Homework 9
Introduction to Quantum Information Processing

Exercise 1 *Partial traces*

a) Prove the equivalence between these two definitions of the partial trace for a bipartite system $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$.

$$\rho_B = \sum_{i=0}^{d_A-1} (\langle i|_A \otimes \mathbf{I}_B)\rho_{AB}(|i\rangle_A \otimes \mathbf{I}_B) \tag{1}$$

$$\rho_B = \sum_{i,j=0}^{d_B-1} \sum_{m=0}^{d_A-1} (\rho_{AB})_{mi;mj} |i\rangle_B \langle j|_B \tag{2}$$

b) Apply the formula you prefer to check that the partial trace over A of the two qubits state $\rho_{AB} = |0\rangle\langle 0|_A \otimes |+\rangle\langle +|_B$ gives $|+\rangle\langle +|$.

Exercise 2 *W states, reduced density matrices, and von Neumann entropy.*

The W_θ state is defined here as

$$|W_\theta\rangle = \frac{\cos\theta}{\sqrt{2}}|100\rangle + \frac{\cos\theta}{\sqrt{2}}|010\rangle + \sin\theta|001\rangle$$

This and similar ones play an important role in quantum communication protocols. In this exercise we look at a few of its properties in order to illustrate the concept of reduced density matrix, partial trace, and von Neumann entropy. In what follows we assume that Alice, Bob and Charlie each have a one-qubit share of the state.

a) Compute the reduced density matrices ρ_C and ρ_{AB}.

b) What is the dimension of these matrices, their eigenvectors and corresponding eigenvalues of each density matrix ? Check that your results are consistent with the Schmidt theorem.

c) Compute the von Neumann entropy associated to the AB and C systems.

Now we want to show that the AB system is "Bell-non-local" in the sense that ρ_{AB} violates the Bell inequality if θ is small enough.

d) Compute the average value of the Bell operator $\mathcal{B} = A\otimes B + A'\otimes B - A\otimes B' + A'\otimes B'$. Give the expression in terms of general angles α, β, α' and β'. Show that if $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$ the Bell inequality is *certainly* violated.
Hint: this is not the best possible condition and can be obtained by a two line simple calculation (without carrying out any detailed optimization of the average value of the Bell operator).

Now imagine that a general bipartite system has density matrix that can be written as a convex combination of product density matrices, i.e., $\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}$. Such states are called "separable". This is the generalization of the notion of product states introduced for pure state vectors. Mixed states that are not separable are called "entangled".

e) Prove that for separable states we have $|\mathrm{Tr}\rho_{AB}\mathcal{B}| \leq 2$.

f) From the above questions what can you conclude about ρ_{AB} when $(\sin\theta)^2 < \frac{\sqrt{2}-1}{\sqrt{2}+1}$?

Remarks: For pure states (vector states) there are two situations: product states satisfy the CHSH inequality and entangled states (i.e. non-product vector states) violate the CHSH inequality. For density matrices the situation is richer. Separable states satisfy the CHSH inequality, but for non-separable states the inequality may be violated or not. When it is violated the state is said to be "Bell-non-local" which is a strong form of entanglement. Non-separable states that do not violate the CHSH inequality are still entangled but this is a weaker form of entanglement (for example you cannot use them for a simple Ekert-like protocol). With the results of this exercise you can check that standard W state has $\sin\theta = \frac{\cos\theta}{\sqrt{2}} = \frac{1}{\sqrt{3}}$ leads to a ρ_{AB} that can be shown to not violate the CHSH inequality but is non-separable (entangled).

Exercise 3 *Entropy*

We study the three qubits state

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{3}$$

a) Consider $\rho_{ABC} = |W\rangle\langle W|$. What are its eigenvalues and entropy $S(\rho_{ABC})$?

b) Compute $\rho_A = \mathrm{Tr}_{BC}|W\rangle\langle W|$. Find the eigenvalues and entropy $S(\rho_A)$. Deduce the eigenvalues and entropy $S(\rho_{BC})$ of $\rho_{BC} = \mathrm{Tr}_A|W\rangle\langle W|$.

We now consider the noisy state

$$\rho_{ABC}^{\varepsilon} = (1-\varepsilon)|W\rangle\langle W| + \frac{\varepsilon}{8}\mathbf{I}\otimes\mathbf{I}\otimes\mathbf{I}, \qquad 0 \leq \epsilon \leq 1. \tag{4}$$

where $\mathbf{I}$ is the 2×2 identity.

c) Deduce from **1)** and **2)** the eigenvalues and entropies of ρ_{ABC}^{ε}, of $\rho_A^{\varepsilon} = \mathrm{Tr}_{BC}\,\rho_{ABC}^{\varepsilon}$, and of $\rho_{BC}^{\varepsilon} = \mathrm{Tr}_A\,\rho_{ABC}^{\varepsilon}$. Be careful to the multiplicities.