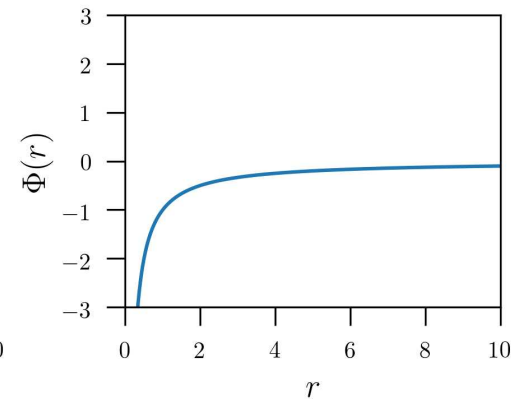
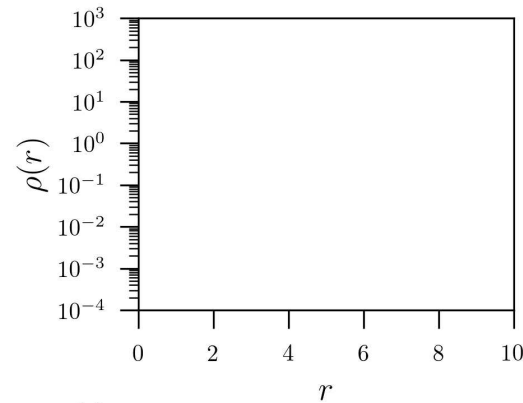
$$\Phi(r) = -\frac{GM}{r}$$

$$\rho(r) = \frac{M\delta(0)}{4\pi r^2}$$

$$M(r) = M$$

$$V_c^2(r) = \frac{GM}{r}$$

$$T(r) = 2\pi\sqrt{\frac{r^3}{GM}}$$



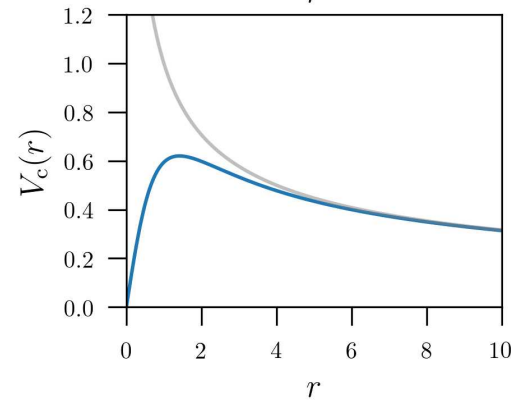
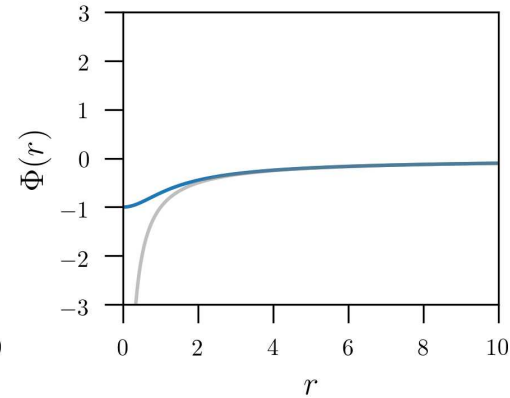
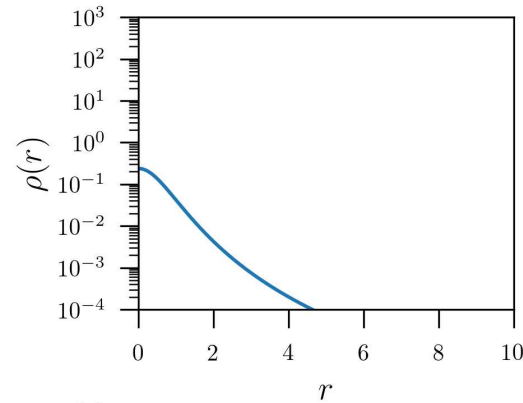
Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}$$

$$\rho(r) = \frac{3M}{4\pi b^3} \left(1 + \frac{r^2}{b^2}\right)^{-5/2}$$

$$M(r) = \frac{Mr^3}{(r^2 + b^2)^{3/2}}$$

$$V_c^2(r) = \frac{GMr^2}{(r^2 + b^2)^{3/2}}$$



- Globular clusters, dwarf spheroidal galaxies

Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2(b + 3\sqrt{b^2 + r^2})}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 + r^2)^{3/2}}$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

