Final exam

Exercise 1. Quiz. (26 points) Answer each short question below. For yes/no questions explicitly say if the statement is true of false and provide a brief justification (proof or counter-example) for your answer. For other questions, provide the result of you computation, as well as a brief justification for your answer.

a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{R}$ be a random variable with cumulative distribution function $F_X(x)$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is a strictly decreasing and continuous function. Define $Y = g(X)$. What is the cumulative distribution function of Y?

b) Let X_1 and X_2 be two Gaussian random variables such that the random vector $X = (X_1, X_2)$ has a covariance matrix

$$\mathrm{Cov}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Does this imply that X_1 and X_2 are independent?

c) Let X, Y be integrable random variables on the measurable space $(\Omega, \mathcal{F}, P)$. Define $\mathcal{G} = \sigma(Y)$. Suppose that $\mathbb{E}[XY|\mathcal{G}] = aY^2 + bY$, where a and b are constants. Compute $\mathbb{E}[X]$ in terms of a, b, and $\mathbb{E}[Y]$.

d) Let X, Y be i.i.d random variables on the measurable space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, also suppose $X + Y$ and $X - Y$ are independent random variables. Is it true that $\phi_X(2t) = (\phi_X(t))^2 |\phi_X(t)|^2$ for all $t \in \mathbb{R}$? Why or Why not?

e) Let $M_0 = 0.4$ and define recursively

$$M_{n+1} = \begin{cases} M_n^2 & \text{with probability} \frac{1}{2} \\ 2M_n - M_n^2 & \text{with probability} \frac{1}{2}. \end{cases}$$

Is it true that the resulting martingale $(M_n, n \in \mathbb{N})$ converges almost surely to some random variable M_∞? Why or why not?

A sequence of biased coins is flipped. The chance that the rth coin shows head is Θ_r, where Θ_r is a random variable taking values in $(0, 1)$. Let X_n be the number of heads after n flips.

f) Does X_n obey the central limit theorem when Θ_r are independent and identically distributed?

g) Does X_n obey the central limit theorem when $\Theta_r = \Theta$ for all r, where Θ is a random variable on $(0, 1)$?

Exercise 2. (26 points)

Let X and Y be random variables on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The total variation distance is defined as

$$d_{TV}(X,Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |(\mathbb{P}(\{X \in A\}) - \mathbb{P}(\{Y \in A\}))|.$$

a) Show that

$$\mathbb{P}(\{X = Y\}) \leq 1 - d_{TV}(X,Y).$$

b) Let X and Y be discrete and supported on $\{0, 1, 2, \dots\}$ with pmfs p_X and p_Y. Show that

$$d_{TV}(X,Y) = \frac{1}{2} \sum_{m \geq 0} |p_X(m) - p_Y(m)|.$$

c) (Optimal coupling) Let $\Omega = \{1, \dots k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\omega) = \frac{1}{k} \quad \forall \omega \in \Omega$. Suppose that X is Bernoulli$(\frac{k-1}{k})$ and Y is Bernoulli$(\frac{2}{k})$. What is $d_{TV}(X,Y)$?

Construct an explicit mappings $X \colon \Omega \to \{0,1\}$ and $Y \colon \Omega \to \{0,1\}$ so that the bound in part a) is satisfied with equality.

d) Let $\mathbb{P}(\{X = 1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ and $Y \sim \mathcal{N}(0,1)$. What is $d_{TV}(X,Y)$?

e) Let $(X_n, n \geq 1)$ be a sequence of random variables and X be another random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Show that if $\lim_{n\to\infty} d_{TV}(X_n, X) = 0$ then $X_n \underset{n\to\infty}{\overset{d}{\to}} X$.

f) Is the converse true? That is, if $X_n \underset{n\to\infty}{\overset{d}{\to}} X$ then $\lim_{n\to\infty} d_{TV}(X_n, X) = 0$. If yes, prove the statement. If no, provide a counter example.

