The gravity force
and Newtonian mechanics

Outlines

Newtonian Mechanics:

- refreshing memory

The gravity : a long distance force

- collision-less systems
- the relaxation time

Potential Theory : general results

- Gravitational field force, gravitational potential
- Gauss Law
- Poisson Equation
- Total potential energy

Refreshing memory...

Newtonian mechanics

Newtonian mechanics : a very short remainder

point mass :

mass m

position $\vec{x}$

velocity $\vec{v} = \frac{d\vec{x}}{dt}$

momentum $\vec{p} = m\vec{v} = m\frac{d\vec{x}}{dt}$

Newtonian mechanics : a very short remainder

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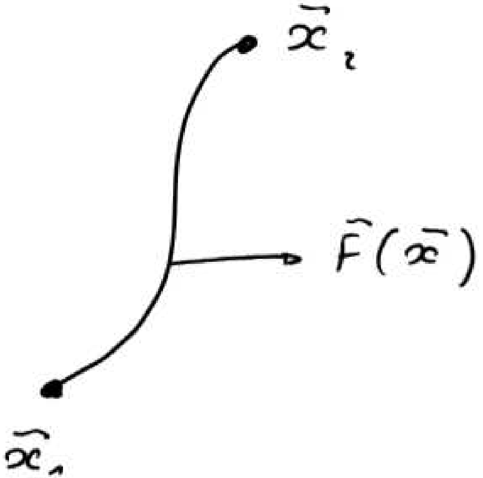
Newton second law

$$\boxed{\frac{d}{dt}(\vec{p}) = m\frac{d^2\vec{x}}{dt^2} = \vec{F}}$$

- $\vec{F}$: a force

$\vec{p}$ is constant in absence of a force

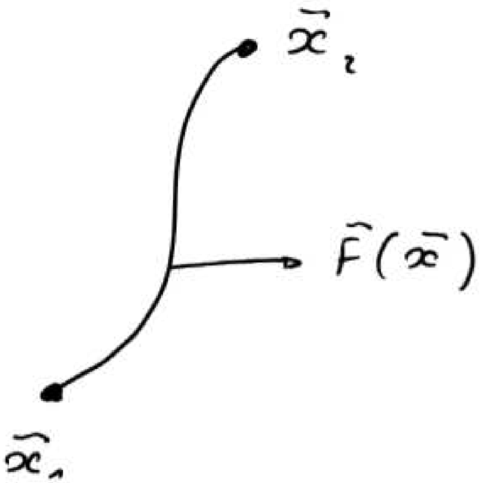
Work : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$



$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g\,\frac{cm^2}{s^2}$$
$$= erg$$

Work : work done by a force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$



$$W_{12} = - \int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

$$[W_{12}] = g \frac{cm^2}{s^2} = erg$$

Power of a force (energy rate) $\frac{erg}{s}$

with $\frac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\frac{d}{dt} W_{12}(x(t)) = - \frac{d}{dt} \int_{\vec{x}_1}^{\vec{x}(t) = \vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} = - \frac{d}{dt} \left(\mathcal{F}(\vec{x}(t)) - \mathcal{F}(\vec{x}_1) \right)$$

$$= - \vec{\nabla}_{\vec{x}} \mathcal{F}(\vec{x}) \cdot \frac{d}{dt}(\vec{x}(t)) = - \vec{F}(\vec{x}) \vec{v}(\vec{x})$$

Kinetic energy

$\vec{x} = \vec{x}(t)$

$d\vec{x} = \frac{d\vec{x}}{dt}\, dt$

$$W_{12} = -\int_{\vec{x}_1}^{\vec{x}_2} \vec{F}(\vec{x}) \cdot d\vec{x} \overset{\text{Newton 2nd law}}{=} -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \frac{d\vec{x}}{dt} \cdot dt$$

$$= -m \int_{\vec{x}_1}^{\vec{x}_2} \frac{d\vec{v}}{dt} \cdot \vec{v} \cdot dt$$

integration by part gives

$$= -m \left[\vec{v}^{\,2} \Big|_{\vec{x}_1}^{\vec{x}_2} - \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt}\, dt \right] = -m \vec{v}_2^{\,2} + m \vec{v}_1^{\,2} + \underbrace{m \int_{\vec{x}_1}^{\vec{x}_2} \vec{v} \frac{d\vec{v}}{dt}\, dt}_{-W_{12}}$$

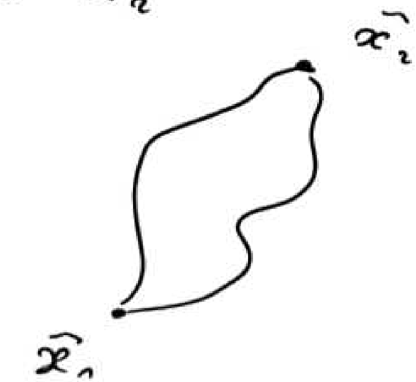
<u>Thus</u> $\quad W_{12} = \frac{1}{2} m \vec{v}_1^{\,2} - \frac{1}{2} m \vec{v}_2^{\,2}$

$$\boxed{W_{12} = K_1 - K_2}$$

$K = \frac{1}{2} m \vec{v}^{\,2}$: Kinetic energy

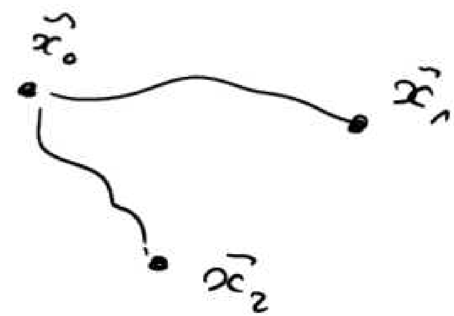
Potential energy and conservative forces

A force $\vec{F}(\vec{x})$ is called conservative if the work done by this force in moving the particle from $\vec{x}_1$ to $\vec{x}_2$ is independent of the path.



Then, for any given point $\vec{x}_0$ we can define the function (potential) $V_0(\vec{x})$

$$V_0(\vec{x}) := V_{0x} = -\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}\,')\, d\vec{x}\,'$$



Then $W_{12} = W_{10} + W_{02}$

$$\boxed{W_{12} = V(\vec{x}_2) - V(\vec{x}_1)}$$

Useful convention : $\vec{x}_0 \to \infty$ (far away from all interacting bodies)

Gradient of the potential

with $\dfrac{\partial \mathcal{F}(\vec{x})}{\partial x_i} = F_i(\vec{x})$

$$\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{\nabla}_{\vec{x}} \cdot \left[\int_{\vec{x}_0}^{\vec{x}} \vec{F}(\vec{x}')\, d\vec{x}' \right] = -\vec{\nabla}_{\vec{x}} \left(\mathcal{F}(\vec{x}) - \underbrace{\mathcal{F}(\vec{x}_0)}_{ct \Rightarrow 0} \right) = -\vec{F}(\vec{x})$$

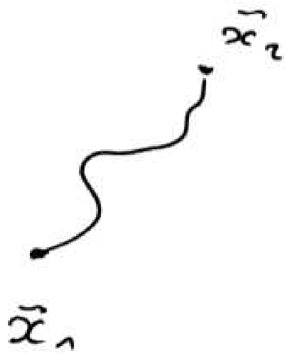
$$\boxed{\vec{\nabla}_{\vec{x}} \cdot V(\vec{x}) = -\vec{F}(\vec{x})}$$

We can represent a conservative force field by its potential

Total Energy E

$$E := K + V = \frac{1}{2} m \vec{v}^{\,2} + V(\vec{x})$$

Theorem The energy E of a system evolving under conservative forces $\vec{F}(\vec{x})$ (associated to a potential $V(\vec{x})$) is constant.



$$E_1 = E(\vec{x}_1) = \frac{1}{2} m v_1^2 + V(\vec{x}_1)$$

$$E_2 = E(\vec{x}_2) = \frac{1}{2} m v_2^2 + V(\vec{x}_2)$$

$$E_1 - E_2 = \underbrace{K_1 - K_2}_{W_{12}} + \underbrace{V(\vec{x}_1) - V(\vec{x}_2)}_{-W_{12}}$$

$$= 0$$

#

Angular momentum and Tork

Angular momentum $\qquad \vec{L} = \vec{x} \times \vec{p}$

Tork $\qquad \vec{N} = \vec{x} \times \vec{F}$

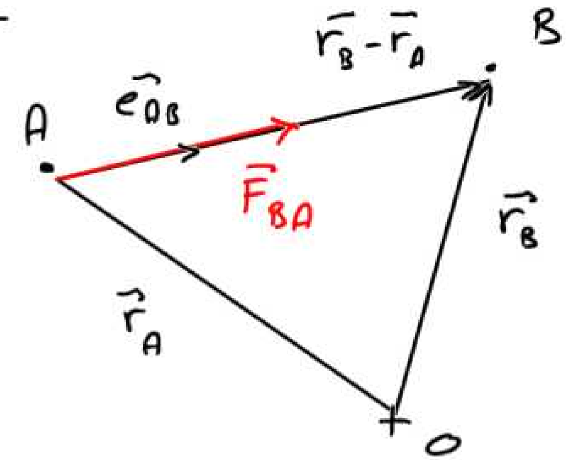
$$\frac{d\vec{L}}{dt} = \frac{d\vec{x}}{dt} \times \vec{p} + \vec{x} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times \vec{p}}_{= 0} + \underbrace{\vec{x} \times \vec{F}}_{\vec{N}}$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{N}}$$

The Gravity :
a long distance force

The gravity : a long range force



$$\boxed{\vec{F}_{BA} = \frac{G\, m_A\, m_B}{|\vec{r}_B - \vec{r}_A|^2}\, \vec{e}_{AB}}$$

$$\vec{e}_{AB} = \frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}$$

$$\boxed{\vec{F}_{BA} = \frac{G\, m_A\, m_B}{|\vec{r}_B - \vec{r}_A|^3}\, \vec{r}_B - \vec{r}_A}$$

$$r_{AB} = |\vec{r}_B - \vec{r}_A|$$

$$\boxed{|\vec{F}_{BA}| = \frac{G\, m_A\, m_B}{r_{AB}^2}}$$

$$[G] = \frac{cm^3}{g\, s} \equiv \frac{erg\ cm}{g^2}$$

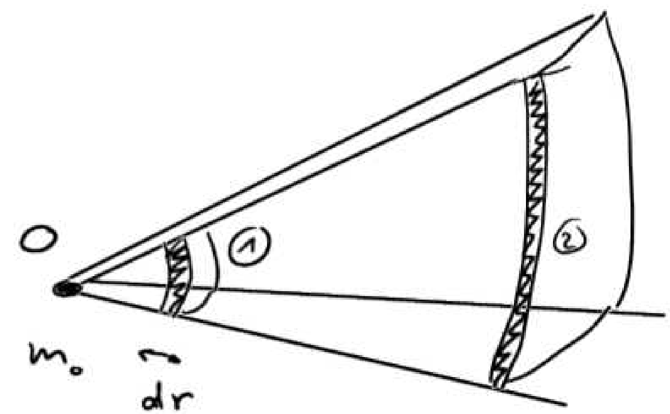
$$[erg] = \frac{cm^2}{s^2}\, g$$

Contrary to, for example, molecular forces, gravity is a long range force, i.e: we cannot neglect distant regions

<u>Illustration</u>: an homogeneous medium ($\rho(\vec{x}) = $ cte)

<u>Force on point O due to a thin shell of mass δm at distance r</u>

$$\delta F = \frac{G\, m_o\, \delta m}{r^2}$$

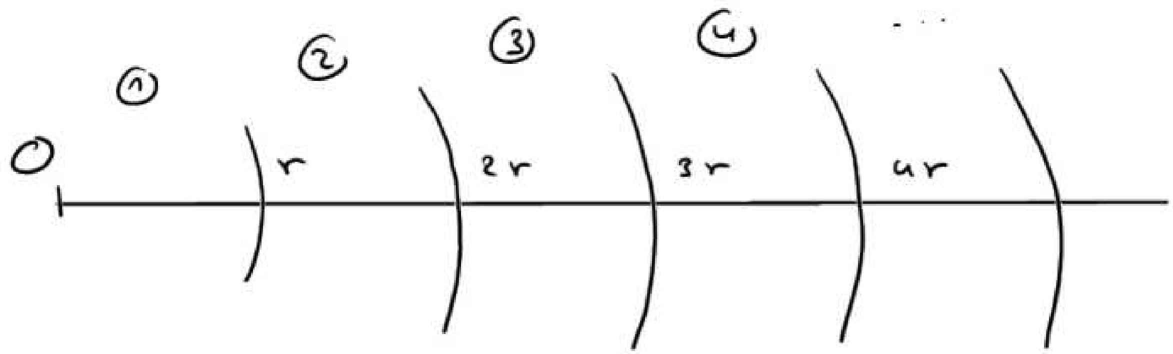


but $\delta m = \rho\, r^2\, \delta\Omega\, dr$

thus

$$\boxed{\delta F = \underbrace{G\, m_o\, \rho\, \delta\Omega}_{\text{cte indep. of } r}\, dr}$$

Split the space in shells of thikness r



$$\delta F_1 = \int_0^r \delta F = \int_0^r dr \, G m_0 \rho \, \delta\Omega = G m_0 \rho \, d\Omega \, r$$

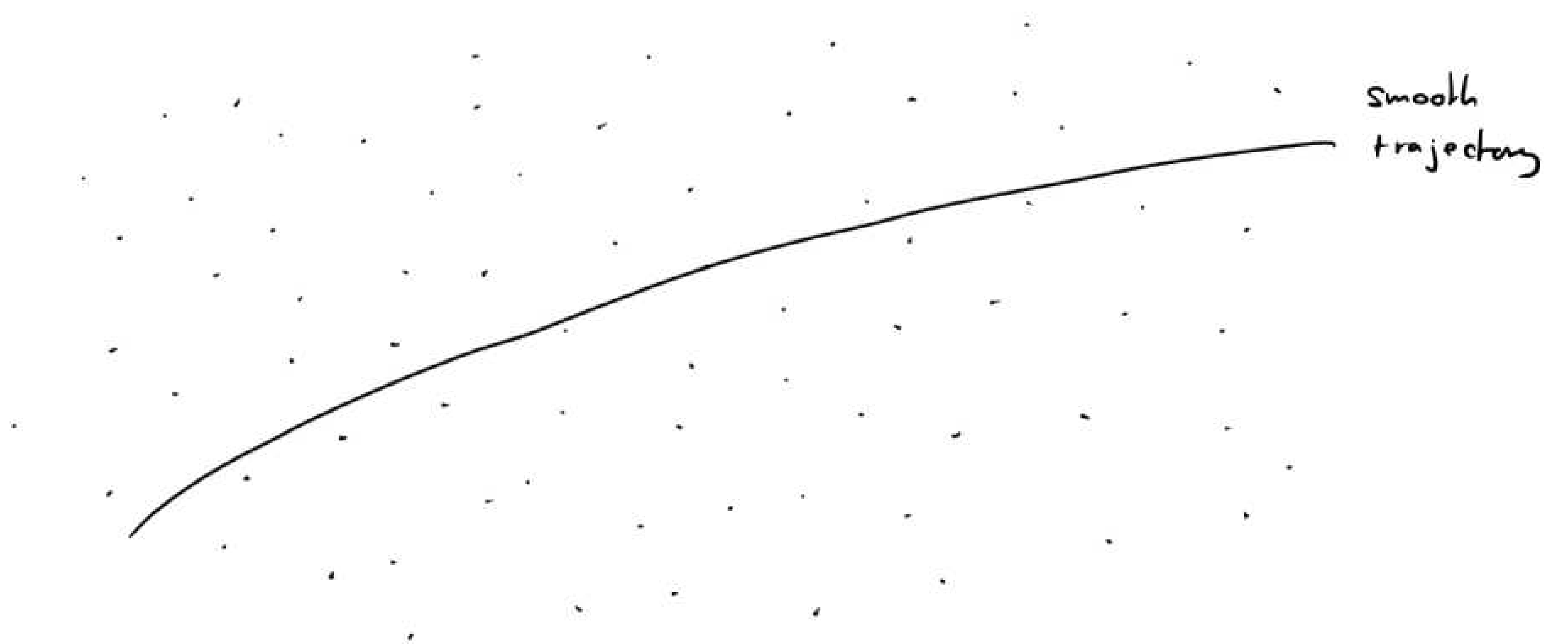
$$\delta F_2 = \int_r^{2r} \delta F = \; \dots \; = G m_0 \rho \, d\Omega \, r$$

$\boxed{\text{As the contribution of all shell is the same, the contribution of the stars with } r' > r \text{ will dominate over the ones with } r' < r}$

$\boxed{\text{We cannot neglect regions at large distances !}}$

Corollary : As the force is dominated by the mass at large scale, the force varies smoothly along the trajectory of a particle (star).