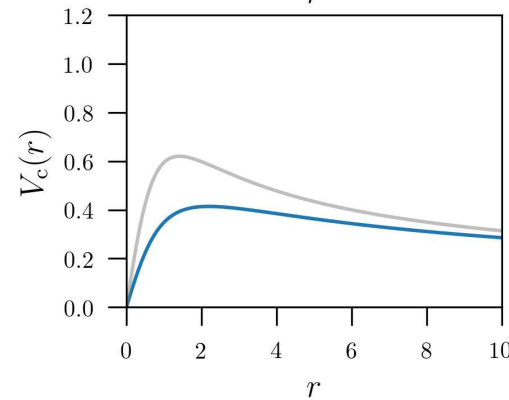
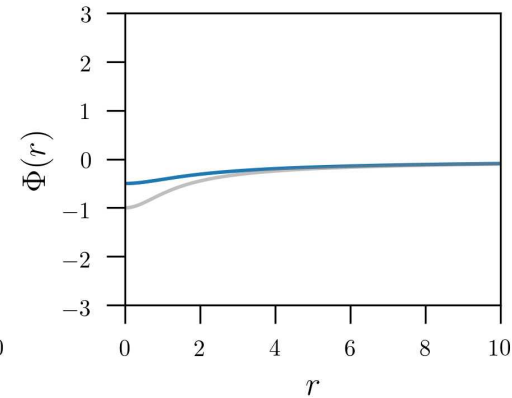
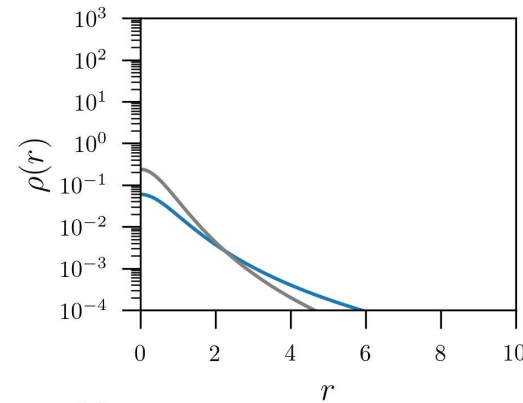
Isochrone potential

$$\Phi(r) = -\frac{GM}{b + \sqrt{r^2 + b^2}}$$

$$\rho(r) = M \frac{3(b + \sqrt{b^2 + r^2})(b^2 + r^2) - r^2}{4\pi(b + \sqrt{b^2 + r^2})^3(b^2 -$$

$$M(r) = \frac{Mr^3}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$

$$V_c^2(r) = \frac{GMr^2}{\sqrt{b^2 + r^2}(b + \sqrt{b^2 + r^2})^2}$$



Orbits are analytical !

Examples of Spherical models

**“Density based”
models**

Homogeneous sphere

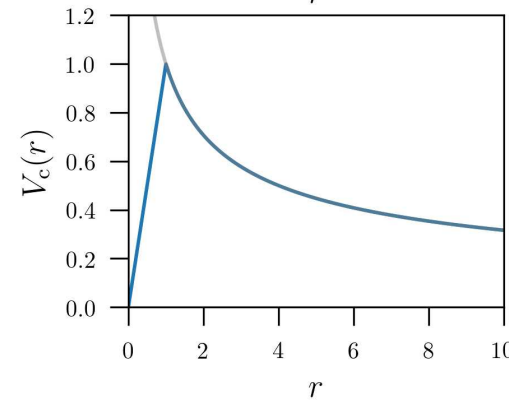
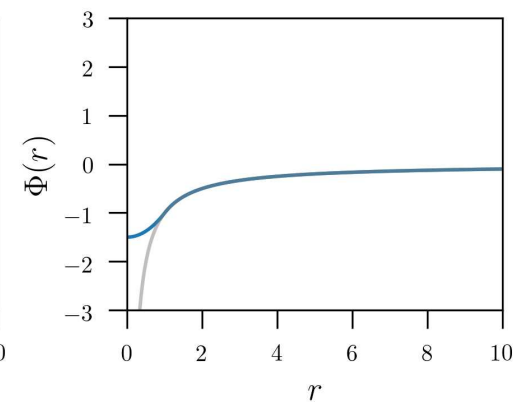
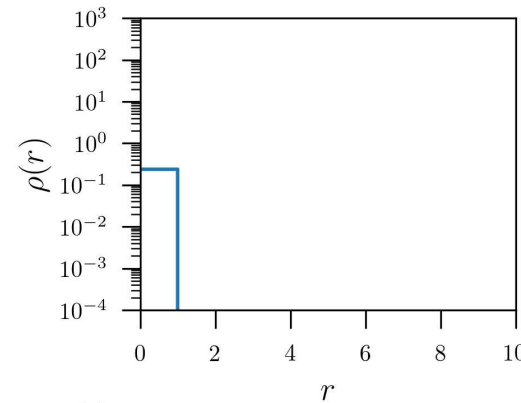
$$\rho(r) = \begin{cases} \rho & r < R \\ 0 & r > R \end{cases}$$

$$M(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_0 & r < R \\ \frac{4}{3}\pi R^3 \rho & r > R \end{cases}$$

$$\Phi(r) = \begin{cases} -2\pi G \rho \left(R^2 - \frac{1}{3}r^2 \right) & r < R \\ -4\pi G \rho R^3 / (3r) & r > R \end{cases}$$

$$V_c^2(r) = \begin{cases} \frac{4}{3}\pi G \rho_0 r^2 & r < R \\ \frac{4}{3}\pi G \rho_0 \frac{R^3}{r} & r > R \end{cases}$$

$$T(r) = \begin{cases} \sqrt{\frac{3\pi}{G\rho_0}} & r < R \\ \sqrt{\frac{3\pi}{G\rho_0 R^3}} r^{3/2} & r > R \end{cases}$$



$$\frac{d^2 r}{dt^2} = -\frac{d\Phi(r)}{dr} = -\frac{GM(r)}{r^2} = -\frac{4}{3}\pi\rho_0 r = -\omega^2 r$$

Harmonic oscillator !

