Final exam

Exercise 1. Quiz. (25 points) Answer each short question below. For yes/no questions explicitly say if the statement is true of false and provide a brief justification (proof or counter-example) for your answer. For other questions, provide the result of you computation, as well as a brief justification for your answer.

a) Let $\Omega = \{1, 2, \dots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 3, 5\}\}$. Let $\mathcal{F} = \sigma(\mathcal{A})$ be the σ-field generated by $\mathcal{A}$. What are the atoms of $\mathcal{F}$?

b) Let $\Omega = [0, 1]^2$, $\mathcal{F} = \mathcal{B}([0, 1]^2)$, and $\mathbb{P}$ be the probability measure on $(\Omega, \mathcal{F})$ defined as

$$\mathbb{P}\left(]a, b[\times]c, d[\right) = (b - a) \cdot (d - c), \text{ for } 0 \leq a < b \leq 1 \text{ and } 0 \leq c < d \leq 1$$

which can be extended uniquely to all Borel sets in $\mathcal{B}([0, 1]^2)$, according to Caratheodory's extension theorem. Let us now consider the following random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$:

$$X(\omega_1, \omega_2) = \frac{\omega_1 - \omega_2}{2}.$$

Compute the cdf F_X of X.

c) Let X be a random variable supported on $\{-1, 1\}$ with $\mathbb{P}(\{X = 1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$. Let $Z \sim \mathcal{N}(0, 1)$ and assume that X and Z are independent. Then, is (XZ, Z) a Gaussian random vector?

d) Let X and Z be as in part (c). Then, is (XZ, Z) a continuous random vector?

e) Let X and Y be integrable random variables. If $Y = g(X)$ for some measurable function $g : \mathbb{R} \to \mathbb{R}$, then is it true that $\mathbb{E}(X|Y) = h(X)$ for some function $h : \mathbb{R} \to \mathbb{R}$?

f) Let X and Y be two independent Bernoulli random variables with parameter $0 \leq p \leq 1$. Let Z be defined as

$$Z = \begin{cases} 1, & \text{if } X + Y = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Are $\mathbb{E}(X|Z)$ and $\mathbb{E}(Y|Z)$ independent?

g) Let $(S_n, n \in \mathbb{N})$ be the simple symmetric random walk and let $(\mathcal{F}_n, n \in \mathbb{N})$ be its natural filtration. Define a random time

$$T = \inf\{n \colon S_n = S_{n-2}, n \geq 2\}.$$

Is T a stopping time?

Exercise 2. (15 points)

Let X and Y be random variables defined on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$d(X,Y) = \mathbb{E}\left(\log_2\left(1 + \frac{|X-Y|}{1+|X-Y|}\right)\right).$$

a) First, we would like to confirm that $d(X,Y)$ is a distance metric. Show that $d(X,Y)$ satisfies the triangle inequality. That is, $d(X,Z) \leq d(X,Y) + d(Y,Z)$ for any X, Y, and Z.

Hint: the function $f(x) = \log_2(1+x)$ *is sub-additive, e.g.* $f(x+y) \leq f(x) + f(y)$.

Next, we would like to check if convergence with respect to $d(X,Y)$ is equivalent to convergence in probability (a distance metric with this property is sometimes called a Ky-Fan metric).

b) Let $(X_n, n \geq 1)$ be sequence of random variables and X be another random variable, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that if $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} X$ then $\lim_{n\to\infty} d(X_n, X) = 0$.

c) Is the converse true? That is, if $\lim_{n\to\infty} d(X_n, X) = 0$ then $X_n \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} X$. If yes, prove the statement. If no, provide a counter example.

Exercise 3. (25 points)

Recall that the moment-generating function of a random variable X is defined for every $t \in \mathbb{R}$ as

$$M_X(t) = \mathbb{E}\left(e^{tX}\right).$$

a) Show that if $X \sim \mathcal{N}(0, \sigma^2)$, then

$$M_X(t) = \exp\left(\frac{1}{2}t^2\sigma^2\right).$$

We now introduce the concept of *sub-gaussianity.* A random variable X is called sub-gaussian if, for every $t > 0$,

$$M_X(t) \leq \exp\left(\frac{1}{2}t^2\eta^2\right)$$

for some $\eta \in \mathbb{R}^+$. (Note that η^2 need not be the variance of X!).

b) Show that if $X \sim \mathcal{U}([-a,a])$ for some $a > 0$, then X is sub-gaussian with $\eta = a$.

Hint: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

c) Show that if X is sub-gaussian for some $\eta \in \mathbb{R}^+$, then for every $t > 0$,

$$\mathbb{P}(|X| \geq t) \leq 2\exp\left(-\frac{t^2}{2\eta^2}\right).$$

d) Prove the following generalization of Hoeffding's inequality. Let $X_i, i \in \{1, 2, \dots, n\}$ be independent random variables, where for each i, $X_i - \mathbb{E}(X_i)$ is sub-gaussian for some $\eta_i \in \mathbb{R}^+$. Let also $S_n = \sum_{i=1}^n X_i$. Show that for every $t > 0$,

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq t) \leq 2 \exp\left(-\frac{t^2}{2\sum_{i=1}^n \eta_i^2}\right).$$

e) Let $X_i, i \in \{1, 2, \dots, n\}$ be sub-gaussian random variables with the same $\eta \in \mathbb{R}^+$. Show that

$$\mathbb{E}\left(\max_i X_i\right) \leq \eta\sqrt{2 \ln n}.$$

Hint: Start by rewriting $\mathbb{E}\left(\max_i X_i\right) = \frac{1}{t}\mathbb{E}\left(\ln \exp\left(t \max_i X_i\right)\right)$.

Exercise 4. (25 points)

a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_n, n \in \mathbb{N}\}$ be a filtration on this space. Let $A \in \mathcal{F}$ and define $Y_n = \mathbb{E}\left(1_A | \mathcal{F}_n\right)$. Show that $(Y_n, n \in \mathbb{N})$ is a martingale with respect to the filtration $\{\mathcal{F}_n, n \in \mathbb{N}\}$.

b) Is it true that

$$Y_n \to Y_\infty, \text{ a.s.}$$

for some random variable Y_∞? Why or why not? Could we say something about convergence in distribution to Y_∞?

Next, we will use this martingale to prove Kolmogorov's zero-one law. Let $X_0, X_1, \dots$ be independent random variables. Recall that the tail σ-field is

$$\mathcal{T} = \bigcap_{n=0}^{\infty} \mathcal{H}_n$$

where $\mathcal{H}_n = \sigma(X_n, X_{n+1}, \dots)$ and assume $A \in \mathcal{T}$. Our goal will be to prove that $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

c) Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ and $\mathcal{F}_\infty$ be the smallest σ-field that contains every $\mathcal{F}_n$. A standard measure-theoretic argument could be used to show that $Y_\infty = \mathbb{E}\left(1_A | \mathcal{F}_\infty\right)$, but we will take it as a fact here.

Assume $Y_\infty = \mathbb{E}\left(1_A | \mathcal{F}_\infty\right)$. Show, furthermore, that for all $A \in \mathcal{T}$,

$$Y_\infty := \mathbb{E}\left(1_A | \mathcal{F}_\infty\right) = 1_A.$$

d) Show that

$$Y_n := \mathbb{E}\left(1_A | \mathcal{F}_n\right) = \mathbb{P}(A).$$

Hint: How are the σ-fields $\mathcal{T}$ and $\mathcal{F}_n$ related to each other?

e) Combine the ingredients above to prove Kolmogorov's zero-one law.