Stellar systems can be modeled by smooth mass distributions



Notes

① **2D case** $\quad \delta F = \frac{G m_0}{r^2} \delta m = \frac{G m_0}{r^2} \Sigma \delta\theta \, r \, dr = G m_0 \Sigma \, \delta\theta \, dr \, \frac{1}{r}$

Σ : surface density

$$\delta F_0 = \int_0^r G m_0 \rho \, \delta\theta \, \frac{1}{r} \, dr = G m_0 \rho \, \delta\theta \left[\ln(r) - \ln(0) \right] = \infty$$

$$\delta F_i = \int_{ir}^{(i+1)r} G m_0 \rho \, \delta\theta \, \frac{1}{r} \, dr = G m_0 \rho \, \delta\theta \left[\underbrace{\ln(i+1) - \ln(i)}_{= 0 \quad (i \to \infty)} \right] < \infty$$

1D case $\quad \delta F = \frac{G m_0}{r^2} \delta m = \frac{G m_0}{r^2} \lambda \, dr \qquad \lambda$: linear density

$$\delta F_0 = \int_0^r G m_0 \lambda \, \frac{1}{r^2} \, dr = G m_0 \lambda \left[-\frac{1}{r} + \frac{1}{0} \right] = \infty$$

$$\delta F_i = \int_{ir}^{(i+1)r} G m_0 \lambda \, \frac{1}{r^2} \, dr = G m_0 \lambda \left[\underbrace{\frac{1}{r} \left(\frac{1}{i} - \frac{1}{i+1} \right)}_{= 0 \ (i \to \infty)} \right] < \infty$$

② Molecular dynamics

"long distance" attraction force between two molecules,

Van der Waals Force $\sim r^{-7}$

$\boxed{\Rightarrow \text{local molecules dominates}}$

Relaxation Time

Question :

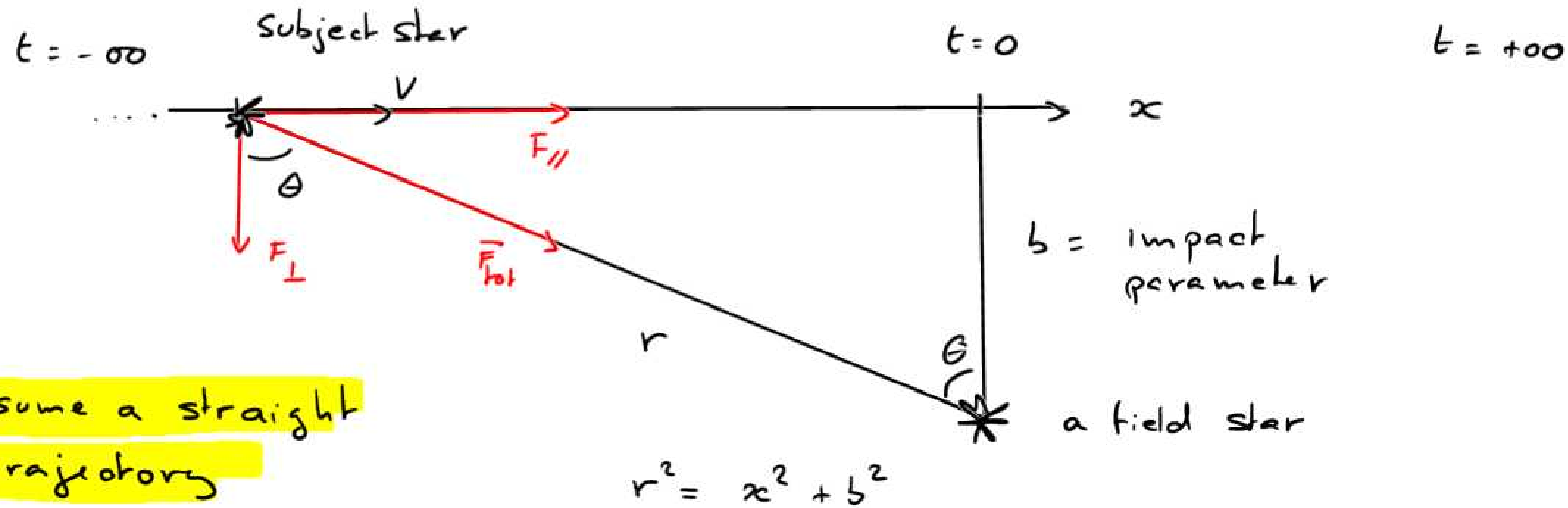
How accurate is the assumption that a galaxy may be modelled as a smooth distribution ?

1) Effect of one star on the orbit of a peculiar star.

2) Effect of all stars of a stellar system on a peculiar star.

3) Under which conditions the orbit of a peculiar star is strongly influenced by the discrete nature of the stellar system (importance of “collision” with other stars).

① Estimate the effect of one star on the trajectrory of a peculiar star



We assume a straight line trajectory

1) acceleration along x ($F_{//}$) does not matter, as it is symetric (the star decelerate after passing the field star

2) acceleration perpendicular to x ($F_{\perp}$)

$$|F_\perp| = |F_{tot}| \cos\theta = \frac{Gmm}{r^2} \cos\theta \qquad \text{but } \cos\theta = \frac{b}{r}$$

$$= \frac{Gmm}{x^2+b^2} \frac{b}{\sqrt{x^2+b^2}} = \frac{Gmm}{(x^2+b^2)^{3/2}} b$$

$r^2 = x^2 + b^2$

$$= \frac{Gmm}{\left(1+\frac{x^2}{b^2}\right)^{3/2}} \frac{1}{b^2} = \frac{Gmm}{b^2} \left(1+\frac{x^2}{b^2}\right)^{-3/2}$$

<u>with $x = vt$</u>

$$\boxed{F_\perp = |F_\perp| = \frac{Gmm}{b^2} \left(1+\left(\frac{vt}{b}\right)^2\right)^{-3/2}}$$

<u>Newton 2nd law</u>

$$F_\perp = m a_\perp = m \frac{dV_\perp}{dt}$$

$$dV_\perp = \frac{F_\perp}{m} dt$$

Integrating over time from $t = -\infty$ to $t = \infty$

$$\delta v_\perp = \frac{1}{m} \int_{-\infty}^{\infty} F_\perp \, dt$$ net velocity increase

$$= \frac{1}{m} \int_{-\infty}^{\infty} \frac{G m' m}{b^2} \left(1 + \left(\frac{vt}{b}\right)^2\right)^{-3/2} dt$$

with : $s = \frac{vt}{b}$ $\qquad ds = \frac{v}{b} \, dt$

$$\delta v_\perp = \frac{G m}{b^2} \int_{-\infty}^{\infty} \left(1 + s^2\right)^{-3/2} dt = 2 \frac{G m}{b v}$$

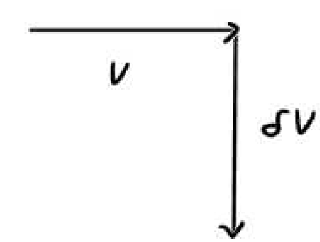
This can be written as :

$$\boxed{\delta v_\perp = \underbrace{\frac{Gm}{b^2}}_{\substack{\text{acceleration}\\ \text{at the closest}\\ \text{approach}}} \cdot \underbrace{\frac{2b}{v}}_{\substack{\text{"duration"}\\ \text{of the closest}\\ \text{approach}}}} \qquad \boxed{= 2\,\frac{Gm}{bv}}$$

$\underline{\text{Note}}$: our hypothesis of a straight line is ok if $\underline{\delta v \ll v}$

$$\Rightarrow \quad \boxed{b \gg \frac{2Gm}{v^2} := b_{90}}$$

b_{90} define as $v = \delta v$



② Effect of all stars of a stellar system on the trajectory of a peculiar star

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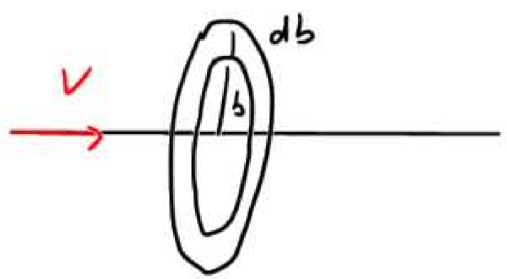
N : total number of stars $\qquad$ R typical size of the system

- number density of stars per unit of surface $\quad n = \dfrac{N}{\pi R^2}$
- number of stars met by the star with $[b, b+db]$

$$\delta N = 2\pi b \, db \cdot n$$
$$= 2\pi b \, db \frac{N}{\pi R^2} = \frac{2 N b \, db}{R^2}$$

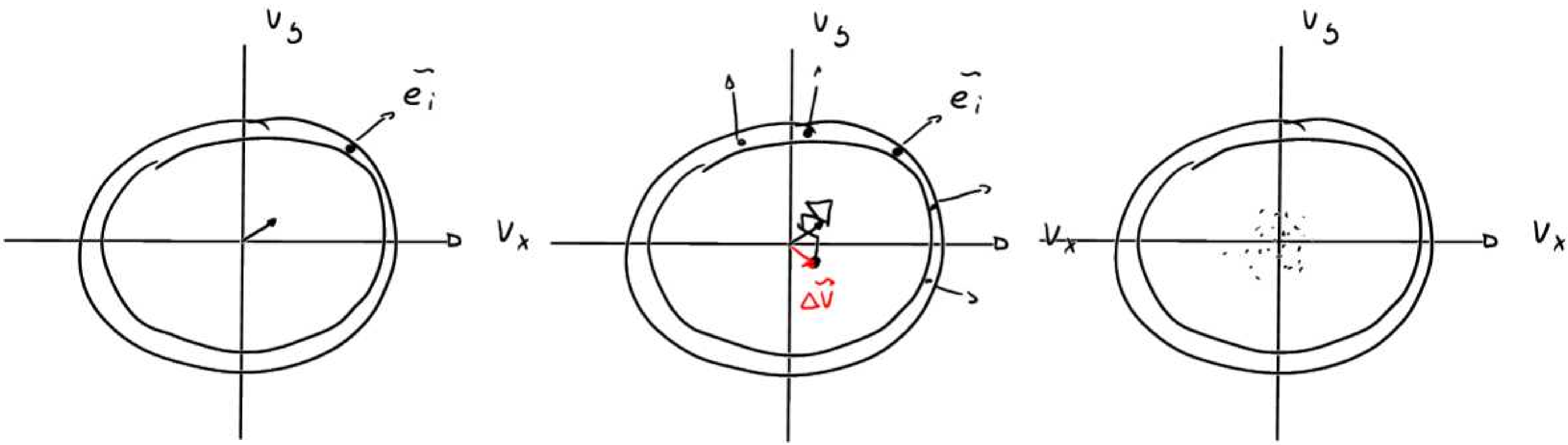


<u>Note</u> : each of those stars (alone) induces a change of velocity

$$\delta V_\perp = \frac{2Gm}{bv}$$

Velocity change due to δN stars in a ring

$$\Delta \vec{v} = \sum_i \delta v_{\perp,i}\, \vec{e}_i$$



velocity change due to one star in the ring

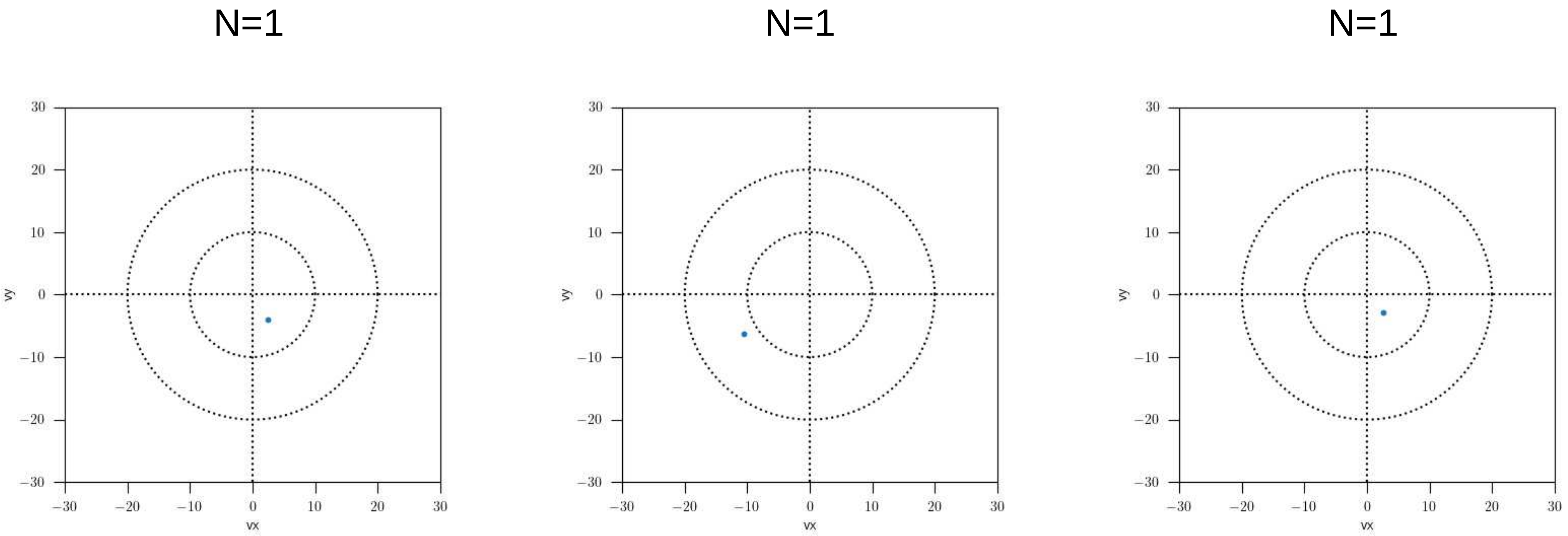
velocity change due to δn star in the ring

