Solution 9

Introduction to Quantum Information Processing

Exercise 1 *Partial traces*

a) Recall that $\rho_{ij;kl} = \langle ij| \rho |kl\rangle$. Then starting from (2) we have:

$$\begin{aligned}
\rho_B &= \sum_{i,j=0}^{d_B-1} \sum_{m=0}^{d_A-1} (\rho_{AB})_{mi;mj} |i\rangle \langle j|_B \\
&= \sum_{i,j=0}^{d_B-1} \sum_{m=0}^{d_A-1} |i\rangle_B (\rho_{AB})_{mi;mj} \langle j|_B \\
&= \sum_{i,j=0}^{d_B-1} \sum_{m=0}^{d_A-1} \Big(\langle m|_A \otimes \big(|i\rangle \langle i|_B \big)\Big) \rho_{AB} \Big(|m\rangle_A \otimes \big(|j\rangle \langle j|_B \big)\Big) \\
&= \sum_{m=0}^{d_A-1} \big(\langle m|_A \otimes \mathbf{I}_B \big) \rho_{AB} \big(|m\rangle_A \otimes \mathbf{I}_B \big)
\end{aligned}$$

since $\sum_{i=0}^{d_B-1} |i\rangle \langle i|_B = \mathbf{I}_B$. This is exactly (1).

b) Using 2 :

$$\begin{aligned}
\rho_B =& \big(\langle 0|_A \otimes \mathbf{I}_B \big) \big(|0\rangle \langle 0|_A \otimes |+\rangle \langle +|_B \big) \big(|0\rangle_A \otimes \mathbf{I}_B \big) \\
&+ \big(\langle 1|_A \otimes \mathbf{I}_B \big) \big(|0\rangle \langle 0|_A \otimes |+\rangle \langle +|_B \big) \big(|1\rangle_A \otimes \mathbf{I}_B \big) \\
=& \big(\langle 0|0\rangle_A \langle 0|0\rangle_A \big) \otimes \big(\mathbf{I}_B |+\rangle \langle +|_B \mathbf{I}_B \big) \\
&+ \big(\langle 1|0\rangle_A \langle 0|1\rangle_A \big) \otimes \big(\mathbf{I}_B |+\rangle \langle +|_B \mathbf{I}_B \big) \\
=& |+\rangle \langle +|_B
\end{aligned}$$

Exercise 2 *W states, reduced density matrix, entropy*

a)

$$\begin{aligned}
|W_\theta\rangle \langle W_\theta| =& \frac{\cos^2\theta}{2} |100\rangle \langle 100| + \frac{\cos^2\theta}{2} |010\rangle \langle 010| + \sin^2\theta |001\rangle \langle 001| \\
&+ \frac{\cos^2\theta}{2} \big(|100\rangle \langle 010| + |010\rangle \langle 100| \big) + \frac{\cos\theta \sin\theta}{\sqrt{2}} \big(|100\rangle \langle 001| + |001\rangle \langle 100| \big) \\
&+ \frac{\cos\theta \sin\theta}{\sqrt{2}} \big(|010\rangle \langle 001| + |001\rangle \langle 010| \big)
\end{aligned}$$

So we find (using cyclicity of trace and inner product for AB system)

$$\rho_C = \mathrm{Tr}_{AB}[|W_\theta\rangle \langle W_\theta|] = \cos^2\theta |0\rangle \langle 0| + \sin\theta^2 |1\rangle \langle 1| \tag{1}$$