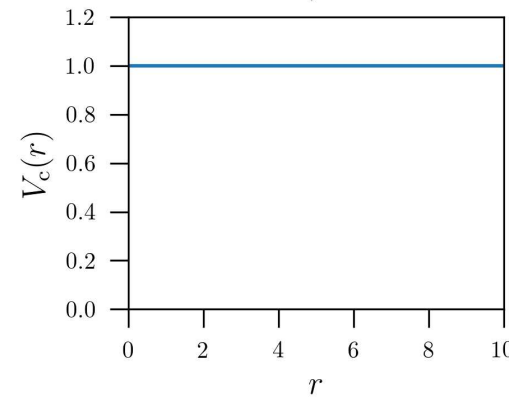
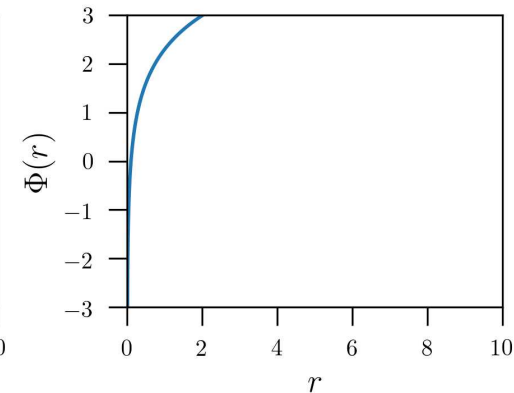
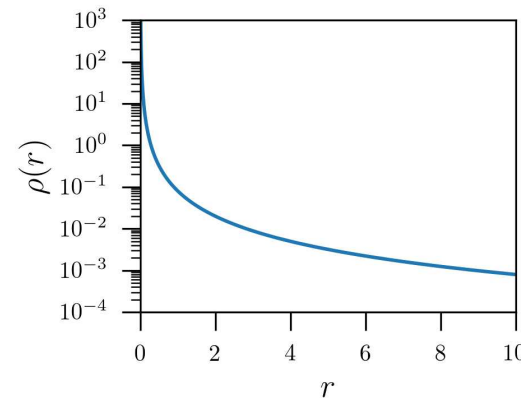
Isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \ln\left(\frac{r}{a}\right)$$

$$M(r) = 4\pi \rho_0 a^2 r$$

$$V_c^2(r) = 4\pi G \rho_0 a^2$$



- often used for gravitational lens models
- But !
 - diverge towards the centre !
 - infinite mass !

Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left(\frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left(1 - \frac{a}{r} \arctan\left(\frac{r}{a}\right) \right)$$

- Avoid the central divergence of the isothermal sphere
 - However, the mass is still not bounded

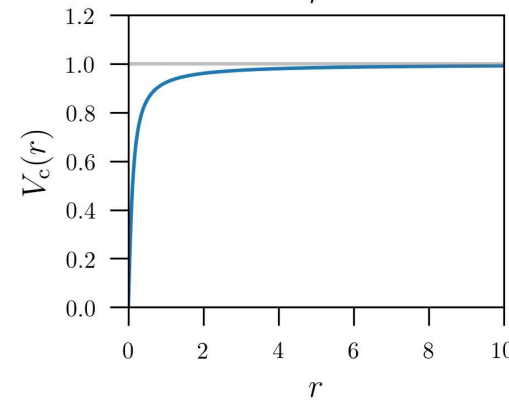
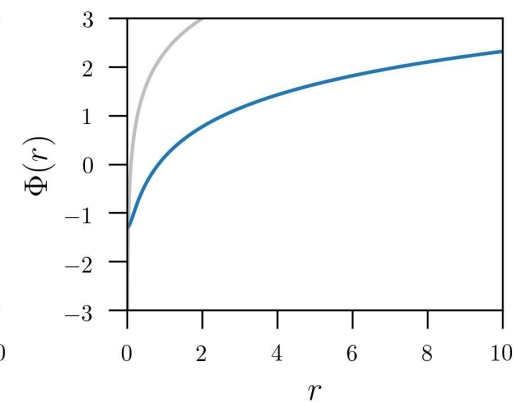
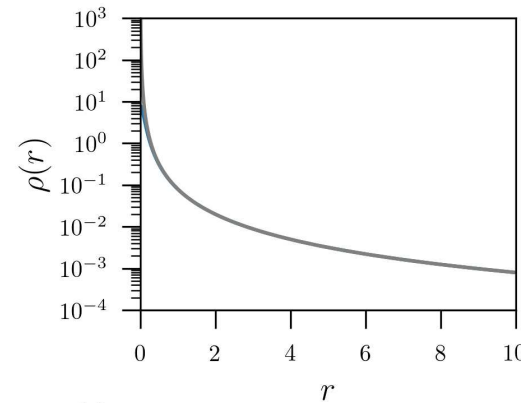
Pseudo-isothermal sphere

$$\rho(r) = \rho_0 \frac{a^2}{a^2 + r^2}$$

$$\Phi(r) = 4\pi G \rho_0 a^2 \left(\frac{1}{2} \ln(a^2 + r^2) + \frac{a}{r} \arctan \frac{r}{a} \right)$$

$$M(r) = 4\pi r \rho_0 a^2 \left(1 - \frac{a}{r} \arctan \left(\frac{r}{a} \right) \right)$$

$$V_c^2(r) = 4\pi G \rho_0 a^2 \left(1 - \frac{a}{r} \arctan \left(\frac{r}{a} \right) \right)$$



- Avoid the central divergence of the isothermal sphere
 - However, the mass is still not bounded