statistical distribution of velocity changes

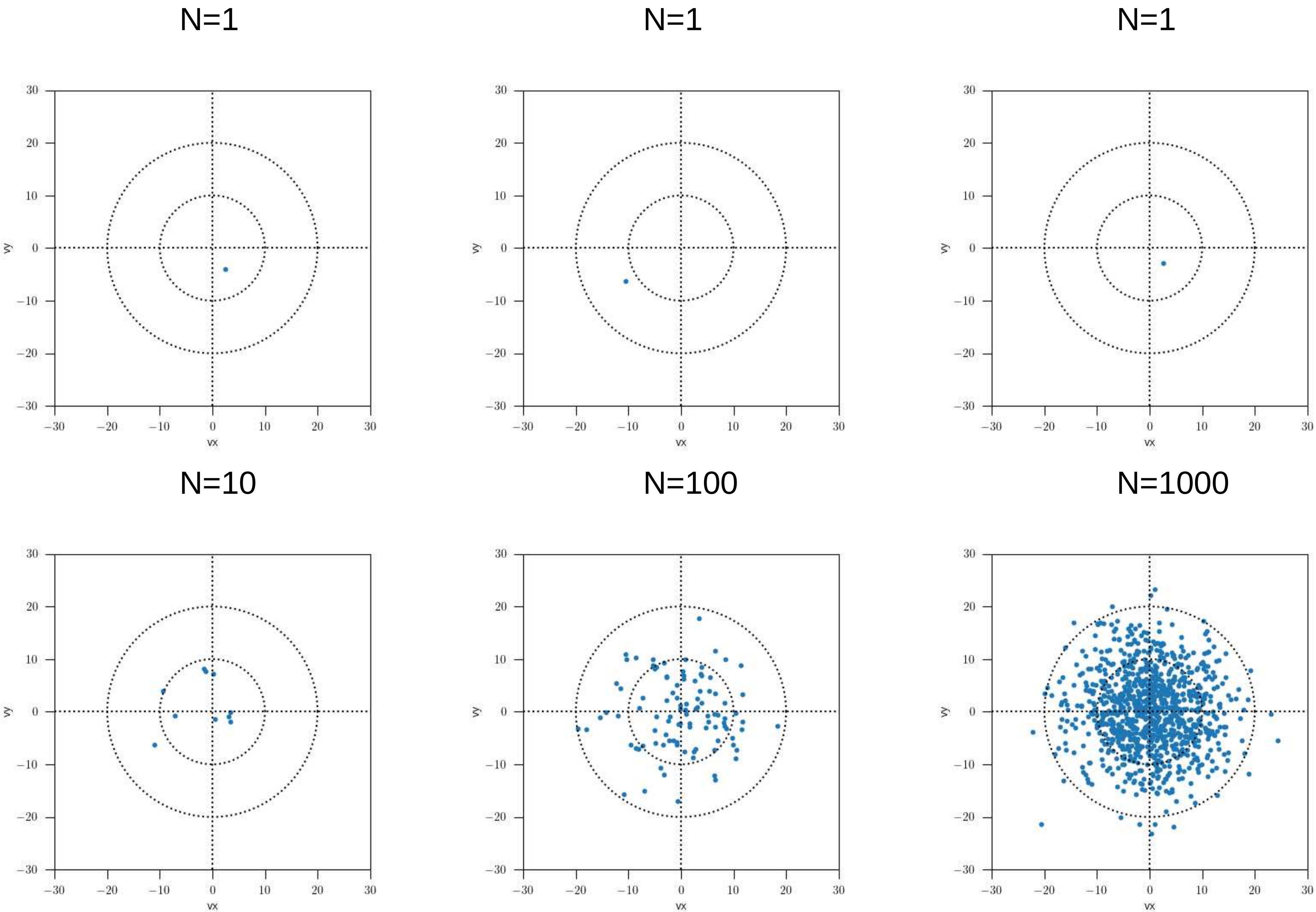
$$\Delta \vec{v} = \sum_i \delta v_{\perp,i}\, \vec{e}_i = 0 \qquad \Delta v^2 = \sum_i \delta v_{\perp,i}^2 \neq 0$$

$$= \delta N\, \delta v^2 = \frac{2N\, b\, db}{R^2}\left(\frac{2Gm}{bv}\right)^2 = \frac{8NG^2m^2}{v^2R^2}\,\frac{db}{b}$$

Spread in the velocity space due to two-body encounters



Spread in the velocity space due to two-body encounters



For all encounters, we integrate over b from b_{min} to b_{max}

$b_{min} := \delta_1 \, b_{90}$ if $b < b_{90}$ $\quad \delta v \sim v \quad$ $\delta_1 \simeq 1$

$b_{max} := \delta_2 \, R$ if $b > R$ $\quad$ the density is no longer constant $\quad$ $\delta_2 \simeq 1$

<u>we get</u>

$\ln \Lambda$: Coulomb logarithm

$$\Delta v^2 = 8N \left(\frac{Gm}{vR} \right)^2 \int_{b_{min}}^{b_{max}} \frac{db}{b} = 8N \left(\frac{Gm}{vR} \right)^2 \overbrace{\ln \left(\frac{b_{max}}{b_{min}} \right)}^{\ln \Lambda}$$

$$\boxed{\Delta v^2 = 8N \left(\frac{Gm}{vR} \right)^2 \left[\ln \left(\frac{R}{b_{90}} \right) + \underbrace{\ln \left(\frac{\delta_2}{\delta_1} \right)}_{\simeq 0} \right]}$$

- variation due to one crossing.

Replacing R with V

Typical velocity of one star (circular orbit)

$$v^2 \sim \frac{GNm}{R} \Rightarrow R = \frac{GNm}{V^2}$$

$$\Delta v^2 = 8N\left(\frac{Gm}{vR}\right)^2 \ln\Lambda = 8N\left(\frac{V}{N}\right)^2 \ln\Lambda = 8\,\frac{V^2}{N}\ln\Lambda$$

$$\boxed{\Delta v^2 = 8\,\frac{V^2}{N}\ln\Lambda}$$

Relaxation

How many crossing of the system (n_{relax}) will lead to a substantial change of the orbit? $\Delta v^2_{tot} \cong v^2$

$$n_{relax} \Delta v^2 \cong v^2$$

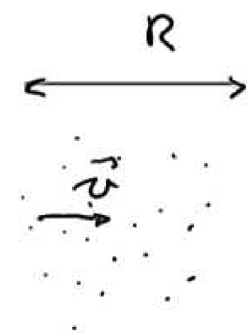
$$n_{relax} \; 8 \frac{v^2}{N} \ln \Lambda \cong v^2$$

$$\boxed{n_{relax} = \frac{8}{N \ln \Lambda}}$$

Crossing time

Time needed for a star to cross the system

$$\boxed{t_{cross} \simeq \frac{R}{V}} \qquad \left(V^2 \sim \frac{GNm}{R} \right)$$



Relaxation time

Time after which the orbit substantially changes

$$\Delta v_{tot}^2 \simeq v^2$$

$$\equiv$$

Time to cross the system n_{relax} times

$$t_{relax} = n_{relax} \cdot t_{cross} = \frac{8}{N \ln \Lambda} t_{cross}$$

if $t \ll t_{relax}$: the perturbation of nearby stars is weak

the system is collision- less

Estimations for stellar systems

N, m, R

using

$$b_{90} = \frac{2Gm}{v^2} \qquad \text{circular orbit: } a = \frac{v^2}{r} = \frac{GNm}{r^2} \Rightarrow v^2 = \frac{GNm}{r}$$

we get

$$\boxed{b_{90} = \frac{2R}{N}} \qquad \boxed{\frac{R}{b_{90}} = \frac{N}{2}} \qquad \boxed{\ln \Lambda \simeq \ln N}$$

$$\boxed{t_{relax} = \frac{N}{8 \ln \Lambda} \cdot t_{cross}} \Rightarrow \boxed{t_{relax} \simeq \frac{0.1\, N}{\ln N}\, t_{cross}}$$

$\ln \Lambda \simeq \ln N$

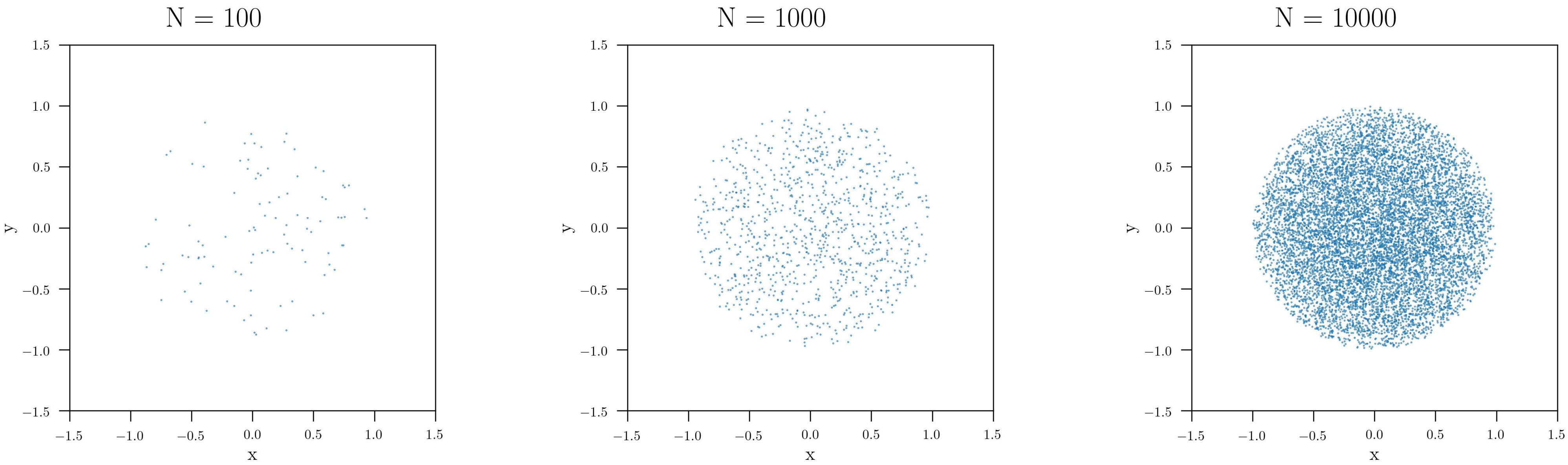
$\frac{1}{8} \simeq 0.1$

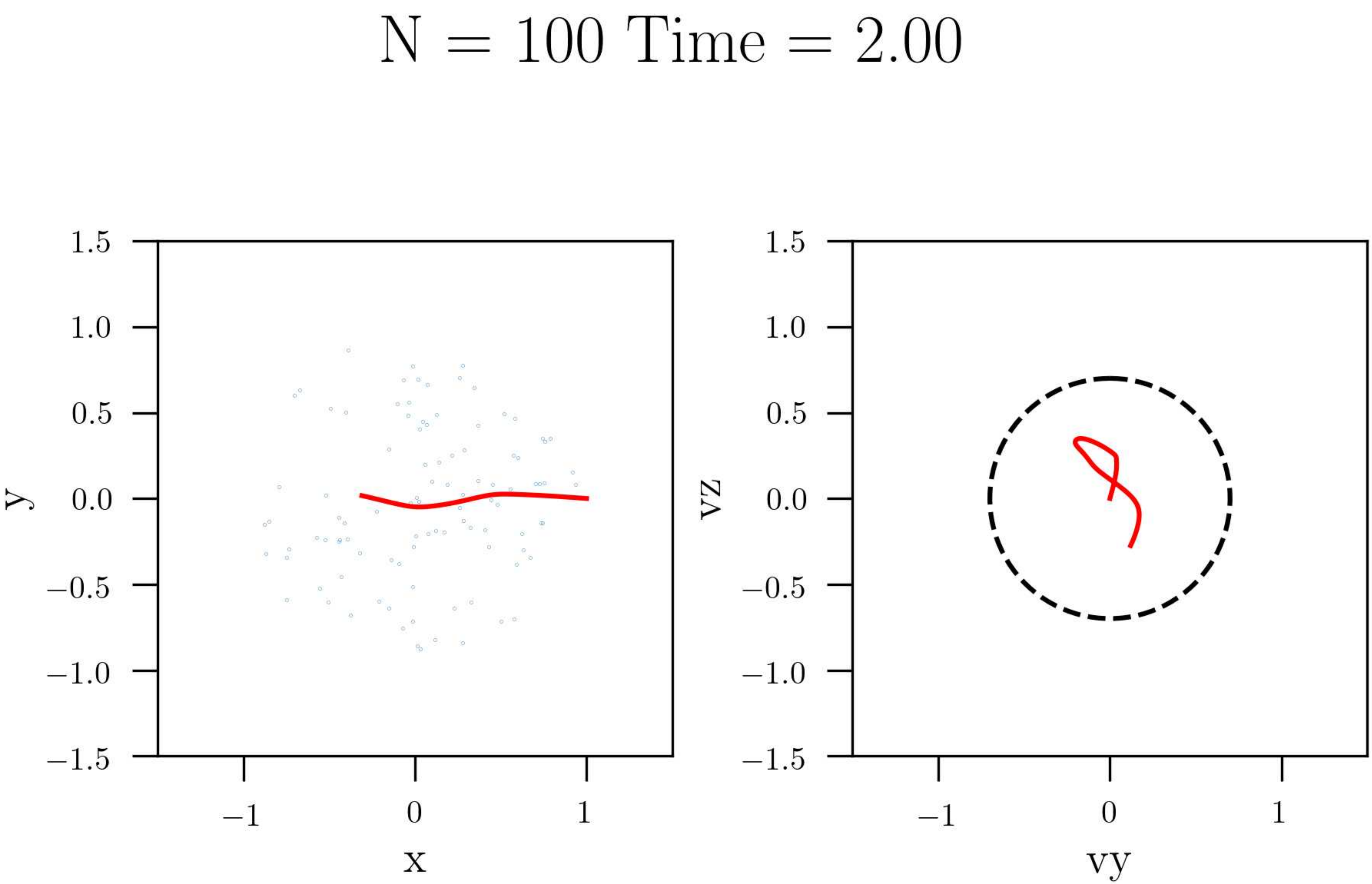
Numerical application / Relaxation Time

	N	R	b_{90}	$\ln(R/b_{90})$	t_{relax}
Globular Cluster	10^5	10 pc	2×10^{-4} pc	~10	~1 Gyr
Dwarf Galaxy	10^6	1 kpc	2×10^{-3} pc	~13	$\gg t_{Hubble}$
Spiral Galaxy	10^{10}	15 kpc	3×10^{-6} pc	~20	$\gg t_{Hubble}$
Galaxy Cluster	10^{13}	1 Mpc	2×10^{-7} pc	~30	$\gg t_{Hubble}$

