Solution 9
Introduction to Quantum Information Processing

Exercise 1 *Dynamics of one-qubit density matrix*

(a) From Homework 5 we can write down with $\alpha_t^2 + \beta_t^2 = 1$ and $n_x^2 + n_z^2 = 1$:

$$U_t = \alpha I + \beta \left(n_x \sigma_x + n_z \sigma_z \right) \tag{1}$$

Now we can compute:

$$\rho_t = U_t \rho_0 U_t^\dagger \tag{2}$$

After a (long) calculation, one finds:

$$a_x(t) = a_x(0) \left(\alpha^2 + \beta^2 n_x^2 - \beta^2 n_z^2 \right) - 2 a_y(0) \alpha \beta n_z + 2 a_z(0) \beta^2 n_x n_z \tag{3}$$
$$a_y(t) = 2 a_x(0) \alpha \beta n_z + a_y(0) \left(\alpha^2 - \beta^2 n_x^2 - \beta^2 n_z^2 \right) - 2 a_z(0) \alpha \beta n_x \tag{4}$$
$$a_z(t) = 2 a_x(0) \beta^2 n_x n_z + 2 a_y(0) \alpha \beta n_x + a_z(0) \left(\alpha^2 - \beta^2 n_x^2 + \beta^2 n_z^2 \right) \tag{5}$$

(b) One can check this after a long calculation using the previous formulas

(c) It suffices to notice that $1 - \|a_t\|^2 = \det(\rho_t) = \det(\rho) = 1 - \|a\|^2$

Exercise 2 *The difference between a Bell state and a statistical mixture of* $|00\rangle$, $|11\rangle$

a) For the Bell state the density matrix is simply

$$\rho_{\text{Bell}} = |B_{00}\rangle\langle B_{00}| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

In array form

$$\rho_{\text{Bell}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank one matrix as it should since ρ_{Bell} is a rank one projector with eigenvalues 1 and $0, 0, 0$. Note also that teh criterion for a pure state is satisfied: $\rho_{\text{Bell}}^2 = \rho_{\text{Bell}}$. Sanity check $Tr \rho_{\text{Bell}} = 1$.

b) For the statistical mixture we have

$$\rho_{\text{stat}} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

In array form

$$\rho_{\text{stat}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank two matrix with eigenvalues $1, 0, 0, 1$. Note also that $\rho_{\text{stat}}^2 \neq \rho_{\text{stat}}$. Sanity check: $Tr\rho_{\text{stat}} = 1$.

c) In the Bell state the average of the observable $\mathcal{B}$ is

$$Tr(\mathcal{B}\rho_{\text{Bell}}) = Tr(\mathcal{B}|B_{00}\rangle\langle B_{00}|) = Tr\langle B_{00}|\mathcal{B}|B_{00}\rangle = \langle B_{00}|\mathcal{B}|B_{00}\rangle$$

The expression as a function of angles is calulated in the course

$$\cos 2(\alpha - \beta) + \cos 2(\alpha - \beta') - \cos 2(\alpha' - \beta) + \cos 2(\alpha' - \beta')$$

and for the optimal choice of angles the values is $2\sqrt{2}$.

In the statistical state we have by linearity and cyclicity of the trace

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \frac{1}{2}\langle 00|\mathcal{B}|00\rangle + \frac{1}{2}\langle 11|\mathcal{B}|11\rangle$$

For $A \otimes B$ we get the contribution

$$\frac{1}{2}\langle 0|A|0\rangle\langle 0|B|0\rangle + \frac{1}{2}\langle 1|A|1\rangle\langle 1|B|1\rangle = (\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) = \cos 2\alpha \cos 2\beta$$

So for the correlation coefficient we have

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \cos 2\alpha \cos 2\beta + \cos 2\alpha \cos 2\beta' - \cos 2\alpha' \cos 2\beta + \cos 2\alpha' \cos 2\beta'$$

For the optimal angles of CSHS we find $\sqrt{2}$. Note that it is possible to prove this expression can never be greater than 2.