And (using cyclicity of trace and inner product for C system)

$$\rho_{AB} = \text{Tr}_C[|W\rangle\langle W|] = (\cos\theta)^2 (|10\rangle\langle 10| + |10\rangle\langle 01|) \tag{2}$$
$$+ (\cos\theta)^2 (|01\rangle\langle 10| + |01\rangle\langle 01|) \tag{3}$$
$$+ (\sin\theta)^2 |00\rangle\langle 00| \tag{4}$$
$$= (\cos\theta)^2 |\beta_{01}\rangle\langle\beta_{01}| + (\sin\theta)^2 |00\rangle\langle 00| \tag{5}$$

b) ρ_{AB} is a matrix of size 4×4 while ρ_C is of size 2×2. They are both rank 2 with non-zero eigenvalues $(\cos\theta)^2$ and $(\sin\theta)^2$ (notice that $|\beta_{01}\rangle$ is orthonormal with $|00\rangle$). Note that the matrix ρ_{AB} has two extra zero eigenvalues.

c) In both cases, the Von Neumann entropy is:

$$S = -(\cos^2\theta)\log\left(\cos^2\theta\right) - (\sin^2\theta)\log\left(\sin^2\theta\right) \tag{6}$$

d) Here we can find the condition $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$ without much calculation in the following way. By linearity and cyclicity

$$\text{Tr}\mathcal{B}\rho_{AB} = (\sin\theta)^2 \langle 00|\mathcal{B}|00\rangle + (\cos\theta)^2 \langle\beta_{01}|\mathcal{B}|\beta_{01}\rangle$$

Let us take the angles that maximize the term $\langle\beta_{01}|\mathcal{B}|\beta_{01}\rangle$ and make it equal to $2\sqrt{2}$. For the other term since it is a product state we must certainly have $\langle 00|\mathcal{B}|00\rangle \geq -2$. Thus we get for these angles:

$$\text{Tr}\mathcal{B}\rho_{AB} \geq -2(\sin\theta)^2 + 2\sqrt{2}(\cos\theta)^2$$

To check violation of the Bell inequality we impose $-2(\sin\theta)^2 + 2\sqrt{2}(\cos\theta)^2 > 2$ which gives the condition $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$.

The computation for al angles of the average of the Bell operator is done as follows.

$$\langle 00| A\otimes B|00\rangle = (\cos(\alpha)^2 - \sin(\alpha)^2)(\cos(\beta)^2 - \sin(\beta)^2) = \cos(2\alpha)\cos(2\beta) \tag{7}$$

On the other hand, notice: $|\beta_{01}\rangle = (X\otimes I)|\beta_{00}\rangle$ So in fact, with $\tilde{A} = XAX$ we have:

$$\langle\beta_{01}| A\otimes B|\beta_{01}\rangle = \langle\beta_{00}|\tilde{A}\otimes B|\beta_{00}\rangle \tag{8}$$

Now it can be checked that with $\tilde{\alpha} = \frac{\pi}{2} - \alpha$ we have:

$$\tilde{A} = XAX = |\tilde{\alpha}\rangle\langle\tilde{\alpha}| - |\tilde{\alpha}^\perp\rangle\langle\tilde{\alpha}^\perp| \tag{9}$$

Hence using the formula from the course:

$$\langle\beta_{01}| A\otimes B|\beta_{01}\rangle = \cos(2(\tilde{\alpha}-\beta)) = \cos(\pi - 2(\alpha+\beta)) = -\cos(2(\alpha+\beta)) \tag{10}$$

Putting things together gives a general expression for $\text{Tr}\mathcal{B}\rho_{AB}$ in terms of θ, α, β, α', β' which however is not easily optimized (if one would like to find angles that maximize it for given θ).

For the last question: note that for a density matrix of the form $\rho_A \otimes \rho_B$ the locality assumption is true i.e. $p(a,b|\alpha,\beta) = p(a|\alpha)p(b|\beta)$. Indeed if say A and B choose the α, β measurement basis the probablity distributions are (by the measurement principle for mixed states)

$$p(a,b|\alpha,\beta) = \langle \alpha,\beta| \rho_A \otimes \rho_B |\alpha,\beta\rangle$$

and

$$p(a|\alpha) = \langle \alpha| \rho_A |\alpha\rangle\,, \quad p(a|\alpha) = \langle \alpha| \rho_A |\alpha\rangle$$