Numerical illustration

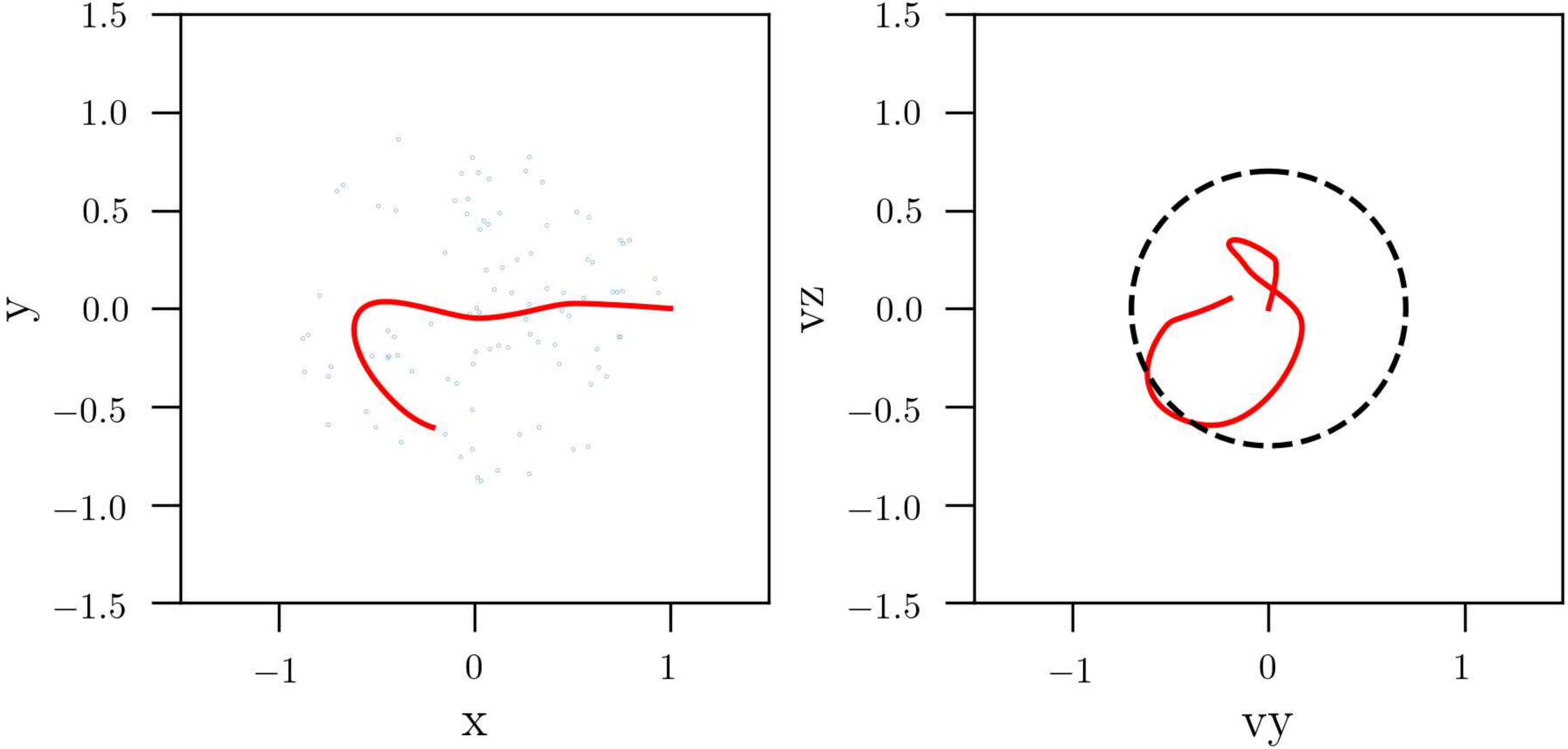
Orbit of a point mass in an homogeneous sphere
sampled with a discrete number of stars.



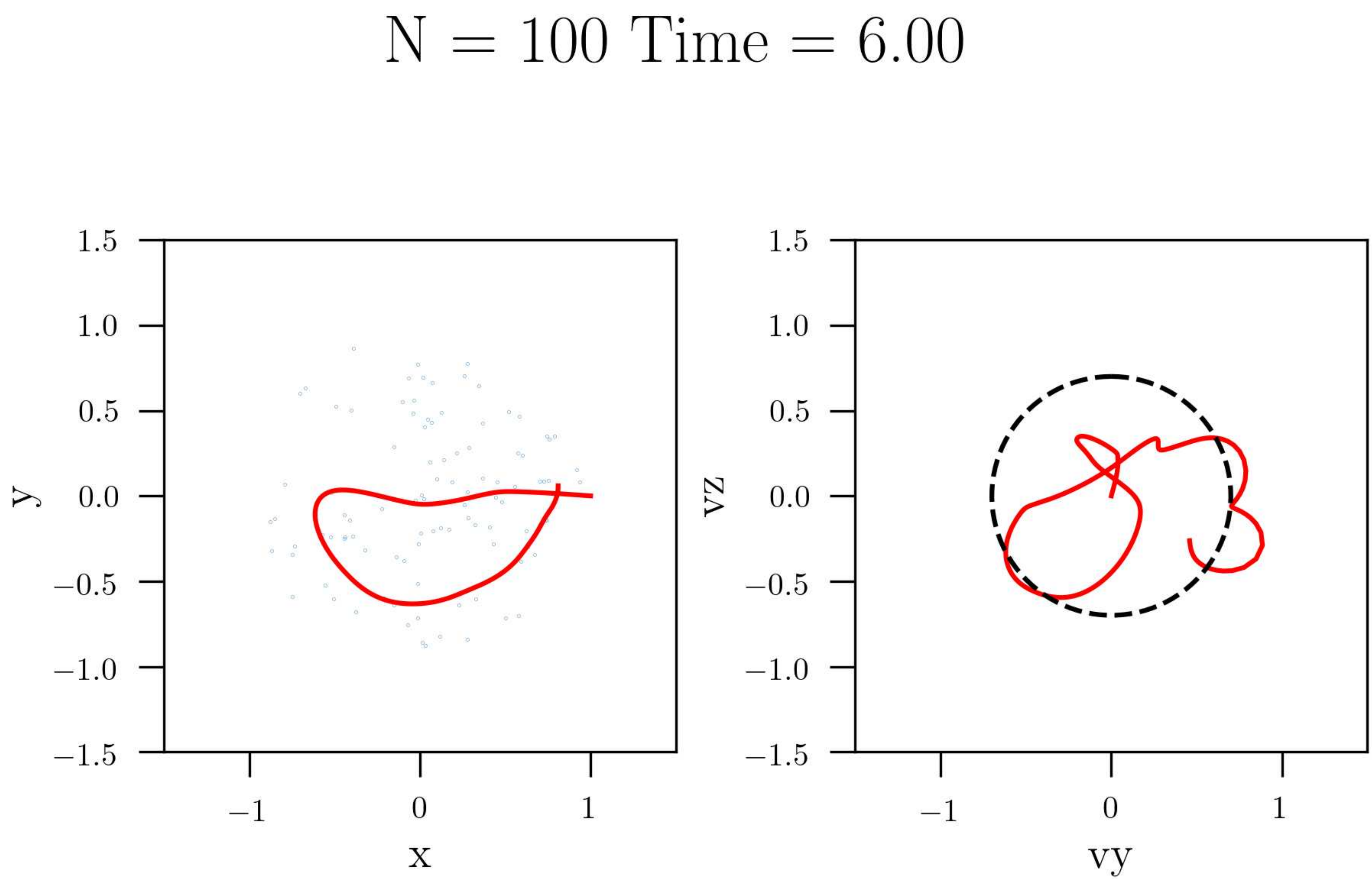

N = 100 Time = 2.00
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
y
−1
0
1
x
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
vz
−1
0
1
vy