Generic two power density models

$$\rho(r) = \frac{\rho_0}{(r/a)^\alpha (1 + r/a)^{\beta-\alpha}}$$

- diverges at the center if $\alpha \neq 0$

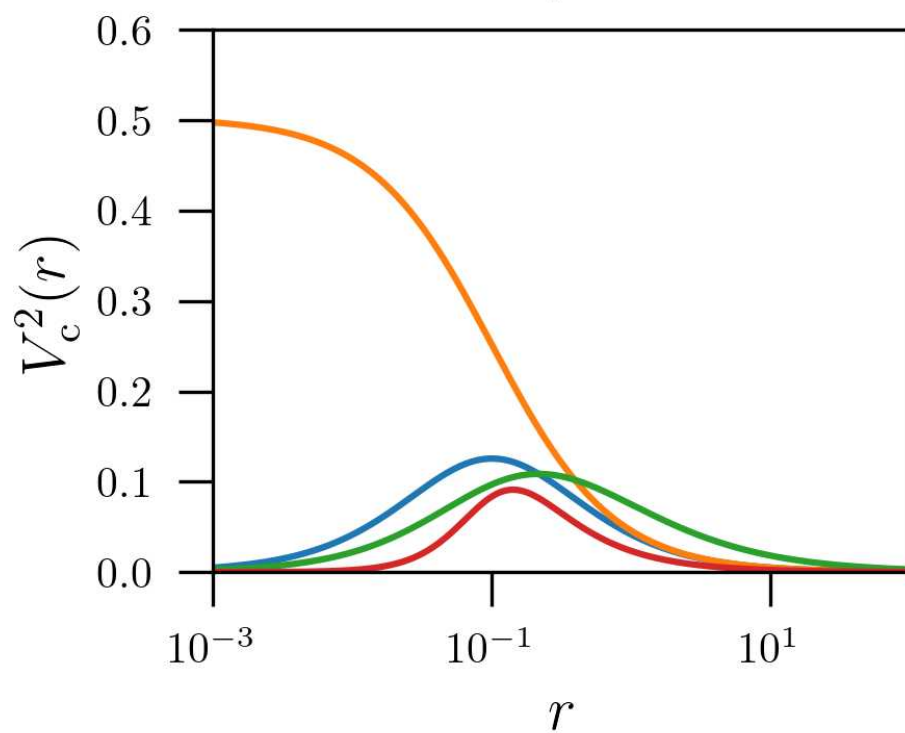