Exercise 3. (22 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X, Y : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable random variables on this space. For this problem, you may wish to recall Hölder's inequality from the mid-term exam:

$$\mathbb{E}(|XY|) \leq (\mathbb{E}(|X|^p))^{1/p} (\mathbb{E}(|Y|^q))^{1/q}$$

for $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

a) (Lyapunov's inequality) Show that for every $1 \leq r < s < \infty$ we have

$$\mathbb{E}(|X|^r))^{1/r} \leq \mathbb{E}(|X|^s))^{1/s}.$$

Recall that $X_n \underset{n\to\infty}{\overset{L^r}{\to}} X$ if $\mathbb{E}(|X_n^r|) < \infty$ for all n and

$$\mathbb{E}(|X_n - X|^r) \underset{n\to\infty}{\to} 0.$$

b) Assume $X_n \underset{n\to\infty}{\overset{L^s}{\to}} X$. Show that $X_n \underset{n\to\infty}{\overset{L^r}{\to}} X$ for every $1 \leq r < s < \infty$.

c) (Minkowski's inequality) Show that for every $r \geq 1$ and any two $\mathcal{F}$-measurable random variables X and Y such that $\mathbb{E}(|X|^r), \mathbb{E}(|Y|^r) < \infty$, we have

$$(\mathbb{E}(|X+Y|^r))^{1/r} \leq (\mathbb{E}(|X|^r))^{1/r} + (\mathbb{E}(|Y|^r))^{1/r}.$$

Hint: Start by writing $|X+Y|^r = |X+Y||X+Y|^{r-1} \leq |X||X+Y|^{r-1} + |Y||X+Y|^{r-1}$. *Then, apply Hölder's inequality .*

d) Assume that $X_n \underset{n\to\infty}{\overset{L^r}{\to}} X$ and $Y_n \underset{n\to\infty}{\overset{L^r}{\to}} Y$ for some $r \geq 1$. Show that $X_n + Y_n \underset{n\to\infty}{\overset{L^r}{\to}} X + Y$.

Exercise 4. (26 points)

Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence such that $\mathbb{E}(|X_1|) < \infty$. Let $\{\mathcal{F}_n, n \in \mathbb{N}\}$ be the natural filtration and τ be a stopping time with respect to this filtration.

a) (Wald's) Given $\mathbb{E}(\tau) < \infty$, show that

$$\mathbb{E}\left(\sum_{n=1}^{\tau} X_n\right) = \mathbb{E}(\tau)\mathbb{E}(X_1).$$

For the remaining parts of the problem, assume that $\{X_n\}$ is uniformly distributed over the discrete alphabet $\{1, 2, \ldots, k\}$.

b) Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$ for $n \geq 1$. Is the process $\{S_n, n \in \mathbb{N}\}$ a martingale with respect to $\{\mathcal{F}_n, n \in \mathbb{N}\}$? Why?

If it is a martingale, set $M_n = S_n$ for all $n \in \mathbb{N}$. If not, find the unique predictable and increasing process $\{A_n, n \in \mathbb{N}\}$ such that the process $\{M_n = S_n - A_n, n \in \mathbb{N}\}$ is a martingale with respect to $\{\mathcal{F}_n, n \in \mathbb{N}\}$.

c) Let $\tau_a = \min\{n \geq 1 : X_n = a\}$ for a fixed $a \in \{1, 2, \ldots, k\}$. Is τ_a a stopping time? Explain why.

d) Find $\mathbb{E}(\tau_a)$. Does it depend on a?

e) Using the previous parts, calculate $\mathbb{E}(S_{\tau_a})$ and $\mathbb{E}(M_{\tau_a})$.

f) Let $\tau_b = \inf\{n \geq 1 : |M_n| \geq b\}$ for a fixed $b \in \{k, k+1, \ldots, 3k\}$. Is τ_b a stopping time? Calculate $\mathbb{E}(M_{\tau_b})$.