t_{relax}=3

$N = 100$ Time $= 4.00$

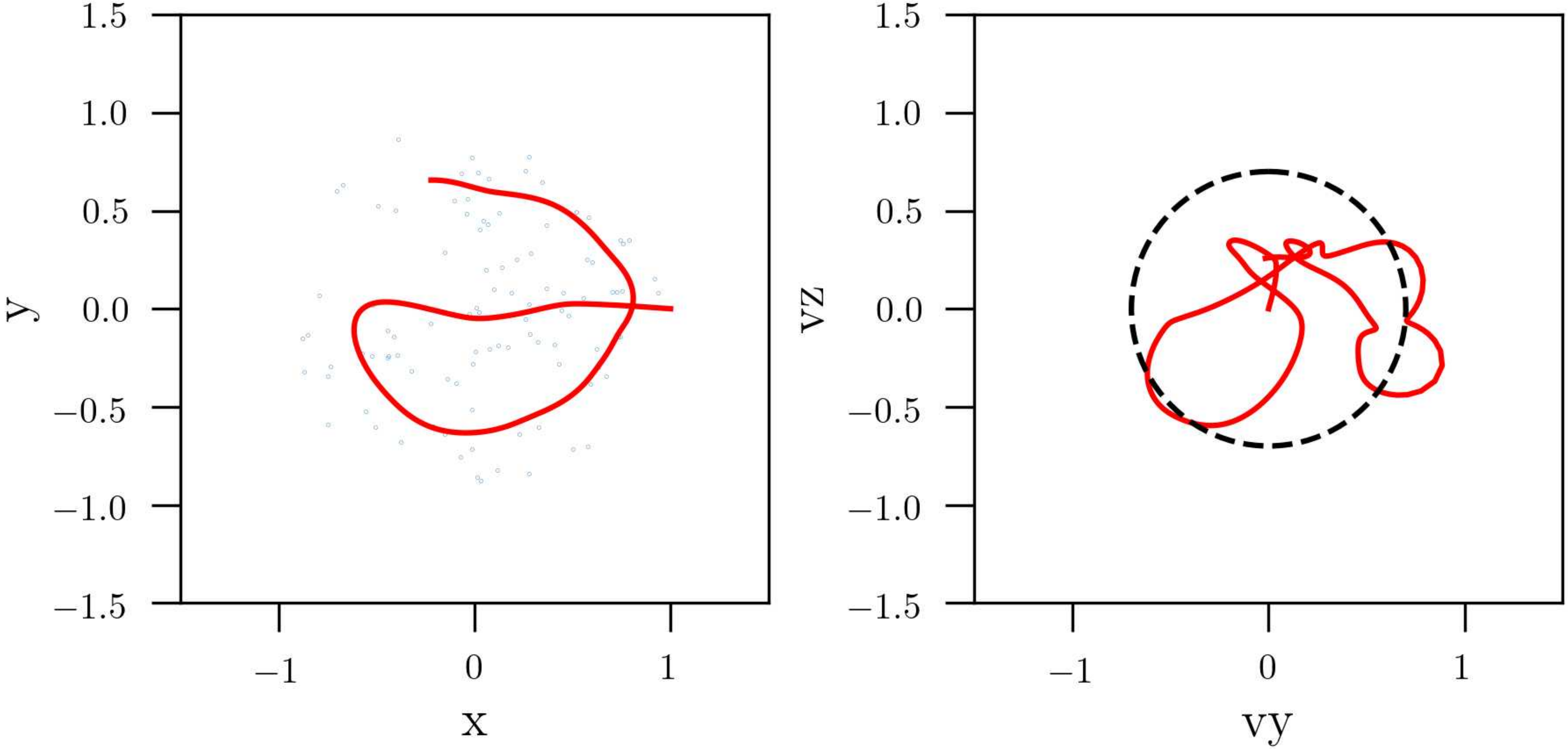


t_{relax}=3



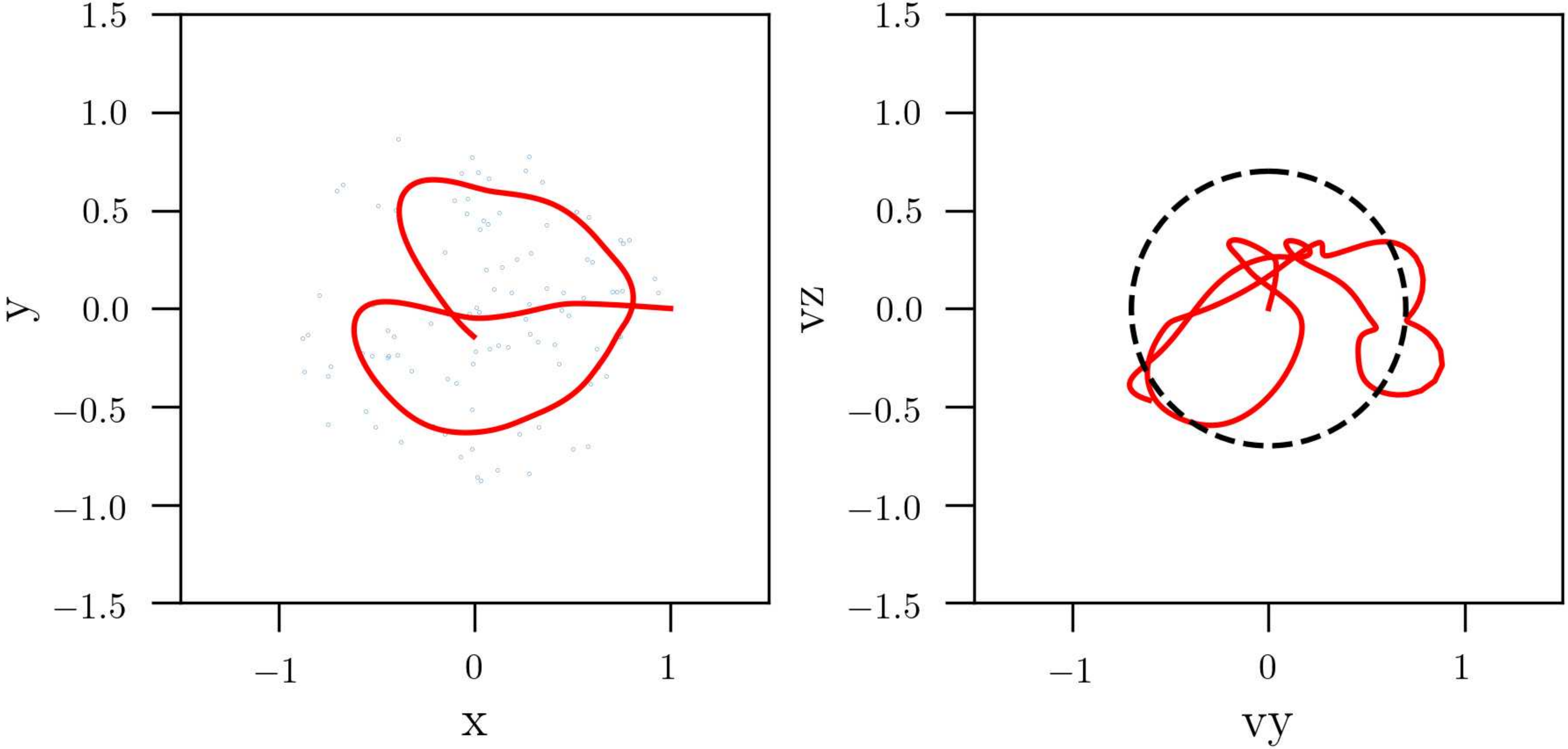
t_{relax}=3

$N = 100$ Time $= 8.00$

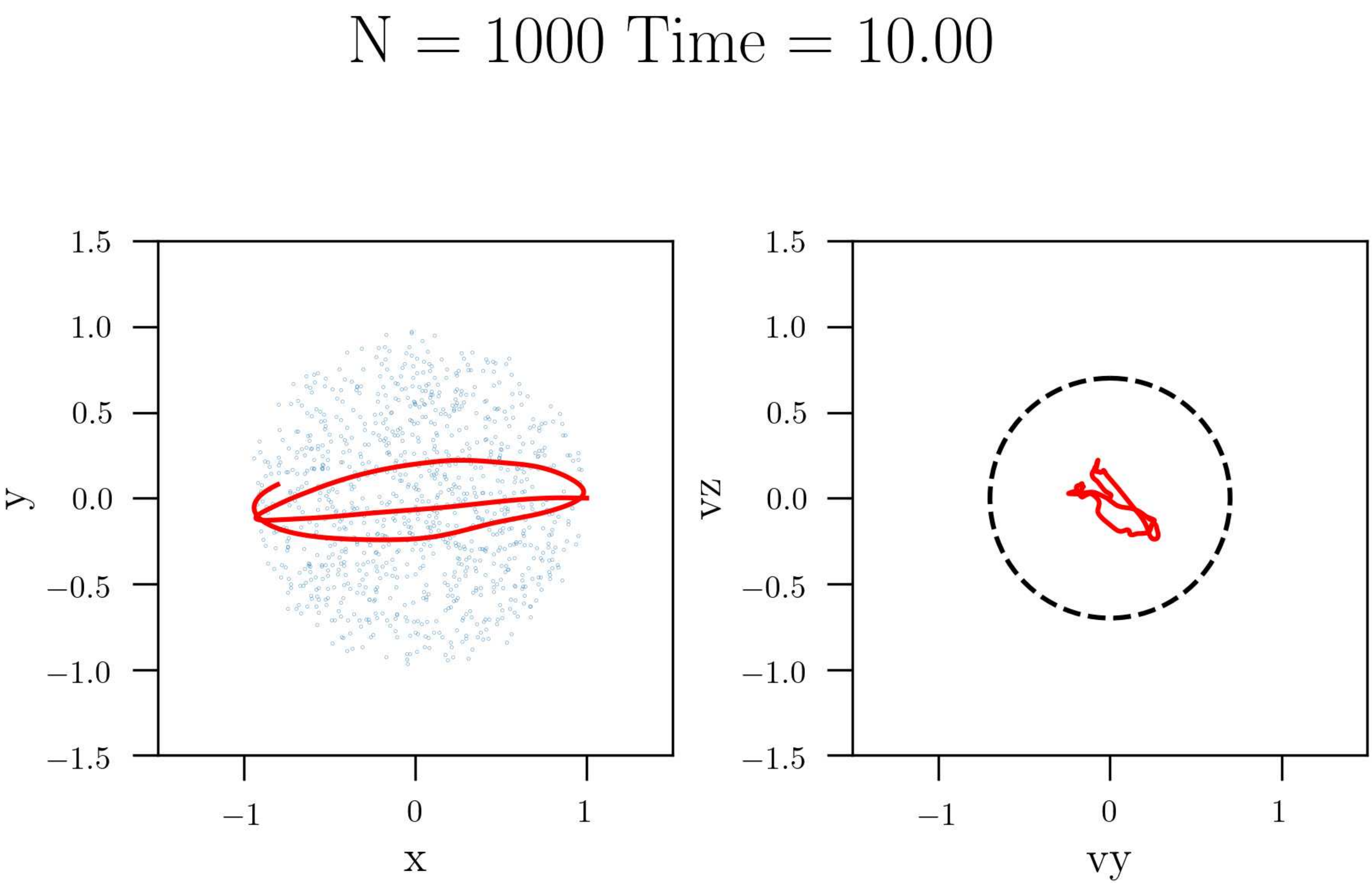


t_{relax}=3

N = 100 Time = 10.00

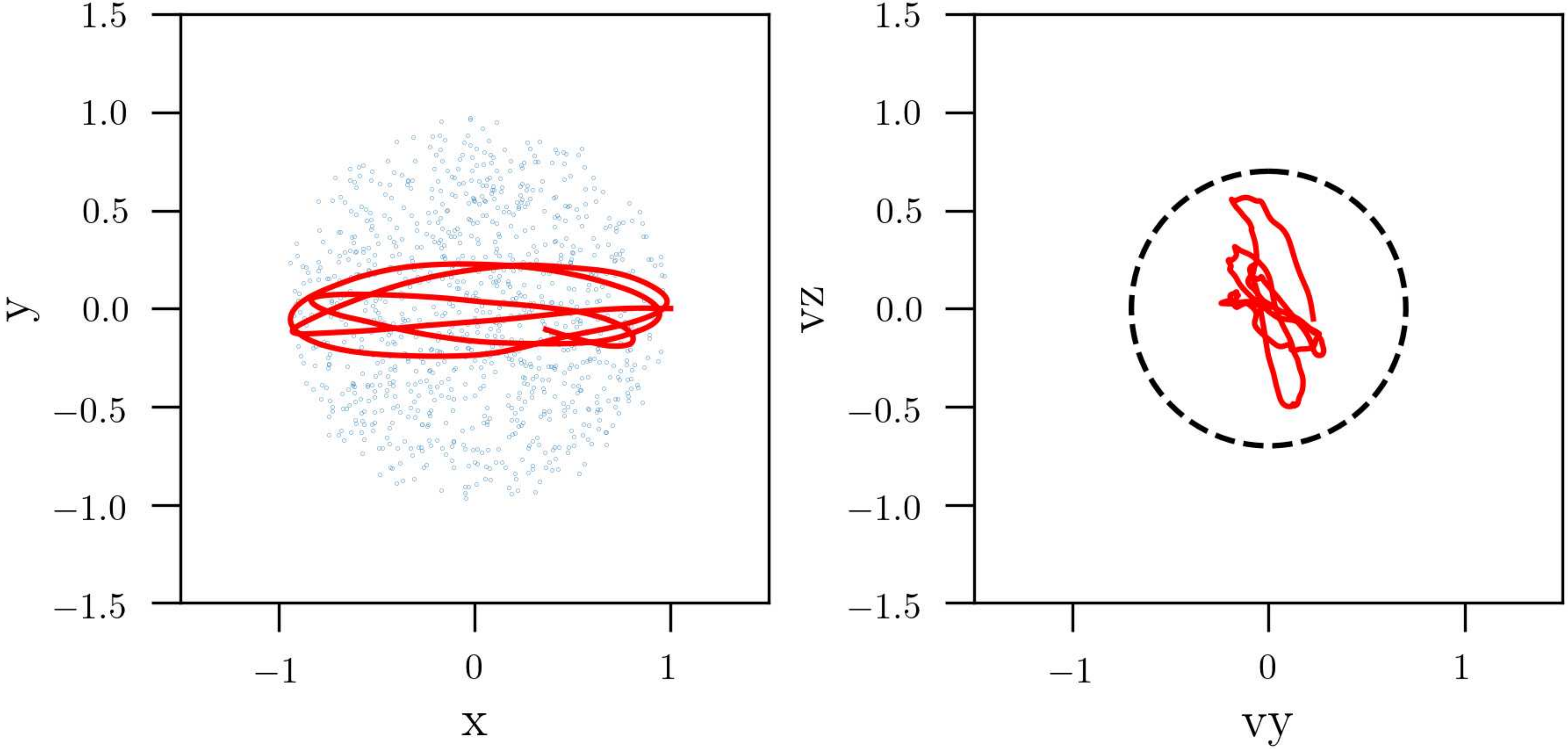


$t_{relax}=3$

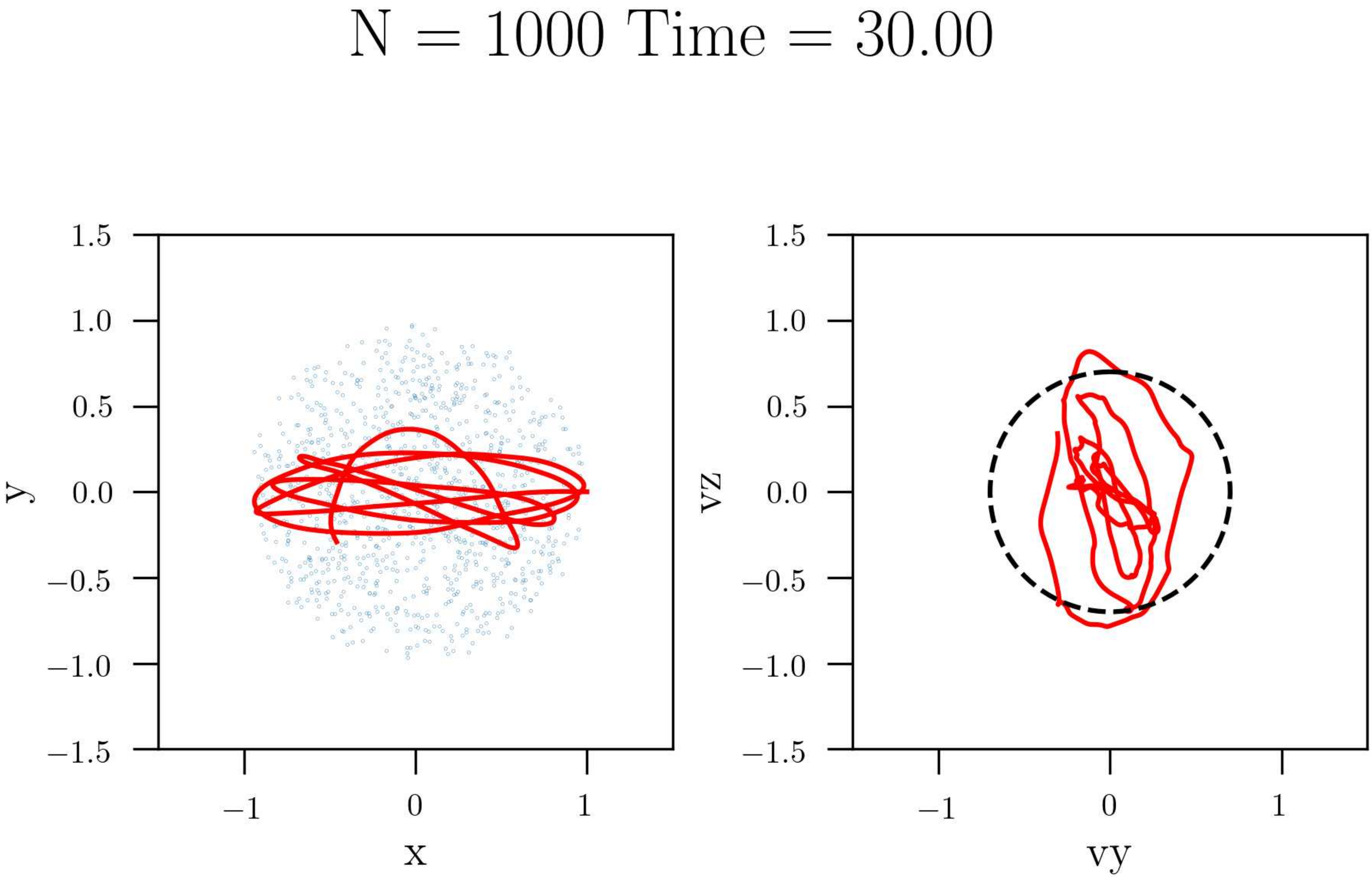


t_{relax}=20

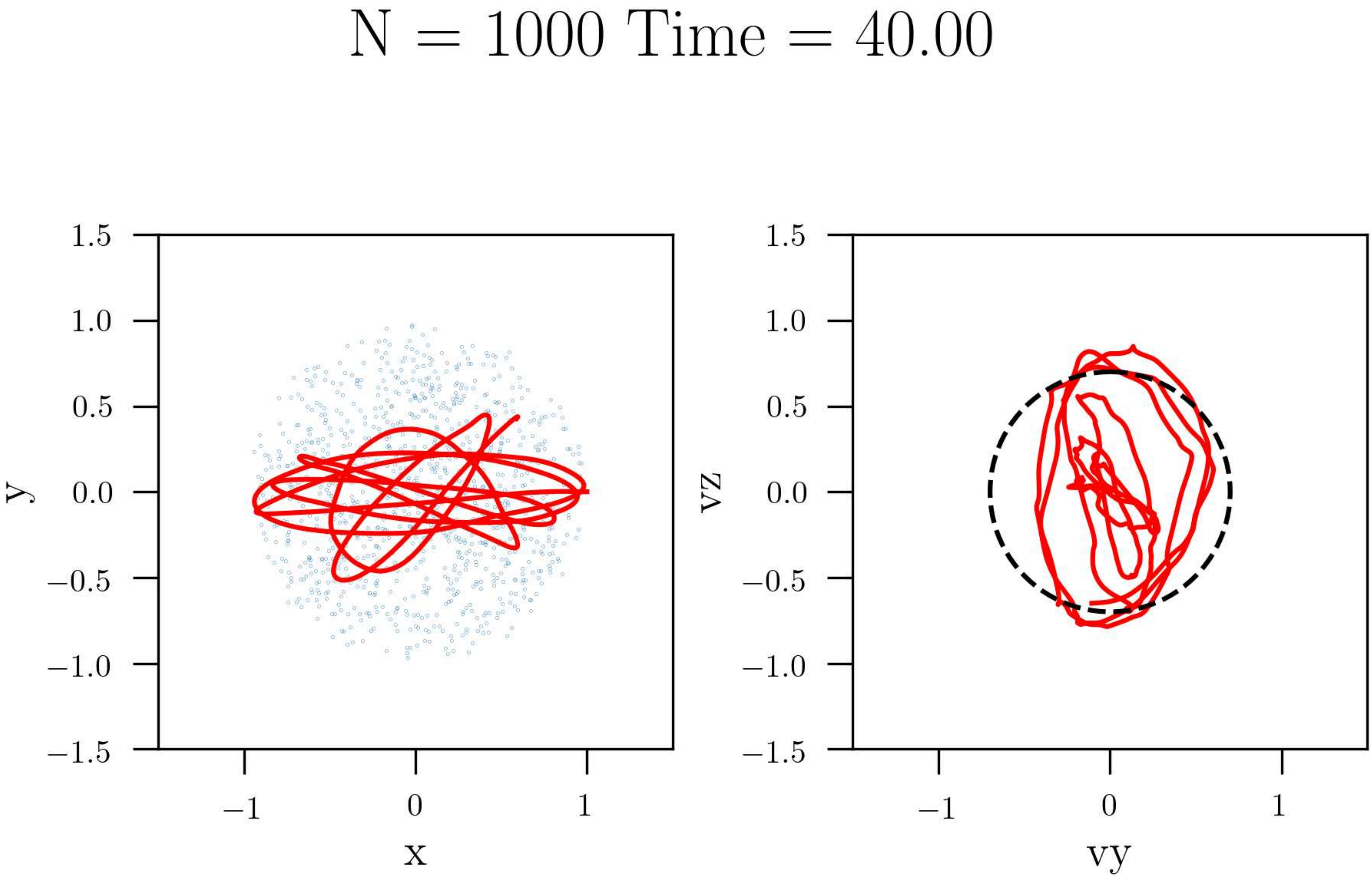
$N = 1000 \text{ Time} = 20.00$



$t_{relax}=20$

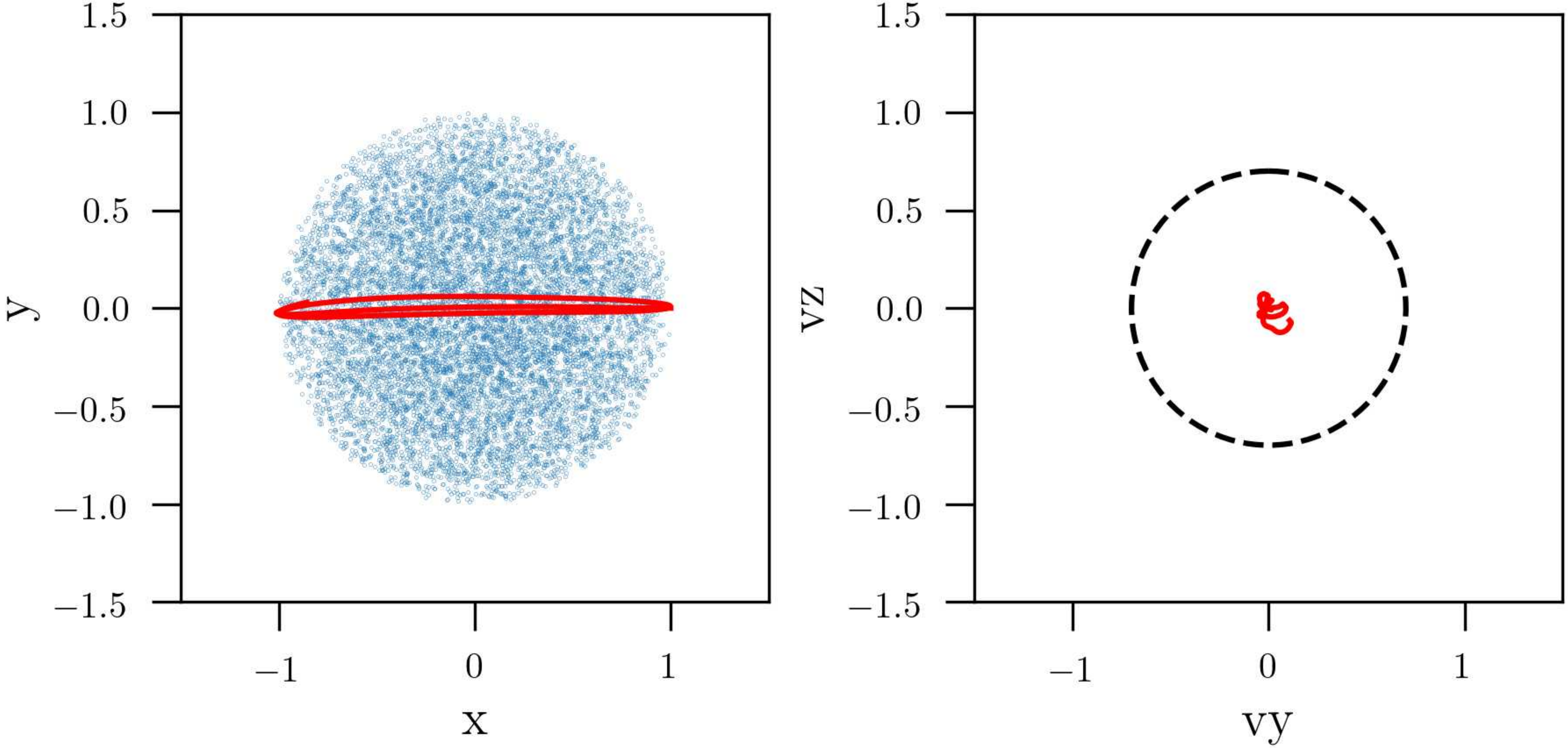


t_{relax}=20



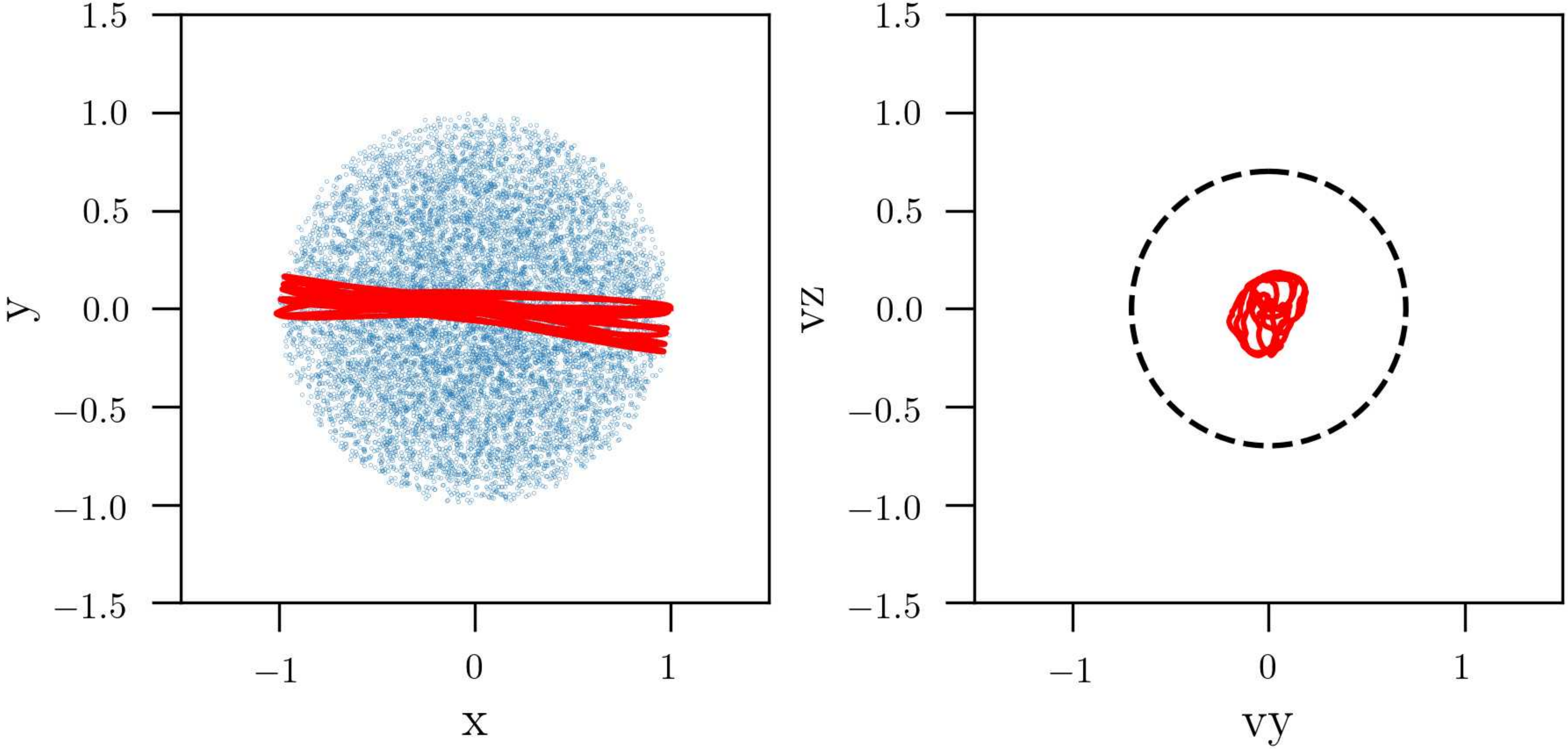
t_{relax}=20

$N = 10000$ Time $= 10.00$



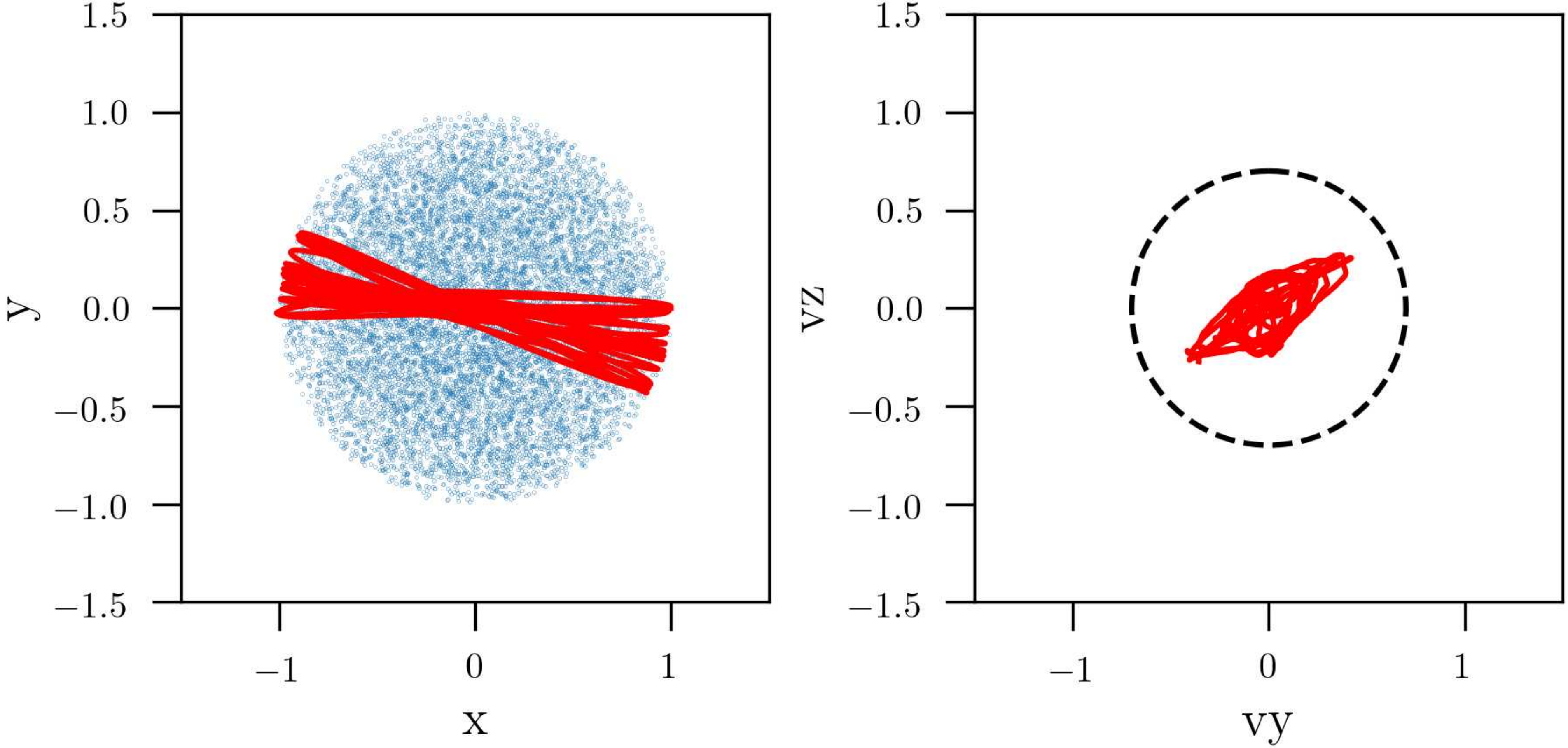
t_{relax}=150

$N = 10000$ Time $= 40.00$

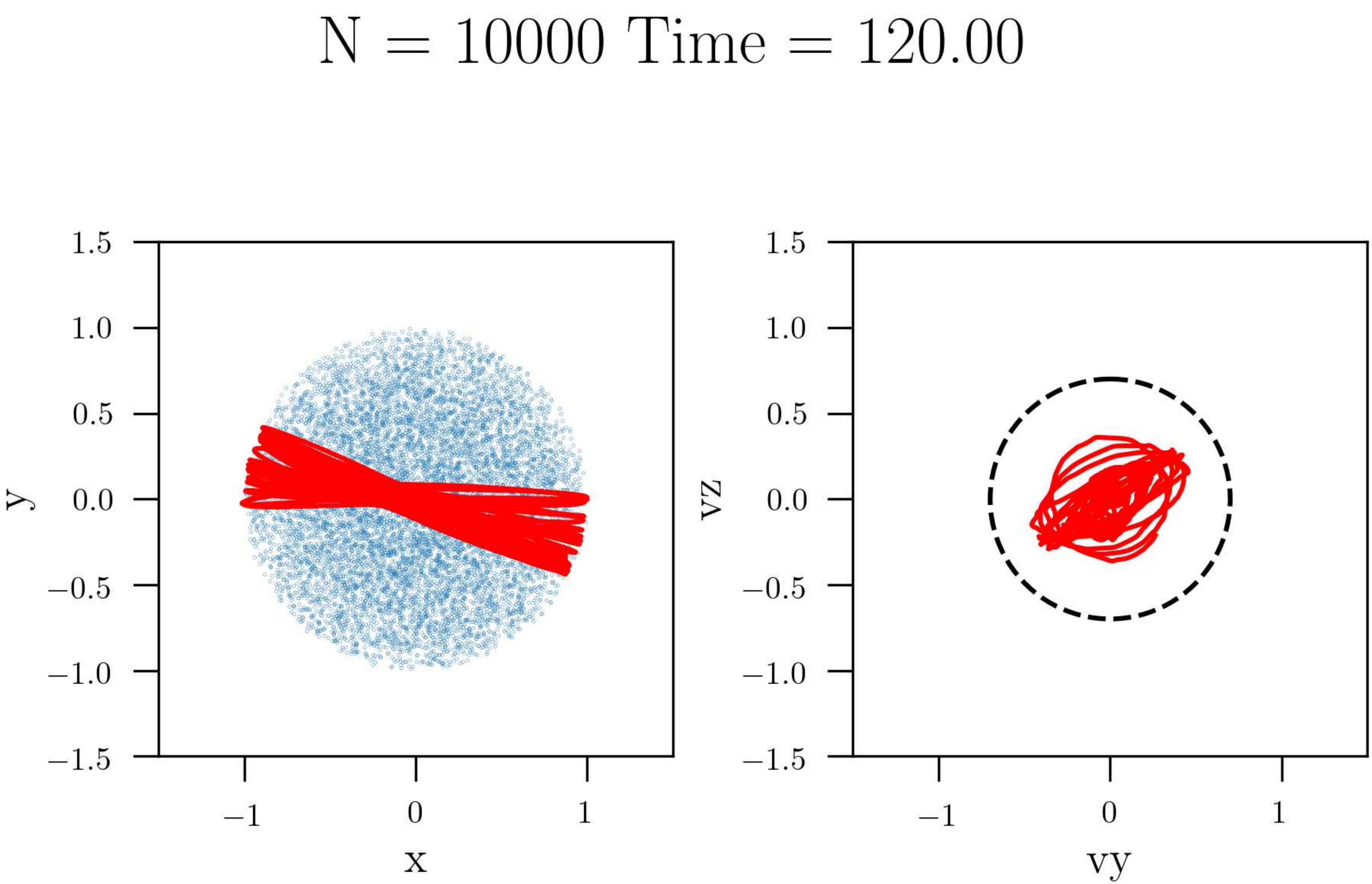


t_{relax}=150

N = 10000 Time = 80.00

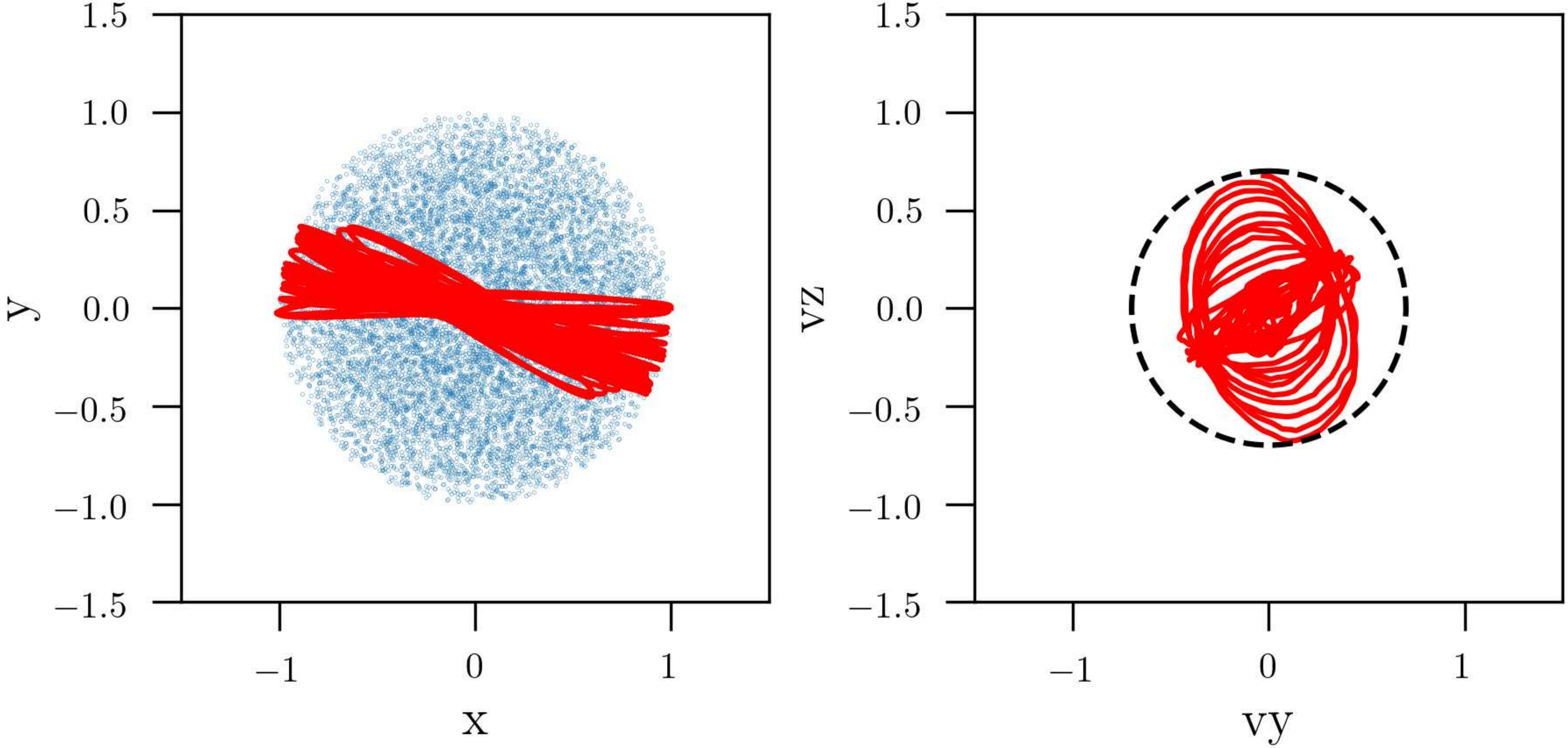


t_{relax}=150



t_{relax}=150

N = 10000 Time = 160.00



t_{relax}=150

Potential Theory I

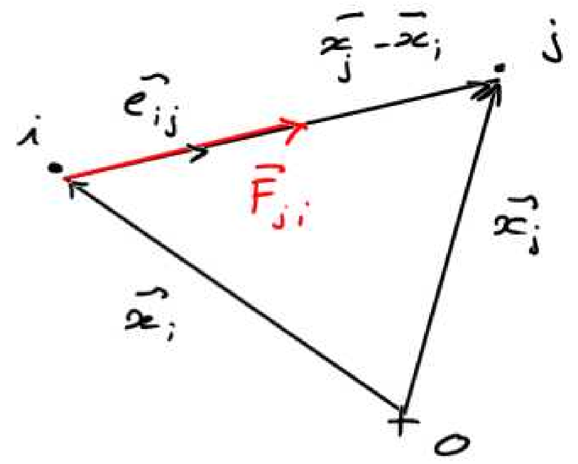
Potential theory : general results

Goal : compute the gravitational potential / forces due to a large number of stars (point masses)

$N \sim 10^{11}$ for a Milky Way like galaxy

As the relaxation time of such system is very large ($\gg$ the age of the Universe) we can describe the system with a smooth analytical potential / density .

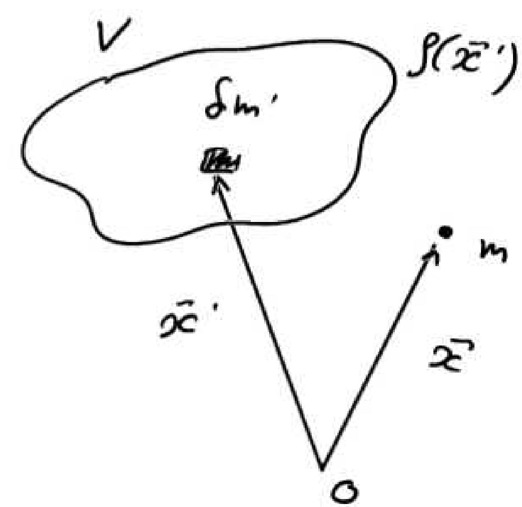
Newton Law



$$\vec{F}_{ji} = \frac{G m_i m_j}{|\vec{x}_j - \vec{x}_i|^2} \vec{e}_{ij} = \frac{G m_i m_j}{|\vec{x}_{ij}|^3} \vec{x}_{ij}$$

$$\vec{x}_{ij} = \vec{x}_j - \vec{x}_j$$

Force on a particle of mass m in $\vec{x}$ due to a distribution of mass $\rho(\vec{x})$



$$\delta \vec{F}(\vec{x}) = \frac{G m \, \delta m'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

$$= \frac{G m \, \rho(\vec{x}') \, d^3\vec{x}'}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})$$

So, the total force writes :

