Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 14 – Solutions

Exercise 1: Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ on N, we have

$$F^*\left(\sum_I{}' \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F) \, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Solution:

(a) Let $\omega, \eta \in \Omega^k(N)$ and $\lambda, \mu \in \mathbb{R}$. Fix $p \in M$ and let $v_1, \ldots, v_k \in T_pM$. We have

$$\begin{aligned}
\big(F^*(\lambda\omega + \mu\eta)\big)_p(v_1, \ldots, v_k) &= (\lambda\omega + \mu\eta)_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda\, \omega_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) + \mu\, \eta_p\big(dF_p(v_1), \ldots, dF_p(v_k)\big) \\
&= \lambda(F^*\omega)_p(v_1, \ldots, v_k) + \mu(F^*\eta)_p(v_1, \ldots, v_k) \\
&= \big(\lambda(F^*\omega)_p + \mu(F^*\eta)_p\big)(v_1, \ldots, v_k),
\end{aligned}$$

which implies that

$$\big(F^*(\lambda\omega + \mu\eta)\big)_p = \lambda(F^*\omega)_p + \mu(F^*\eta)_p,$$

and whence $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) Assume that ω resp. η is a k- resp. l-covector. Fix $p \in M$ and let $v_1, \ldots, v_{k+l} \in T_pM$. We have

$$\begin{aligned}
F^*(\omega \wedge \eta)_p\,(v_1, \ldots, v_{k+l}) &= (\omega \wedge \eta)_{F(p)}\big(dF_p(v_1), \ldots, dF_p(v_{k+l})\big) \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}\sigma)\, \omega\left(dF_p\big(v_{\sigma(1)}\big), \ldots, dF_p\big(v_{\sigma(k)}\big)\right) \eta\left(dF_p\big(v_{\sigma(k+1)}\big), \ldots, dF_p\big(v_{\sigma(k+l)}\big)\right)
\end{aligned}$$

and

$$
\begin{aligned}
&[(F^*\omega) \wedge (F^*\eta)]_p (v_1, \dots, v_{k+l}) = \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) (F^*\omega)_p \left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) (F^*\eta)_p \left(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}\right) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \omega \left(dF_p\big(v_{\sigma(1)}\big), \dots, dF_p\big(v_{\sigma(k)}\big)\right) \eta \left(dF_p\big(v_{\sigma(k+1)}\big), \dots, dF_p\big(v_{\sigma(k+l)}\big)\right).
\end{aligned}
$$

As the two expressions agree, we conclude that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) The assertion follows immediately from (a), (b) and *Proposition 8.13*.

Exercise 2:

(a) Let V be a finite-dimensional real vector space and let $\omega^1, \dots, \omega^k \in V^*$. Show that the covectors $\omega^1, \dots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \dots \wedge \omega^k = 0$.

(b) Let M be a smooth n-manifold. Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. Assume that ω is closed and that η is exact. Show that $\omega \wedge \eta$ is closed and exact.

(c) Consider the smooth manifolds

$$M = \left\{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\right\} \qquad \text{and} \qquad N = \mathbb{R}^3 \setminus \{0\},$$

the smooth map

$$F\colon M \to N,\ (u, v) \mapsto \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

and the differential forms

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x\, dx + y\, dy + z\, dz}{(x^2 + y^2 + z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.
(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.
(iii) Compute $F^*\omega$ and $F^*(d\eta)$.
(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.

Solution:

(a) Assume first that the covectors $\omega^1, \dots, \omega^k$ are linearly dependent. Then there exist $j \in \{1, \dots, k\}$ and $\lambda_1, \dots, \widehat{\lambda_j}, \dots, \lambda_k \in \mathbb{R}$ such that $\omega^j = \sum_{i \neq j} \lambda_i\, \omega^i$. Therefore,

$$
\begin{aligned}
\omega^1 \wedge \dots \wedge \omega^{j-1} \wedge \omega^j \wedge \omega^{j+1} \wedge \dots \wedge \omega^k &= \omega^1 \wedge \dots \wedge \omega^{j-1} \wedge \sum_{i \neq j} \lambda_i\, \omega^i \wedge \omega^{j+1} \wedge \dots \wedge \omega^k \\
&= \sum_{i \neq j} \omega^1 \wedge \dots \wedge \omega^{j-1} \wedge \omega^i \wedge \omega^{j+1} \wedge \dots \wedge \omega^k \\
&= 0
\end{aligned}
$$

by [*Multilinear Algebra, Proposition C.25*(d)].