Thus by the general theory $|\mathrm{Tr}\mathcal{B}\rho_A \otimes \rho_B| \leq 2$. Hence this is also true for any convex combination $\sum_i p_i \rho_A^i \otimes \rho_B^i$

Exercise 3 *Entropy*

a)

$$\begin{aligned}\rho_{ABC} =& |W\rangle\langle W| = \frac{1}{3}(|001\rangle + |010\rangle + |100\rangle)(\langle 001| + \langle 010| + \langle 100|)\\ =&\frac{1}{3}\Big(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100| \\ &+ |001\rangle\langle 010| + |010\rangle\langle 001| + |001\rangle\langle 100| + |100\rangle\langle 001| + |010\rangle\langle 100| + |100\rangle\langle 010| \Big)\end{aligned}$$

The eigenvalues are 1 and 0 (with multiplicity 7) and $S(\rho_{ABC}) = 0$ since $|W\rangle$ is a pure state.

b)

$$\rho_A = \mathrm{Tr}_{BC}|W\rangle\langle W| = \frac{1}{3}(2\,|0\rangle\langle 0| + |1\rangle\langle 1|)$$

The eigenvalues are $\frac{2}{3}$ and $\frac{1}{3}$ and the entropy is $S(\rho_A) = -\frac{2}{3}\log\left(\frac{2}{3}\right) - \frac{1}{3}\log\left(\frac{1}{3}\right) = \frac{2}{3}(\log(3) - \log(2)) + \frac{1}{3}\log(3) = \log(3) - \frac{2}{3}$. Applying the Schmidt theorem, the eigenvalues of ρ_{BC} are $\frac{2}{3}$, $\frac{1}{3}$ and 0 (with multiplicity 2) and $S(\rho_{BC}) = S(\rho_A)$.

c) Starting with ρ^{ε}_{ABC}:
After scaling by $1-\varepsilon$ and shifting by $\frac{\varepsilon}{8}$, the eigenvalues of ρ^{ε}_{ABC} are $1-\varepsilon+\frac{\varepsilon}{8} = 1-\frac{7\varepsilon}{8}$ and $\frac{\varepsilon}{8}$(with multiplicity 7). Thus

$$\begin{aligned}S\left(\rho^{\varepsilon}_{ABC}\right) =& -\left(1-\frac{7\varepsilon}{8}\right)\log\left(1-\frac{7\varepsilon}{8}\right) - \frac{7\varepsilon}{8}\log\left(\frac{\varepsilon}{8}\right)\\ =&\left(1-\frac{7\varepsilon}{8}\right)(3-\log(8-7\varepsilon)) + \frac{7\varepsilon}{8}(3-\log(\varepsilon))\\ =&3-\left(1-\frac{7\varepsilon}{8}\right)\log(8-7\varepsilon) - \frac{7\varepsilon}{8}\log(\varepsilon)\end{aligned}$$

For $\rho^{\varepsilon}_A = \mathrm{Tr}_{BC}\,\rho^{\varepsilon}_{ABC}$,

$$\begin{aligned}\rho^{\varepsilon}_A =& \mathrm{Tr}_{BC}\,\rho^{\varepsilon}_{ABC} = \mathrm{Tr}_{BC}((1-\varepsilon)\,|W\rangle\langle W| + \frac{\varepsilon}{8}\mathbf{I})\\ =&(1-\varepsilon)\,\mathrm{Tr}_{BC}\,|W\rangle\langle W| + \frac{\varepsilon}{2}\mathbf{I}_A = (1-\varepsilon)\rho_A + \frac{\varepsilon}{2}\mathbf{I}_A\end{aligned}$$