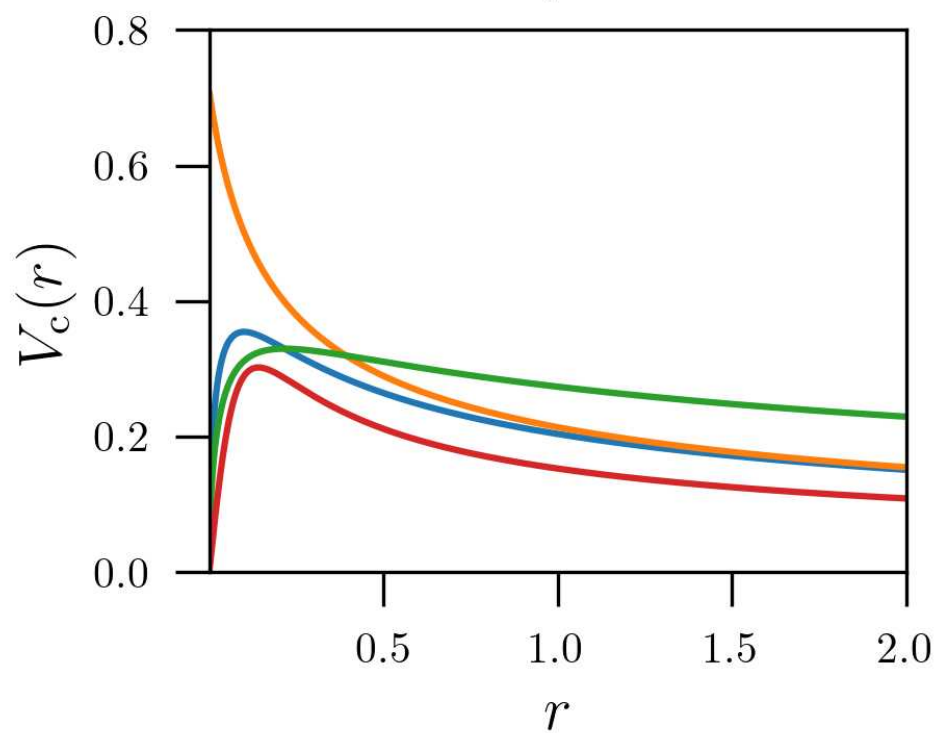
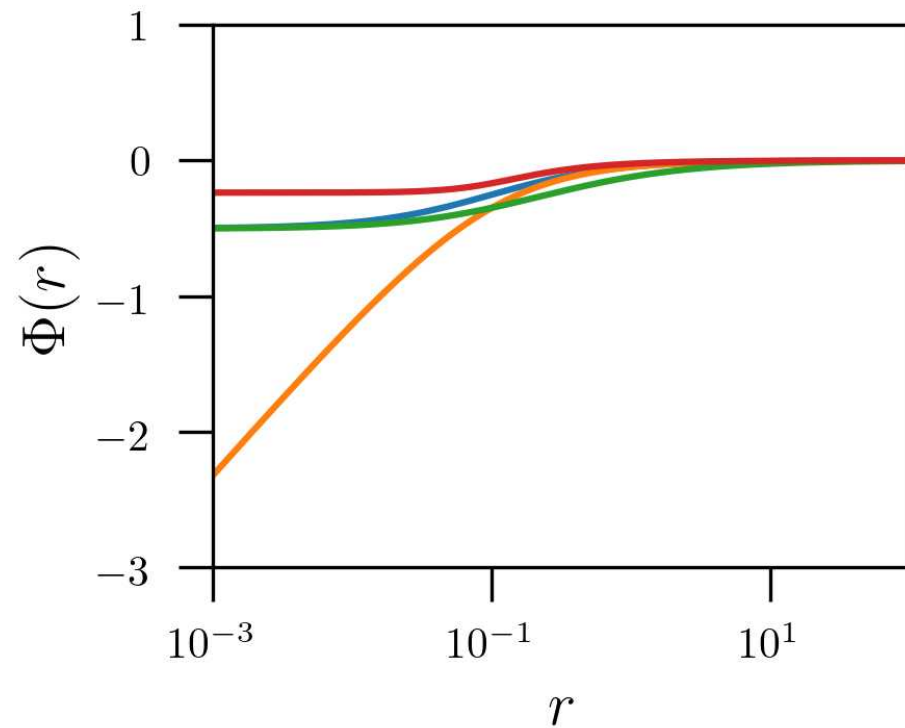
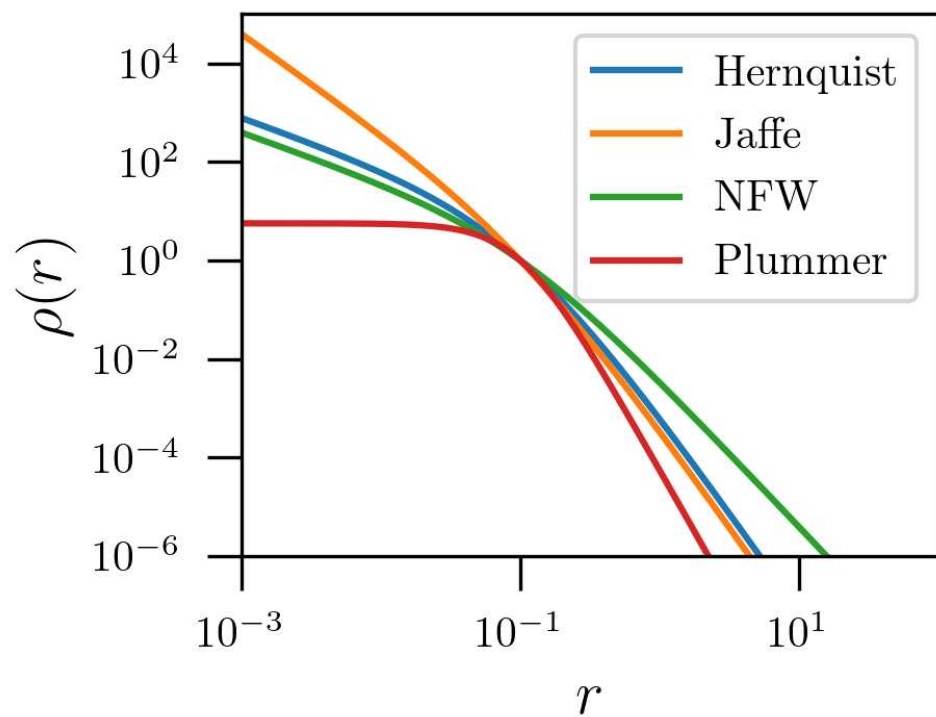
$$M(r) = 4\pi\rho_0a^3 \int_0^{r/a} s \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}$$