Assume now that the covectors $\omega^1, \dots, \omega^k$ are linearly independent. We will show below that (the alternating k-multilinear function) $\eta := \omega^1 \wedge \dots \wedge \omega^k \neq 0$. It suffices to find $v_1, \dots, v_k \in V$ such that $\eta(v_1, \dots, v_k) \neq 0$. To this end, set $n = \dim_{\mathbb{R}} V$ and note that $n \geq k$. Since $\omega^1, \dots, \omega^k$ are linearly independent elements of V^*, we can complete them to a basis $\{\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n\}$ of V^*, and consider subsequently the basis $\{v_1, \dots, v_n\}$ of V dual to $\{\omega^j\}$; see (the second paragraph after) [*Multilinear Algebra, Proposition C.5*]. By [*Multilinear Algebra, Proposition C.25*(d)] we then obtain

$$\eta(v_1, \dots, v_k) = \det\Big(\big(\omega^j(v_i)\big)\Big) = \det\big(\delta_i^j\big) = 1,$$

and thus $\eta \neq 0$, as desired.

(b) Since ω is a closed k-form, we have $d\omega = 0$. Since η is an exact ℓ-form, there exists $\theta \in \Omega^{\ell-1}(M)$ such that $\eta = d\theta$, and thus $d\eta = 0$. Hence, we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = 0;$$

in other words, $\omega \wedge \eta$ is a closed $(k+\ell)$-form. Moreover, since

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^k\, \omega \wedge d\theta = (-1)^k\, \omega \wedge \eta,$$

we infer that

$$\omega \wedge \eta = d\big((-1)^k\, \omega \wedge \theta\big);$$

in other words, $\omega \wedge \eta$ is an exact $(k+\ell)$-form.

(c)(i) Using the facts that

$$dx \wedge dx = dy \wedge dy = dz \wedge dz = 0 \tag{1}$$

and

$$dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy, \tag{2}$$

we compute that

$$\begin{aligned} d\omega &= d\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dy \wedge dz + d\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dz \wedge dx + \\ &\quad + d\left(\frac{z}{(x^2+y^2+z^2)^{3/2}}\right) \wedge dx \wedge dy \\ &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dx \wedge dy \wedge dz + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{3/2}}\right) dy \wedge dz \wedge dx + \\ &\quad + \frac{\partial}{\partial z}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) dz \wedge dx \wedge dy \\ &= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz + \\ &\quad + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{5/2}}\, dx \wedge dy \wedge dz \\ &= 0. \end{aligned}$$

Consider now the smooth function

$$f\colon N \to \mathbb{R},\ (x,y,z) \mapsto \sqrt{x^2+y^2+z^2}$$

and observe that $\eta = df$. Therefore,

$$d\eta = 0.$$

(c)(ii) Using (1) and (2) we compute that

$$\begin{aligned}\omega \wedge \eta &= \frac{x^2}{(x^2+y^2+z^2)^2}\, dy \wedge dz \wedge dx + \frac{y^2}{(x^2+y^2+z^2)^2}\, dz \wedge dx \wedge dy + \\ &\quad + \frac{z^2}{(x^2+y^2+z^2)^2}\, dx \wedge dy \wedge dz \\ &= \frac{1}{x^2+y^2+z^2}\, dx \wedge dy \wedge dz.\end{aligned}$$

Moreover, since $d\eta = 0$ by (c)(i), we infer that

$$\eta \wedge d\eta = 0.$$

(c)(iii) Since

$$du \wedge du = dv \wedge dv = 0, \quad du \wedge dv = -dv \wedge du$$

and

$$d\left(\sqrt{1-u^2-v^2}\right) = -\frac{u}{\sqrt{1-u^2-v^2}}\, du - \frac{v}{\sqrt{1-u^2-v^2}}\, dv,$$

we compute that

$$\begin{aligned}F^*\omega &= \frac{-u^2}{\sqrt{1-u^2-v^2}}\, dv \wedge du + \frac{-v^2}{\sqrt{1-u^2-v^2}}\, dv \wedge du + \sqrt{1-u^2-v^2}\, du \wedge dv \\ &= \frac{1}{\sqrt{1-u^2-v^2}}\, du \wedge dv.\end{aligned}$$