$$\boxed{\vec{F}(\vec{x}) = \int_V \frac{G\, m\, \rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$$\equiv m \underbrace{G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x})\, d^3\vec{x}'}$$

$\vec{g}(\vec{x})$: gravitational field

$$[\vec{g}] = \frac{cm}{s^2} = \frac{erg}{g} \frac{1}{cm}$$

Gravitational Potential

It is easy to see that the function

$$\delta V(\bar{x}) = -\frac{G\, m\, \delta m}{|\bar{x}' - \bar{x}|} \qquad \text{is such that}$$

$$\vec{\nabla}\, \delta V(\bar{x}) = -\frac{G\, m\, \delta m}{|\bar{x}' - \bar{x}|^2} \frac{(\bar{x}' - \bar{x})}{|\bar{x}' - \bar{x}|} = -\delta \vec{F}(\bar{x})$$

so, by defining

$$\boxed{V(\bar{x}) = -G \int_V \frac{m\, \rho(\bar{x}')}{|\bar{x}' - \bar{x}|}\, d^3\bar{x}'}$$

we ensure that $\quad \vec{\nabla} V(\bar{x}) = -\vec{F}(\bar{x})$

We define the specific potential $\phi(\bar{x}) = \dfrac{V(\bar{x})}{m}$

which writes

$$\boxed{\phi(\bar{x}) = -G \int_V \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} d^3\bar{x}'}$$

$[\phi] = \dfrac{erg}{g}$

$\equiv$ specific energy

The gravitational field writes :

$$\boxed{\bar{g}(\bar{x}) = -\bar{\nabla}\phi(\bar{x})}$$

Notes

- The gravity is a conservative force

- $\phi(\bar{x})$: scalar field
  $\vec{g}(\bar{x})$: vector field
  } contain the same information

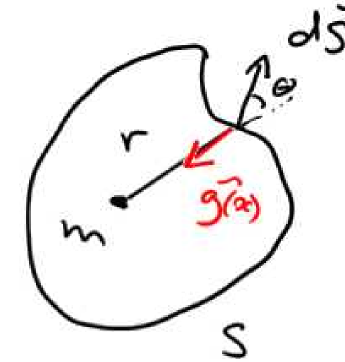
- we will always use "specific" quantities

$$V(\bar{x}) \rightarrow \phi(\bar{x})$$

$$K = \frac{1}{2} m \hat{v}^2 \rightarrow \frac{1}{2} \hat{v}^2$$

$$\frac{1}{2} v^2 + \phi(\bar{x}) = \text{specific energy (conserved quantity)}$$

The Gauss's Law



Consider: • a single point mass m

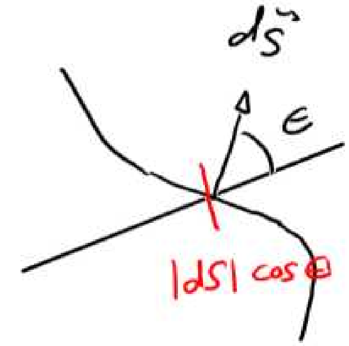
• a surface S around this point

• a point $\vec{x}$ on the surface at a distance r

• $\vec{g}(\vec{x})$ the gravitational field

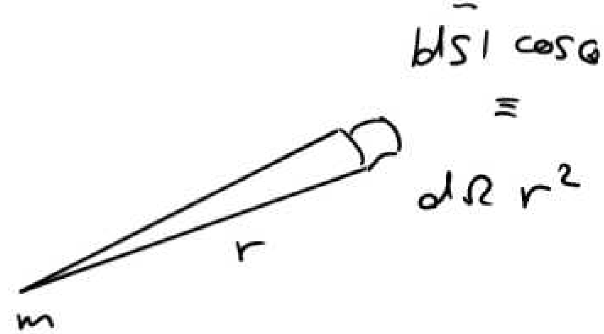
• $d\vec{S}$, the normal at the surface

• θ the angle between $\vec{g}(\vec{x})$ and $d\vec{S}$