model name	inner slope α	outer slope β	
Plummer	0	5	• globular clusters
Dehnen	any	4	
Hernquist	1	4	• bulges, elliptic. gal.
Jaffe	2	4	• elliptic. galaxies
NFW	1	3	• dark haloes

Generic two power density model

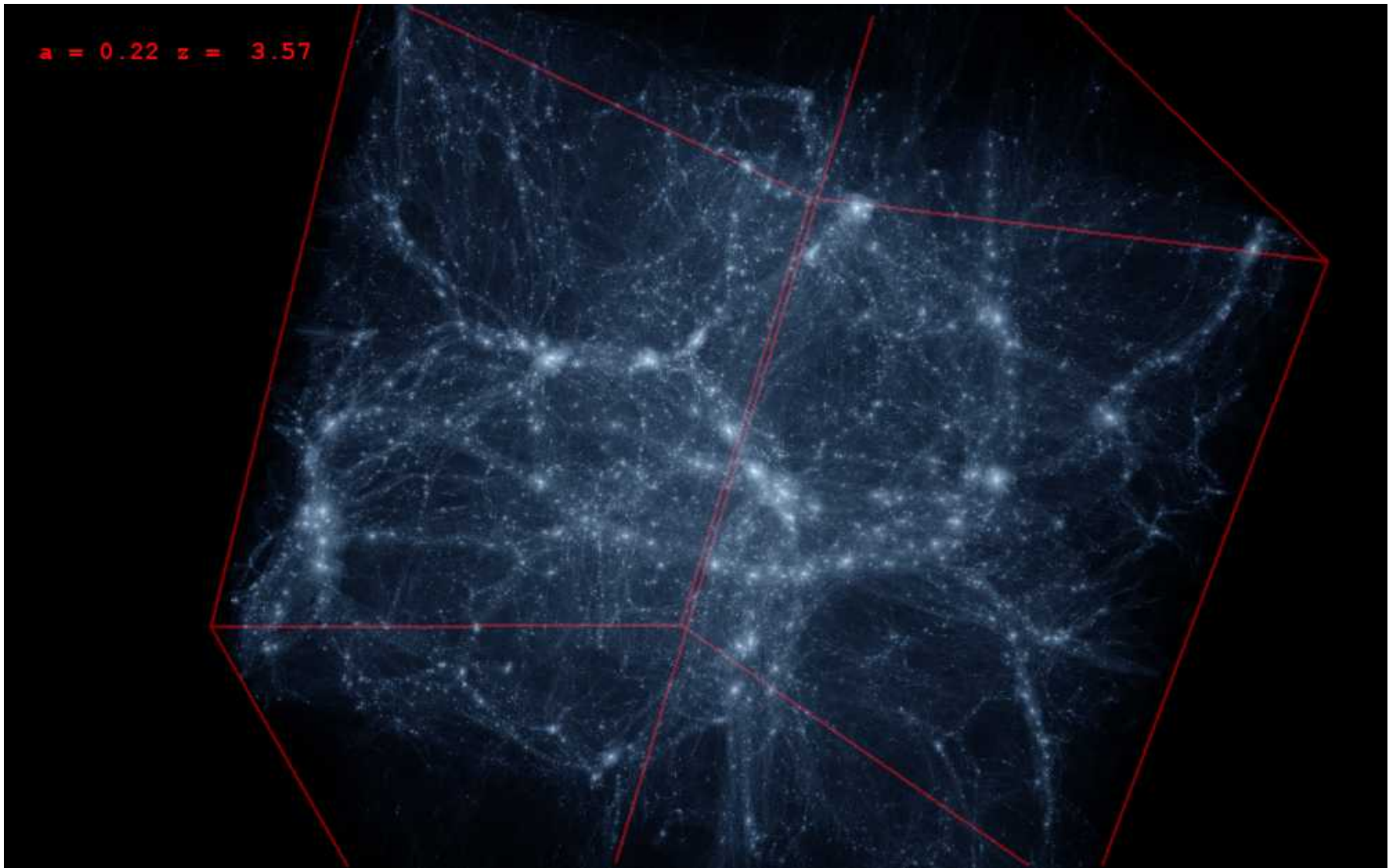
$$M(r) = 4\pi\rho_0 a^3 \times \begin{cases} \frac{r/a}{1+r/a} & \text{(Jaffe)} \\ \frac{(r/a)^2}{2(1+r/a)^2} & \text{(Hernquist)} \\ \ln(1 + r/a) - \frac{r/a}{1+r/a} & \text{(NFW)} \end{cases} \quad \bullet \text{ diverges !!}$$

$$\Phi(r) = -4\pi G\rho_0 a^2 \times \begin{cases} \ln(1 + a/r) & \text{(Jaffe)} \\ \frac{1}{2(1+r/a)} & \text{(Hernquist)} \\ \frac{\ln(1+r/a)}{r/a} & \text{(NFW)} \end{cases}$$



NFW (Navarro, Frenk & White 1995, 1996)

- Density profile that fit dark matter haloes formed in LCDM numerical simulations



NFW (Navarro, Frenk & White 1995, 1996)

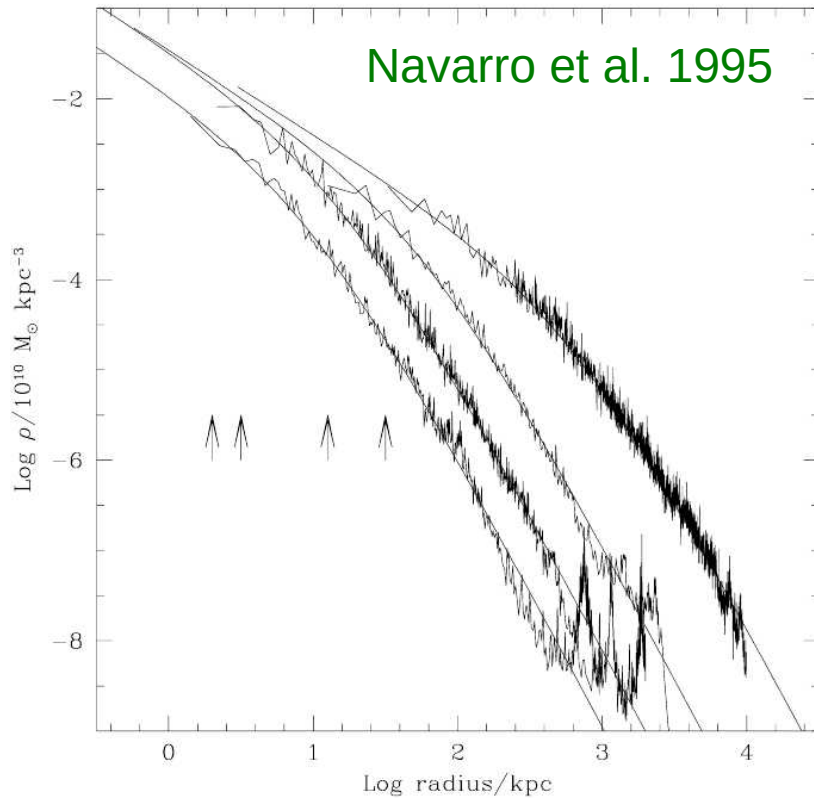


Fig. 3.— Density profiles of four halos spanning four orders of magnitude in mass. The arrows indicate the gravitational softening, h_g , of each simulation. Also shown are fits from eq.3. The fits are good over two decades in radius, approximately from h_g out to the virial radius of each system.