Moreover, since $d\eta = 0$ by (c)(i), we have

$$F^*(d\eta) = 0.$$

(c)(iv) Note that both $d(F^*\omega)$ and $F^*(d\omega)$ are 3-forms on the 2-dimensional manifold M. Hence they are both equal to 0; in particular, we have

$$d(F^*\omega) = F^*(d\omega) = 0.$$

Exercise 3: Let (r,θ) be polar coordinates on the right half-plane $H = \big\{(x,y) \mid x > 0\big\}$. Compute the polar coordinate expression for the smooth 1-form $x\,dy - y\,dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x,y) = (r\cos\theta, r\sin\theta)$ as the coordinate expression for the identity map of H, but using (r,θ) as coordinates for the domain and (x,y) as coordinates for the codomain.]

Solution: We have

$$\begin{aligned}\mathrm{Id}^*(x\,dy - y\,dx) &= r\cos\theta\, d(r\sin\theta) - r\sin\theta\, d(r\cos\theta) \\ &= r\cos\theta(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta(\cos\theta\, dr - r\sin\theta\, d\theta) \\ &= r^2\cos^2\theta\, d\theta + r^2\sin^2\theta\, d\theta \\ &= r^2\, d\theta\end{aligned}$$

and

$$\begin{aligned}\mathrm{Id}^*(dx\wedge dy) &= d(r\cos\theta)\wedge d(r\sin\theta) \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta) \\ &= r\cos^2\theta\, dr\wedge d\theta - r\sin^2\theta\, d\theta\wedge dr \\ &= r\, dr\wedge d\theta,\end{aligned}$$

since $dr\wedge dr = 0 = d\theta\wedge d\theta$ and $dr\wedge d\theta = -d\theta\wedge dr$.

Exercise 4: Consider the smooth 2-form

$$\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota\colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Solution:

(a) We have

$$\begin{aligned}dx &= d(\rho\sin\varphi\cos\theta) = \sin\varphi\cos\theta\ d\rho + \rho\cos\varphi\cos\theta\ d\varphi - \rho\sin\varphi\sin\theta\ d\theta, \\ dy &= d(\rho\sin\varphi\sin\theta) = \sin\varphi\sin\theta\ d\rho + \rho\cos\varphi\sin\theta\ d\varphi + \rho\sin\varphi\cos\theta\ d\theta, \\ dz &= d(\rho\cos\varphi) = \cos\varphi\ d\rho - \rho\sin\varphi\ d\varphi.\end{aligned}$$

Therefore, one computes that

$$\begin{aligned}dy\wedge dz &= \rho^2\sin^2\varphi\cos\theta\ d\varphi\wedge d\theta + \rho\sin\varphi\cos\varphi\cos\theta\ d\theta\wedge d\rho - \rho\sin\theta\ d\rho\wedge d\varphi, \\ dz\wedge dx &= \rho^2\sin^2\varphi\sin\theta\ d\varphi\wedge d\theta + \rho\sin\varphi\cos\varphi\sin\theta\ d\theta\wedge d\rho + \rho\cos\theta\ d\rho\wedge d\varphi, \\ dx\wedge dy &= \rho^2\cos\varphi\sin\varphi\ d\varphi\wedge d\theta - \rho\sin^2\varphi\ d\theta\wedge d\rho.\end{aligned}$$

By combining these expressions, we thus obtain

$$\begin{aligned}
\omega &= x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy\\
&= (\rho^3\sin^3\varphi\cos^2\theta + \rho^3\sin^3\varphi\sin^2\theta + \rho^3\cos^2\varphi\sin\varphi)\ d\varphi\wedge d\theta\\
&\quad + \underbrace{(\rho^2\sin^2\varphi\cos\varphi\cos^2\theta + \rho^2\sin^2\varphi\cos\varphi\sin^2\theta - \rho^2\sin^2\varphi\cos\varphi)}_{=0}\ d\theta\wedge d\rho\\
&\quad + \underbrace{(-\rho^2\sin\varphi\sin\theta\cos\theta + \rho^2\sin\varphi\sin\theta\cos\theta)}_{=0}\ d\rho\wedge d\varphi\\
&= \rho^3\sin\varphi\underbrace{(\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi)}_{=1}\ d\varphi\wedge d\theta\\
&= \rho^3\sin\varphi\ d\varphi\wedge d\theta.
\end{aligned}$$