Exercise 3 *Decoherence model*

(a) The global state of the system is:

$$|\psi_0\rangle = |\mathcal{E}\rangle \otimes |\phi_0\rangle$$

(b) First of all:

$$U^\dagger = \left(\sum_{i=1}^{\infty} |i\rangle\langle i| \otimes \mathcal{D}(\theta_i)\right)^\dagger = \sum_{i=1}^{\infty} |i\rangle\langle i| \otimes \mathcal{D}(\theta_i)^\dagger$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^\dagger U = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i)$$

Now it is easy to show that $\mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i) = I$ so that $U^\dagger U = I$
(c) We have:

$$\mathcal{D}(\theta_i)^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_i} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{ni\theta_i} \end{pmatrix} = \mathcal{D}(n\theta_i)$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^n = \left(\sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i) \right)^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes (\mathcal{D}(\theta_i))^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(n\theta_i)$$

Finally, because $\mathcal{P}(|\mathcal{E}\rangle \to |i\rangle) = |\langle i|\mathcal{E}\rangle|^2 = \mu(\theta_i)^2$ then $\langle i|\mathcal{E}\rangle = \sqrt{\mu(\theta_i)} e^{i \arg\langle i|\mathcal{E}\rangle}$ and thus we have:

$$|\psi_n\rangle = \sum_{i=1}^{\infty} |i\rangle \langle i|\mathcal{E}\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle = \sum_{i=1}^{\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle$$

(d) Using question (a) we find:

$$\rho_0 = |\phi_0\rangle \langle \phi_0| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

The Von Neumann entropy is $S_0 = 0$ as this is a rank-one matrix with one eigenvalue (=1).
(e) We find:

$$\mathcal{D}(\theta) |\phi_0\rangle = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix}$$

Thus:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix} \begin{pmatrix} \alpha^* & e^{-i\theta}\beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & e^{-i\theta}\alpha\beta^* \\ e^{i\theta}\alpha^*\beta & |\beta|^2 \end{pmatrix}$$

(f) Using question (c) we have:

$$|\psi_n\rangle \langle \psi_n| = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} e^{i \arg\langle \mathcal{E}|\theta_i|\mathcal{E}|\theta_i\rangle - i \arg\langle \mathcal{E}|\theta_j|\mathcal{E}|\theta_j\rangle} \sqrt{\mu(\theta_i)\mu(\theta_j)}\, |j\rangle \langle i| \otimes \mathcal{D}(n\theta_j) |\phi_0\rangle \langle \phi_0| \mathcal{D}(n\theta_i)^\dagger$$

Therefore, using (e):

$$\rho_n = \sum_{i=1}^{\infty} \mu(\theta_i)\mathcal{D}(n\theta_i)\rho_0\mathcal{D}(n\theta_i)^\dagger \tag{6}$$

$$= \begin{pmatrix} \sum_{i=1}^{\infty} \mu(\theta_i)|\alpha|^2 & \sum_{i=1}^{\infty} \mu(\theta_i)\alpha\beta^* e^{-in\theta} \\ \sum_{i=1}^{\infty} \mu(\theta_i)\alpha\beta^* e^{in\theta} & \sum_{i=1}^{\infty} \mu(\theta_i)|\beta|^2 \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} \mathbb{E}_{\hat{\theta}}[|\alpha|^2] & \mathbb{E}_{\hat{\theta}}[\alpha\beta^* e^{-in\hat{\theta}}] \\ \mathbb{E}_{\hat{\theta}}[\alpha\beta^* e^{in\hat{\theta}}] & \mathbb{E}_{\hat{\theta}}[|\beta|^2] \end{pmatrix} \tag{8}$$

(g) This is a direct application of the MGF of $\hat{\theta}$. The limit is thus:

$$\rho_\infty = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{9}$$

(h) The entropy initially is $S_0 = 0$ since ρ_0 is in fact pure (or a rank one matrix) and increases to attain its maximum $S_\infty = -|\alpha|^2 \ln |\alpha|^2 - |\beta|^2 \ln |\beta|^2$.