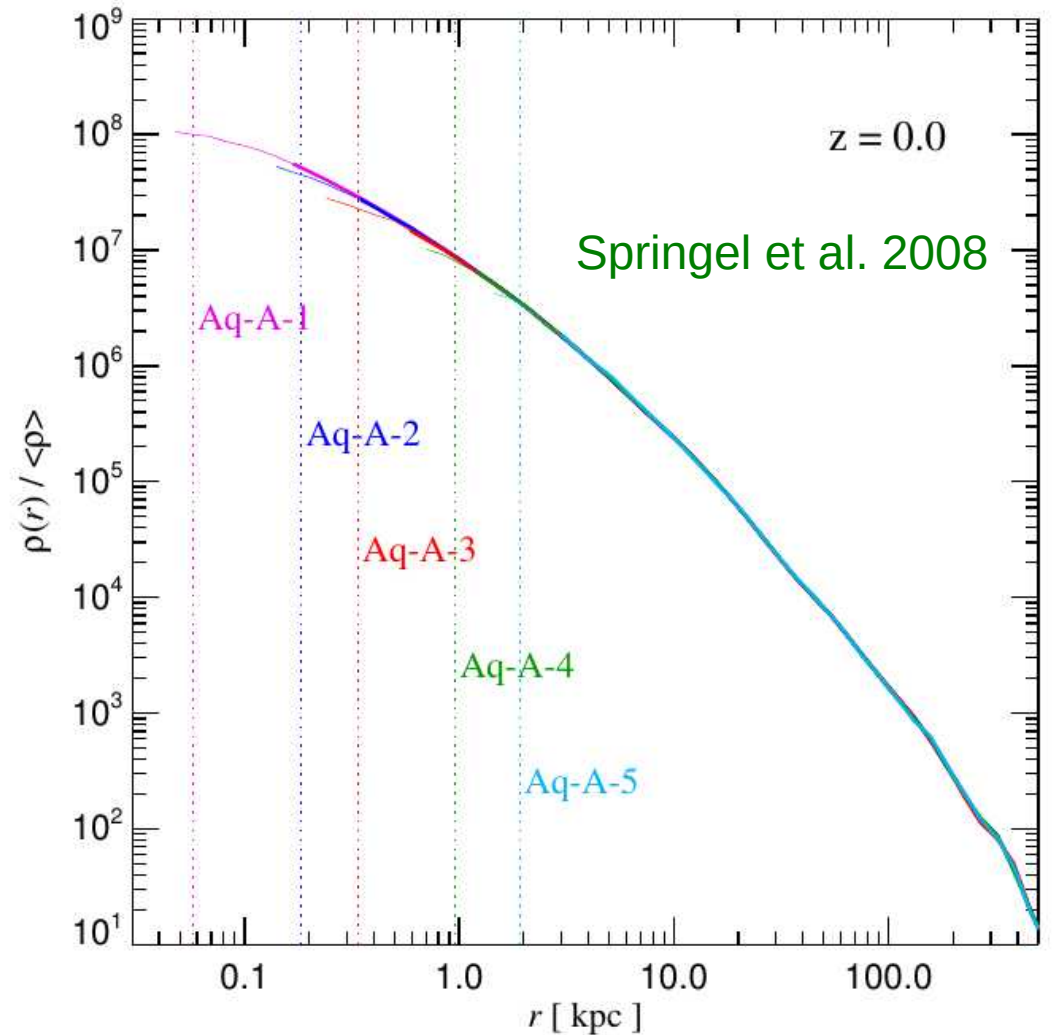
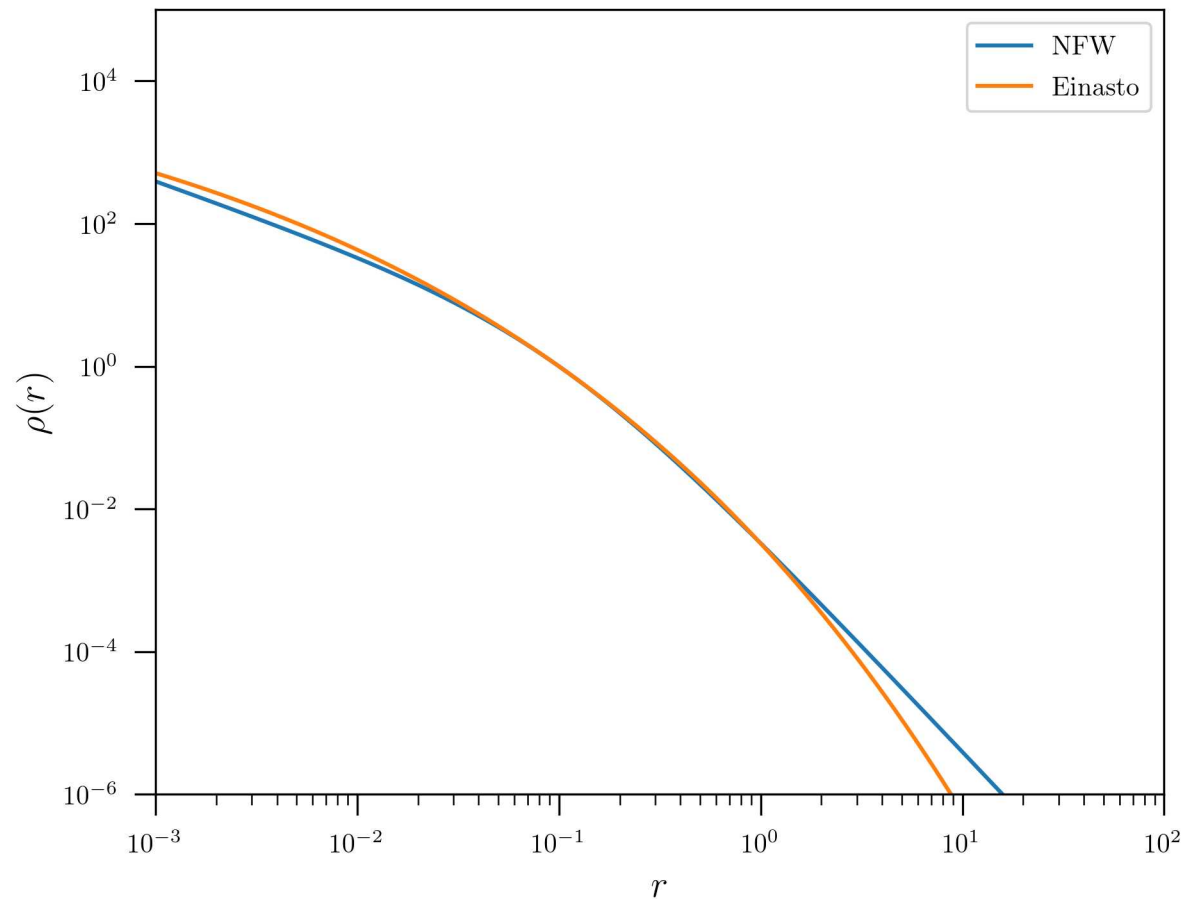