(b) We have

$$d\big(\rho^3\sin\varphi\big) = 3\rho^2\sin\varphi\ d\rho + \rho^3\cos\varphi\ d\varphi,$$

so we obtain

$$d\omega = d\big(\rho^3\sin\varphi\big)\wedge d\varphi\wedge d\theta = 3\rho^2\sin\varphi\ d\rho\wedge d\varphi\wedge d\theta.$$

Another way to compute $d\omega$ would be to note that

$$d\omega = dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy = 3dx\wedge dy\wedge dz.$$

For the standard top differential form $dx\wedge dy\wedge dz$ on $\mathbb{R}^3$, a change of coordinates induces a factor given by the determinant of the Jacobian. You may remember or look up (or compute) that the determinant of the Jacobian of spherical coordinates is $\rho^2\sin\varphi$, so we obtain $d\omega = 3\rho^2\sin\varphi\ d\rho\wedge d\varphi\wedge d\theta$ as well.

(c) We just have to put $\rho = 1$ in the result of part (a). To justify precisely what is going on, let us spell this out in detail. Note that the change into spherical coordinates is provided by the diffeomorphism

$$\begin{aligned}
G\colon \mathbb{R}_{>0}\times(0,\pi)\times(0,2\pi) &\to U\subseteq\mathbb{R}^3\\
(\rho,\varphi,\theta) &\mapsto (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi),
\end{aligned}$$

where $V = \{(x,y,z)\mid y\neq 0\}$. So what we computed above is $G^*(\omega|_V)$. Note that spherical coordinates on the sphere are provided by the diffeomorphism

$$\begin{aligned}
F\colon (0,\pi)\times(0,2\pi) &\to V\subseteq\mathbb{S}^2\\
(\varphi,\theta) &\mapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),
\end{aligned}$$

where $U = \mathbb{S}^2\cap V$. If we denote by j the embedding

$$\begin{aligned}
j\colon (0,\pi)\times(0,2\pi) &\to \mathbb{R}_{>0}\times(0,\pi)\times(0,2\pi)\\
(\varphi,\theta) &\mapsto (1,\varphi,\theta),
\end{aligned}$$

then this is precisely set up so that $G\circ j = \iota\circ F$. What we want to compute is $F^*\iota^*(\omega|_V)$, and this is given by

$$\begin{aligned}
F^*\iota^*(\omega|_V) &= (\iota\circ F)^*(\omega|_V) = (G\circ j)^*(\omega|_V) = j^*G^*(\omega|_V)\\
&= j^*(\rho^3\sin\varphi\ d\varphi\wedge d\theta)\\
&= \sin\varphi\ d\varphi\wedge d\theta.
\end{aligned}$$

(d) As $\sin\varphi \neq 0$ for $\varphi \in (0,\pi)$, we infer that $F^*\iota^*(\omega|_V) = \sin\varphi \; d\varphi \wedge d\theta$ is nowhere vanishing on $(0,\pi)\times(0,2\pi)$. As F is an isomorphism, we obtain that $\iota^*(\omega|_V) = (\iota^*\omega)|_U$ is nowhere vanishing on U, i.e., at the points of $\mathbb{S}^2$ where $y \neq 0$. To conclude, note that we can do the exact same calculations for spherical coordinates around the x- and y-axes, and obtain that then $\iota^*\omega$ is non-zero also at all points where $z \neq 0$ resp. $x \neq 0$. Hence, $\iota^*\omega$ is nowhere zero.

Exercise 5:

(a) *Exterior derivative of a smooth* 1*-form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X,Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X,Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \qquad \text{and} \qquad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.

Solution:

(a) Let $p \in M$ be arbitrary. Choose local coordinates $\big(U, (x^i)\big)$ around p and write

$$\omega = \sum_i c_i\, dx^i, \qquad X = \sum_i f_i \,\frac{\partial}{\partial x^i}, \qquad Y = \sum_i g_i \,\frac{\partial}{\partial x^i}.$$