$$\vec{g}(\vec{x}) \cdot d\vec{S} = -|\vec{g}(\vec{x})| \cdot |d\vec{S}| \cos\theta$$

But $|d\vec{S}| \cos\theta = r^2 \, d\Omega$



$$|\vec{g}(x)| = \frac{Gm}{r^2}$$

$$\boxed{\vec{g}(\vec{x}) \cdot d\vec{S} = -Gm\, d\Omega}$$

integrating over any surface

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = \begin{cases} -4\pi G m & \text{if } m \text{ inside } S \\ 0 & \text{instead} \end{cases}$$

For multiple masses m_i:

$$\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \sum_{i \text{ in } S} m_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{\int_S \vec{g}(\vec{x}) \cdot d\vec{S} = -4\pi G \int_V \rho(\vec{x})\, d\vec{x} = -4\pi G M}$$

Gauss's Law

Divergeance of the specific force (A) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

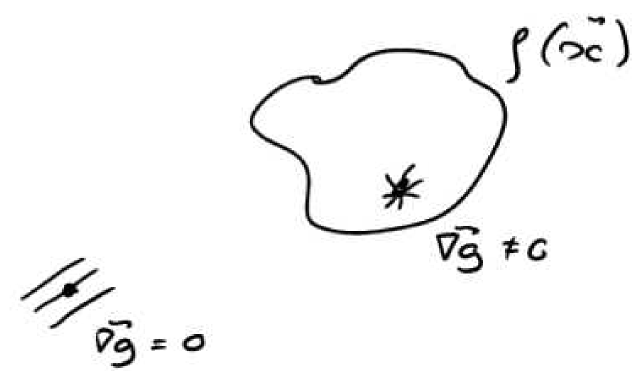
div. theorem

$$\int_V \vec{\nabla} \cdot \vec{g}(\vec{x}) \, d^3\vec{x} = \int_S \vec{g}(\vec{x}) \, d\vec{S}$$

Gauss's Law

$$= -4\pi G \int_V \rho(\vec{x}) \, d\vec{x}$$

$$\boxed{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \, \rho(\vec{x})}$$



The Poisson Equation

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = -4\pi G \rho(\vec{x})$$

with : $\vec{\nabla}_x \phi(\vec{x}) = -\vec{g}(\vec{x})$ $\qquad\qquad$ $\vec{\nabla}_x \cdot (\vec{\nabla}_x) = \vec{\nabla}_x^2$

$$\boxed{\vec{\nabla}_x^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x})}$$

Poisson Equation

Note : To ensure a unique solution, boundary conditions are necessary $\qquad$ (2^{nd} order diff. eqn.)

ex : $\phi(\infty) = 0$

$\vec{\nabla}\phi(\infty) = \vec{g}(x) = 0$

COMPLEMENT

Divergence of the specific force (B) $\vec{\nabla}_x \cdot \vec{g}(\vec{x})$

$$\vec{g}(\vec{x}) = G \int_V \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \, d^3\vec{x}'$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|^3} (\vec{x}' - \vec{x}) \right) d^3\vec{x}'$$