Figure 4. Spherically averaged density profile of the Aq-A halo at $z = 0$, at different numerical resolutions. Each of the pro-

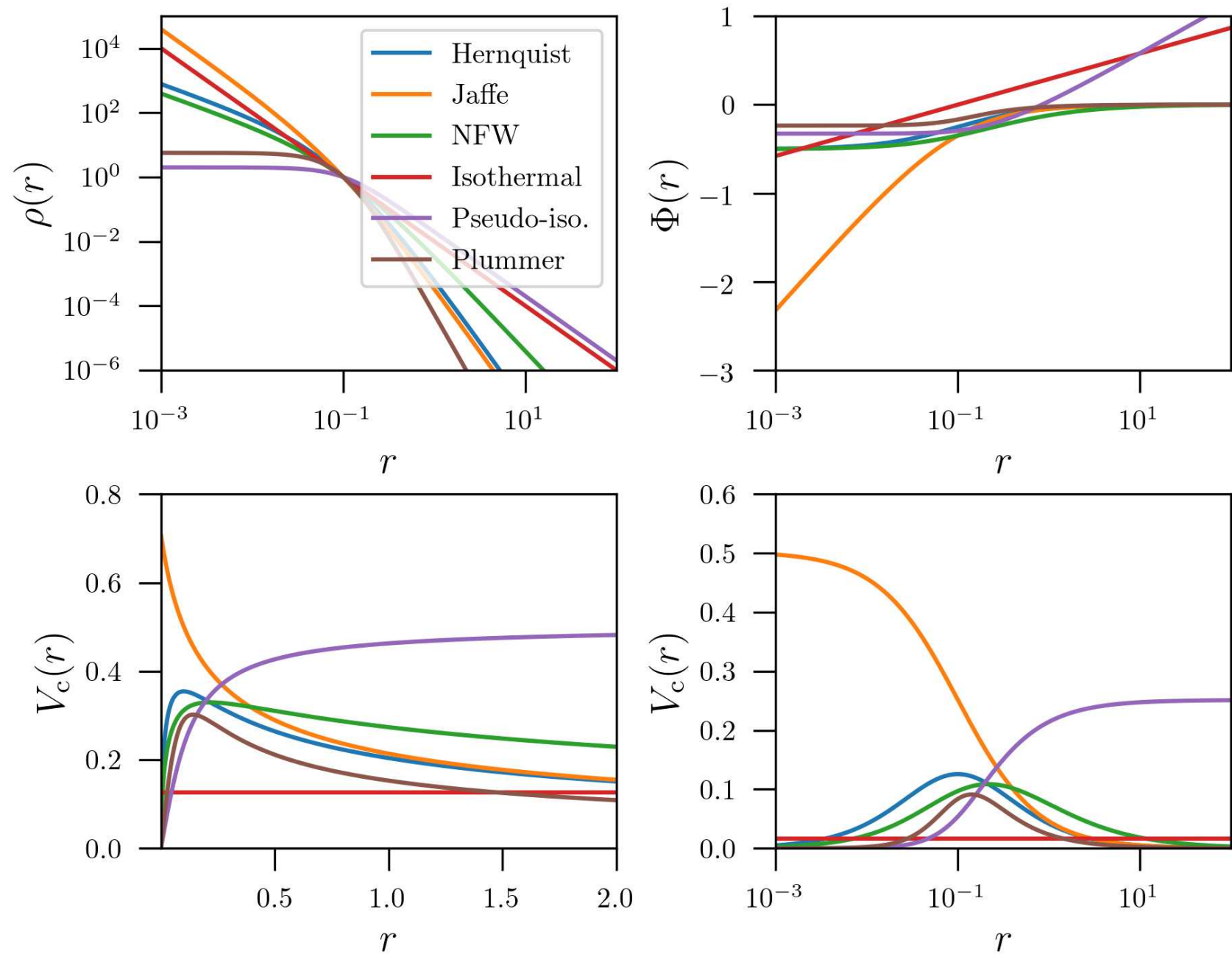
Einasto model

$$\rho(r) = \rho_0 \exp \left[- (r/a)^{1/m} \right] \quad (m \cong 6)$$



- Alternative to NFW

Spherical systems model comparison



The End