Then

$$\begin{aligned}
[d\omega(X,Y)](p) = (d\omega)_p(X_p,Y_p) &= \sum_i [(dc_i)_p \wedge (dx^i)_p](X_p,Y_p) \\
&= \sum_i \big[(dc_i)_p(X_p)(dx^i)_p(Y_p) - (dc_i)_p(Y_p)(dx^i)_p(X_p)\big] \\
&= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) - f_i(p) g_j(p) \frac{\partial c_i}{\partial x^j}(p) \right].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\big[X\big(\omega(Y)\big)\big](p) &= \sum_i [X(c_i g_i)](p) = \sum_i g_i(p)[X(c_i)](p) + c_i(p)[X(g_i)](p) \\
&= \sum_{i,j} \left[g_i(p) f_j(p) \frac{\partial c_i}{\partial x^j}(p) + c_i(p) f_j(p) \frac{\partial g_i}{\partial x^j}(p) \right]
\end{aligned}$$

and

$$\begin{aligned}\left[Y\big(\omega(X)\big)\right](p) &= \sum_i [Y(c_i f_i)](p) = \sum_i f_i(p)[Y(c_i)](p) + c_i(p)[Y(f_i)](p)\\ &= \sum_{i,j}\left[f_i(p)g_j(p)\frac{\partial c_i}{\partial x^j}(p) + c_i(p)g_j(p)\frac{\partial f_i}{\partial x^j}(p)\right]\end{aligned}$$

as well as

$$\left[\omega\big([X,Y]\big)\right](p) = \left[\sum_{i,j} c_i(p)f_j(p)\frac{\partial g_i}{\partial x^j}(p) - c_i(p)g_j(p)\frac{\partial f_i}{\partial x^j}(p)\right],$$

where we used [*Exercise Sheet* 11*, Exercise* 4(*a*)]. By combining these expressions, we obtain

$$\begin{aligned}\left[X\big(\omega(Y)\big) - Y\big(\omega(X)\big)\right](p) &= \left(\sum_{i,j} g_i(p)f_j(p)\frac{\partial c_i}{\partial x^j}(p) + c_i(p)f_j(p)\frac{\partial g_i}{\partial x^j}(p)\right)\\ &\quad - \left(\sum_{i,j} f_i(p)g_j(p)\frac{\partial c_i}{\partial x^j}(p) + c_i(p)g_j(p)\frac{\partial f_i}{\partial x^j}(p)\right)\\ &= \left(\sum_{i,j} g_i(p)f_j(p)\frac{\partial c_i}{\partial x^j}(p) - f_i(p)g_j(p)\frac{\partial c_i}{\partial x^j}(p)\right)\\ &\quad + \left(\sum_{i,j} c_i(p)f_j(p)\frac{\partial g_i}{\partial x^j}(p) - c_i(p)g_j(p)\frac{\partial f_i}{\partial x^j}(p)\right)\\ &= [d\omega(X,Y)](p) + \left[\omega\big([X,Y]\big)\right](p).\end{aligned}$$

(b) Let us compute $d\varepsilon^i(E_j,E_k)$ for some i,j,k with $j<k$. By part (a) we obtain

$$\begin{aligned}d\varepsilon^i(E_j,E_k) &= E_j\big(\varepsilon^i(E_k)\big) - E_k\big(\varepsilon^i(E_j)\big) - \varepsilon^i\big([E_j,E_k]\big)\\ &= \underbrace{E_j(\delta_{ik})}_{=0} - \underbrace{E_k(\delta_{ij})}_{=0} - c^i_{jk} = -c^i_{jk},\end{aligned}$$

where in the last step we used that a derivation evaluated at a constant function gives 0. On the other hand, we have

$$d\varepsilon^i(E_j,E_k) = \sum_{j'<k'} b^i_{j'k'}\left[\varepsilon^{j'}\wedge\varepsilon^{k'}\right](E_j,E_k) = b^i_{jk},$$

where we used that $\varepsilon^{j'}\wedge\varepsilon^{k'} = \varepsilon^{(j',k')}$; see [*Multilinear Algebra, Lemma C.20*(c) and *Proposition C.25*(c)]. Hence, $b^i_{jk} = -c^i_{jk}$.

Remark.

(1) *Exercise* 5(b) shows that the exterior derivative is in a certain sense dual to the Lie bracket. In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

(2) There is an analogue of *Exercise* 5(a) for smooth k-forms as well, which is referred to as the *invariant formula for the exterior derivative* in the literature. Specifically, if $\omega \in \Omega^k(M)$, then for any $X_1, \dots, X_k \in \mathfrak{X}(M)$ it holds that

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{1 \leq i \leq k+1} (-1)^{i-1} X_i\big(\omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\big) + \\ + \sum_{1 \leq i < j \leq k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\big),$$

where the hats indicate omitted arguments. It is worthwhile to mention that the above formula can be used to give an *invariant* definition of d, as well as an alternative proof of *Theorem 8.25* on the existence, uniqueness, and properties of d.