After scaling by $1-\varepsilon$ and shifting by $\frac{\varepsilon}{2}$, the eigenvalues of ρ^{ε}_{ABC} are $(1-\varepsilon)\frac{2}{3}+\frac{\varepsilon}{2}=\frac{2}{3}-\frac{\varepsilon}{6}$ and $(1-\varepsilon)\frac{1}{3}+\frac{\varepsilon}{2}=\frac{1}{3}+\frac{\varepsilon}{6}$. Thus

$$\begin{aligned} S\left(\rho^{\varepsilon}_{A}\right) &= -\left(\frac{2}{3}-\frac{\varepsilon}{6}\right)\log\left(\frac{2}{3}-\frac{\varepsilon}{6}\right)-\left(\frac{1}{3}+\frac{\varepsilon}{6}\right)\log\left(\frac{1}{3}+\frac{\varepsilon}{6}\right) \\ &= \left(\frac{2}{3}-\frac{\varepsilon}{6}\right)\left(\log\left(6\right)-\log\left(4-\varepsilon\right)\right)+\left(\frac{1}{3}+\frac{\varepsilon}{6}\right)\left(\log\left(6\right)-\log\left(2+\varepsilon\right)\right) \\ &= \log\left(6\right)-\left(\frac{2}{3}-\frac{\varepsilon}{6}\right)\log\left(4-\varepsilon\right)-\left(\frac{1}{3}+\frac{\varepsilon}{6}\right)\log\left(2+\varepsilon\right) \end{aligned}$$

For $\rho^{\varepsilon}_{BC}=\mathrm{Tr}_A\,\rho^{\varepsilon}_{ABC}$,

$$\begin{aligned} \rho^{\varepsilon}_{BC} = \mathrm{Tr}_A\,\rho^{\varepsilon}_{ABC} &= \mathrm{Tr}_A((1-\varepsilon)\left|W\right\rangle\left\langle W\right|+\frac{\varepsilon}{8}\mathbf{I}) \\ =&(1-\varepsilon)\,\mathrm{Tr}_A\left|W\right\rangle\left\langle W\right|+\frac{\varepsilon}{4}\mathbf{I}_{BC}=(1-\varepsilon)\rho_{BC}+\frac{\varepsilon}{2}\mathbf{I}_{BC} \end{aligned}$$

After scaling by $1-\varepsilon$ and shifting by $\frac{\varepsilon}{4}$, the eigenvalues of ρ^{ε}_{ABC} are $(1-\varepsilon)\frac{2}{3}+\frac{\varepsilon}{4}=\frac{2}{3}-\frac{5\varepsilon}{12}$, $(1-\varepsilon)\frac{1}{3}+\frac{\varepsilon}{4}=\frac{1}{3}-\frac{\varepsilon}{12}$ and $\frac{\varepsilon}{4}$ with multiplicity 2.

$$\begin{aligned} S\left(\rho^{\varepsilon}_{BC}\right) &= -\left(\frac{2}{3}-\frac{5\varepsilon}{12}\right)\log\left(\frac{2}{3}-\frac{5\varepsilon}{12}\right)-\left(\frac{1}{3}-\frac{\varepsilon}{12}\right)\log\left(\frac{1}{3}-\frac{\varepsilon}{12}\right)-\frac{2\varepsilon}{4}\log\left(\frac{\varepsilon}{4}\right) \\ &= \left(\frac{2}{3}-\frac{5\varepsilon}{12}\right)\left(\log\left(12\right)-\log\left(8-5\varepsilon\right)\right)+\left(\frac{1}{3}-\frac{\varepsilon}{12}\right)\left(\log\left(12\right)-\log\left(4-\varepsilon\right)\right)-\frac{2\varepsilon}{4}\log\left(3\varepsilon\right) \\ &= \log\left(12\right)-\left(\frac{2}{3}-\frac{5\varepsilon}{12}\right)\log\left(8-5\varepsilon\right)-\left(\frac{1}{3}-\frac{\varepsilon}{12}\right)\log\left(4-\varepsilon\right)-\frac{2\varepsilon}{4}\log\left(3\varepsilon\right) \end{aligned}$$