- $$\vec{\nabla}_x \cdot \left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) = \frac{d}{dx_1}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_2}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right) + \frac{d}{dx_3}\left(\frac{\vec{x}' - \vec{x}}{|\vec{x}' - \vec{x}|^3} \right)$$

$$= -\frac{3}{|\vec{x}' - \vec{x}|^3} + \frac{3(\vec{x}' - \vec{x}) \cdot (\vec{x}' - \vec{x})}{|\vec{x}' - \vec{x}|^5}$$

$$= 0 \quad \text{if} \quad \vec{x}' \neq \vec{x}$$

$$\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = G \int_V \vec{\nabla}_x \cdot \left(\frac{\rho(\vec{x}\,')}{|\vec{x}\,' - \vec{x}|^3} (\vec{x}\,' - \vec{x}) \right) d^3\vec{x}\,'$$

$\rho(\vec{x}) = \text{cte}$

$$= G \rho(\vec{x}) \int_{|\vec{x}\,' - \vec{x}| \le h} \vec{\nabla}_x \cdot \left(\frac{\vec{x}\,' - \vec{x}}{|\vec{x}\,' - \vec{x}|^3} \right) d^3\vec{x}\,'$$

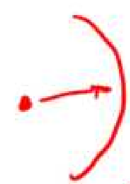
variable exchange

$\vec{\nabla}_x f(\vec{x} - \vec{x}\,') = - \vec{\nabla}_{x'} f(x - x')$

$$= - G \rho(\vec{x}) \int_{|\vec{x}\,' - \vec{x}| \le h} \vec{\nabla}_{x'} \cdot \left(\frac{\vec{x}\,' - \vec{x}}{|\vec{x}\,' - \vec{x}|^3} \right) d^3\vec{x}\,'$$

divergence theorem

$$= - G \rho(\vec{x}) \underbrace{\int_{|\vec{x}\,' - \vec{x}| = h} \frac{\vec{x}\,' - \vec{x}}{|\vec{x}\,' - \vec{x}|^3} d^2\vec{S}\,'}_{4\pi h^2 \cdot \left.\frac{1}{r^2}\right|_{h=r} = 4\pi}$$



$r = |\vec{x}\,' - \vec{x}| = h$

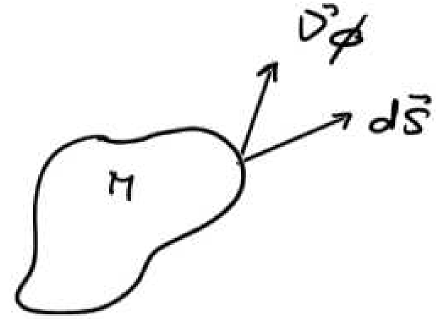
$$\frac{\vec{x}\,' - \vec{x}}{|\vec{x}\,' - \vec{x}|^3} = \frac{1}{r^2}$$

$$\underline{\vec{\nabla}_x \cdot \vec{g}(\vec{x}) = - 4\pi G \rho(\vec{x})}$$

Gauss theorem

(B) integrete the Poisson eqration over a volume V that contains a mass M

$$\int_V \vec{\nabla}^2 \phi(\vec{x})\, d^3\vec{x} = \underbrace{\int_V 4\pi G \rho(\vec{x})\, d^3\vec{x}}$$



div. theorem

$$\boxed{\int_S d^2\vec{S} \cdot \vec{\nabla}\phi = 4\pi G M}$$

Gauss theorem

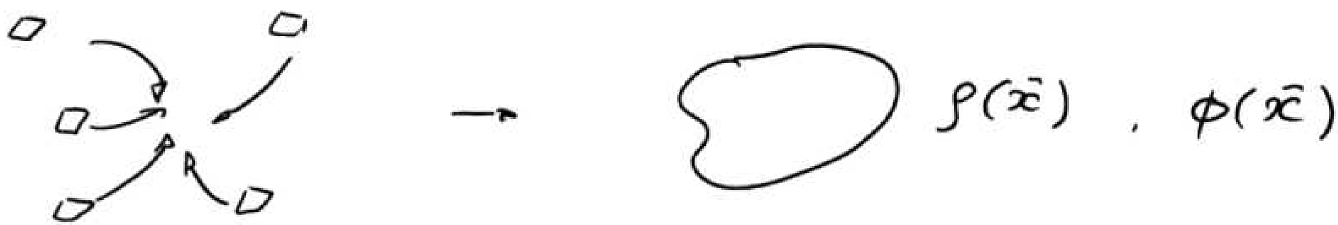
Equivalently :

$$\boxed{\int_S d^2\vec{S} \cdot \vec{g}(\vec{x}) = -4\pi G M}$$

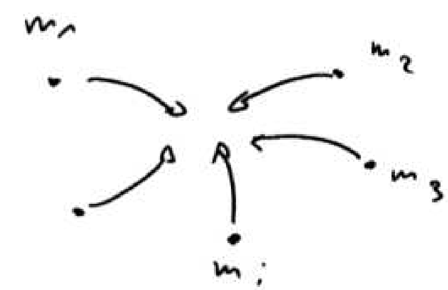
Gauss's Law

Total potential energy (1.0)

Total work needed to assemble a density distribution $\rho(\bar{x})$



Assume a set of discrete points



- The work to bring the 1^{st} point from ∞ to $\bar{x}_1$ is 0
- The work to bring the 2^{nd} point from ∞ to $\bar{x}_2$ is $-\dfrac{G m_1 m_2}{r_{12}}$
- The work to bring the 3^{dn} point from ∞ to $\bar{x}_3$ is $-\dfrac{G m_1 m_3}{r_{13}} - \dfrac{G m_2 m_3}{r_{23}}$

The total work is thus

$$W = -\frac{G m_1 m_2}{r_{12}} - \frac{G m_1 m_3}{r_{13}} - \frac{G m_2 m_3}{r_{23}} - \dots - \sum_{j=1}^{N-1} \frac{G m_{jN}}{r_{jN}}$$

$$= -\sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{G m_i m_j}{r_{ij}} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{G m_i m_j}{r_{ij}}$$

With $\phi_i = -\sum_{\substack{j=1 \\ j \neq i}} \frac{G m_j}{r_{ij}}$ (potential on i)

$$W = \frac{1}{2} \sum_{i=1}^{N} m_i \phi_i \equiv \frac{1}{2} \sum_{i=1}^{N} V_i$$

For a continuous mass distribution $\rho(\vec{x})$

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x}) \phi(\vec{x}) \, d^3\vec{x}}$$

Total potential energy (1.1)

From $W = \frac{1}{2}\int \rho(\vec{x})\,\phi(\vec{x})\,d^3\vec{x}$

- replace $\rho(\vec{x})$ with the Poisson equation $\rho(\vec{x}) = \frac{1}{4\pi G}\nabla^2\phi$

$$W = \frac{1}{8\pi G}\int \nabla^2\phi \cdot \phi(\vec{x})\,d^3\vec{x} = \frac{1}{8\pi G}\int \vec{\nabla}\cdot(\vec{\nabla}\phi)\cdot\phi(\vec{x})\,d^3\vec{x}$$

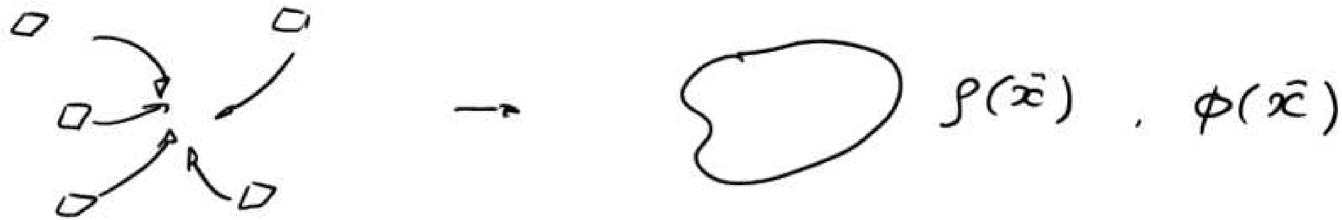
- divergence theorem $\int d^3x\; g\cdot\vec{\nabla}\cdot\vec{F} = \int_S g\cdot\vec{F}\,d\vec{S} - \int d^3x\;\vec{F}\cdot\vec{\nabla}g$

$$W = \frac{1}{8\pi G}\left[\underbrace{\int \phi\,\vec{\nabla}\phi\,d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla}\phi(\infty) = 0} - \int d^3\vec{x}\;\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right]$$

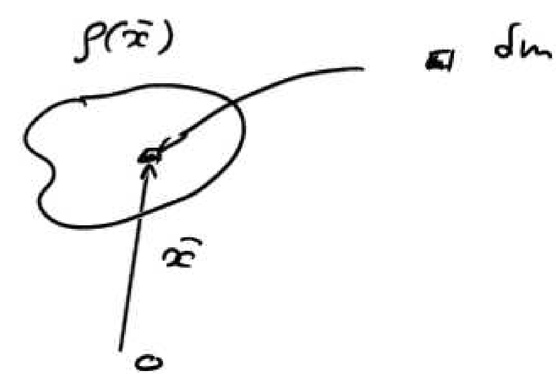
$$\boxed{W = -\frac{1}{8\pi G}\int d^3\vec{x}\,|\vec{\nabla}\phi|^2}$$

Total potential energy (2.0)

Total work needed to assemble a density distribution $\rho(\vec{x})$



(1) Work done to assemble a piece of mass $\delta m = \delta\rho \, d\vec{x}^3$ from ∞ to $\vec{x}$ assuming an existing mass distribution $\rho(\vec{x})$, $\phi(\vec{x})$

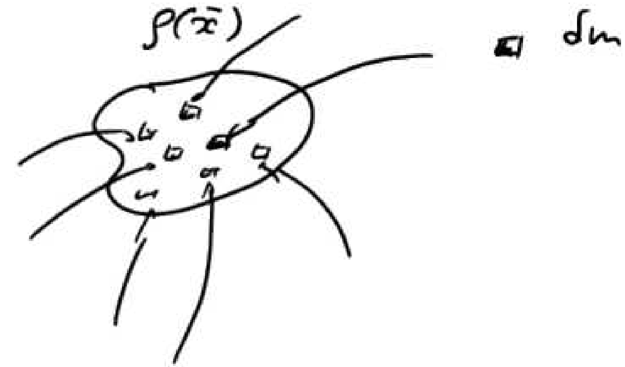


$$\delta W_{\vec{x}} = V(\vec{x}) - \underbrace{V(\infty)}_{=0}$$

$$= \delta m \, \phi(\vec{x}) = \delta\rho(\vec{x}) \, d^3\vec{x} \, \phi(\vec{x})$$

To increase everywhere the mass distribution by $\delta\rho$

$$\rho(\bar{x}) \to \rho(\bar{x}) + \delta\rho(\bar{x})$$



$$\delta W = \int \delta\rho(\bar{x})\, d^3\bar{x}\ \phi(\bar{x})$$

Poisson: $\delta\rho(\bar{x}) = \frac{1}{4\pi G} \vec{\nabla}^2 \delta\phi(x)$

$$\delta W = \frac{1}{4\pi G} \int \vec{\nabla}^2 \delta\phi(x)\ \phi(\bar{x})\ d^3\bar{x}$$

divergeance theorem

$$\int_V d^3x\, g\, \vec{\nabla}\cdot\vec{F} = \int_S g\, \vec{F}\cdot d^2\vec{S} - \int_V d^3x\, \vec{F}\cdot\vec{\nabla}g$$

$$= \underbrace{\frac{1}{4\pi G} \int_{S \text{ at } \infty} \phi(\bar{x})\, \vec{\nabla}\delta\phi(\bar{x})}_{=0} - \frac{1}{4\pi G} \int \vec{\nabla}\phi(\bar{x}) \cdot \vec{\nabla}(\delta\phi(\bar{x}))\ d^3\bar{x}$$

as $\phi(\infty) = 0$

$\vec{\nabla}\delta\phi(\infty) = \delta g(\infty) = 0$

$$\delta W = -\frac{1}{4\pi G}\int \vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}(\delta\phi(\vec{x}))\, d^3\vec{x}$$

with

$$\frac{1}{2}\,\delta|\vec{\nabla}\phi(\vec{x})|^2 = \delta\vec{\nabla}\phi(\vec{x}) \cdot \vec{\nabla}\phi(\vec{x}) = \vec{\nabla}(\delta\phi(\vec{x})) \cdot \vec{\nabla}\phi(\vec{x})$$

$$\delta W = -\frac{1}{8\pi G}\int \delta|\vec{\nabla}\phi|^2\, d^3x \equiv -\frac{1}{8\pi G}\,\delta\int |\vec{\nabla}\phi|^2\, d^3x$$

2 Contribution of all δW to W

$$\boxed{W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2\, d^3x}$$

COMPLEMENT

Total potential energy (2.2)

From $W = -\frac{1}{8\pi G}\int |\vec{\nabla}\phi|^2 d^3x = -\frac{1}{8\pi G}\int \vec{\nabla}\phi \cdot \vec{\nabla}\phi \, d^3\vec{x}$

- divergence theorem $\int d^3x \, \vec{F}\cdot\vec{\nabla} g = \int_S g\,\vec{F}\, d\vec{S} - \int d^3x \, g \, \vec{\nabla}\vec{F}$

$$W = -\frac{1}{8\pi G}\left[\underbrace{\int_S \phi \vec{\nabla}\phi \, d\vec{S}}_{=0 \text{ as } \phi(\infty) = \vec{\nabla}\phi(\infty) = 0} - \int d^3\vec{x} \, \phi \, \vec{\nabla}\underbrace{(\vec{\nabla}\phi)}_{4\pi G\rho \text{ (Poisson)}} \right]$$

$$= \frac{1}{8\pi G} 4\pi G \int d^3\vec{x} \, \phi(\vec{x}) \, \rho(\vec{x})$$

$$\boxed{W = \frac{1}{2}\int \rho(\vec{x})\,\phi(\vec{x})\, d^3\vec{x}}$$

COMPLEMENT

Total potential energy : Summary

$$\boxed{W = \frac{1}{2} \int \rho(\vec{x})\, \phi(\vec{x})\, d^3\vec{x}}$$

$$\boxed{W = -\frac{1}{8\pi G} \int |\vec{\nabla}\phi|^2\, d^3x}$$

Other useful expression

$$\boxed{W = -\int \rho(\vec{x})\, \vec{x} \cdot \vec{\nabla}\phi(\vec{x})\, d^3\vec{x}}$$

Relation between the potential energy and the Poisson equation

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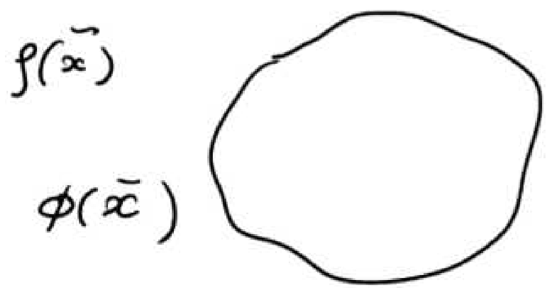
What is the relation that must hold between the density $\rho(\bar{x})$ and potential $\phi(\bar{x})$ in order to minimize the potential energy of a system ?



W : potential energy

<u>Answer</u> : the Poisson equation $\nabla^2\phi = 4\pi G\rho$

The End