Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 14

Exercise 1:
Let $F\colon M \to N$ be a smooth map. Prove the following assertions:

(a) $F^*\colon \Omega^k(N) \to \Omega^k(M)$ is an $\mathbb{R}$-linear map.

(b) It holds that $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart $\big(V, (y^i)\big)$ for N, we have

$$F^*\left({\sum_I}' \omega_I dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = {\sum_I}' (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \ldots \wedge d\left(y^{i_k} \circ F\right).$$

Exercise 2 (to be submitted by Thursday, 19.12.2024, 16:00):

(a) Let V be a finite-dimensional real vector space and let $\omega^1, \ldots, \omega^k \in V^*$. Show that the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent if and only if $\omega^1 \wedge \ldots \wedge \omega^k = 0$.

(b) Let M be a smooth n-manifold. Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$. Assume that ω is closed and that η is exact. Show that $\omega \wedge \eta$ is closed and exact.

(c) Consider the smooth manifolds

$$M = \left\{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\right\} \quad \text{and} \quad N = \mathbb{R}^3 \setminus \{0\},$$

the smooth map

$$F\colon M \to N,\ (u, v) \mapsto \left(u, v, \sqrt{1 - u^2 - v^2}\right),$$

and the differential forms

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}} \in \Omega^2(N)$$

and

$$\eta = \frac{x\, dx + y\, dy + z\, dz}{(x^2 + y^2 + z^2)^{1/2}} \in \Omega^1(N).$$

(i) Compute $d\omega$ and $d\eta$.
(ii) Compute $\omega \wedge \eta$ and $\eta \wedge d\eta$.
(iii) Compute $F^*\omega$ and $F^*(d\eta)$.
(iv) Verify that $d(F^*\omega) = F^*(d\omega)$.

Exercise 3:
Let (r, θ) be polar coordinates on the right half-plane $H = \{(x, y) \mid x > 0\}$. Compute the polar coordinate expression for the smooth 1-form $x\,dy - y\,dx \in \Omega^1(\mathbb{R}^2)$ and for the smooth 2-form $dx \wedge dy \in \Omega^2(\mathbb{R}^2)$.

[Hint: Think of the change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ as the coordinate expression for the identity map of H, but using (r, θ) as coordinates for the domain and (x, y) as coordinates for the codomain.]

Exercise 4:
Consider the smooth 2-form

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

on $\mathbb{R}^3$ with standard coordinates (x, y, z).

(a) Compute ω in spherical coordinates for $\mathbb{R}^3$ defined by

$$(x, y, z) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi).$$

(b) Compute $d\omega$ in spherical coordinates.

(c) Consider the inclusion map $\iota \colon \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ and compute the pullback $\iota^*\omega$ to $\mathbb{S}^2$, using coordinates (φ, θ) on the open subset where these coordinates are defined.

(d) Show that $\iota^*\omega$ is nowhere zero.

Exercise 5:

(a) *Exterior derivative of a smooth* 1*-form*: Show that for any smooth 1-form ω and any smooth vector fields X and Y on a smooth manifold M it holds that

$$d\omega(X, Y) = X\big(\omega(Y)\big) - Y\big(\omega(X)\big) - \omega\big([X, Y]\big).$$

(b) Let M be a smooth n-manifold, let (E_i) be a smooth local frame for M and let (ε^i) be the dual coframe. For each i, denote by b^i_{jk} the component functions of the exterior derivative of ε^i in this frame, and for each j, k, denote by c^i_{jk} the component functions of the Lie bracket $[E_j, E_k]$:

$$d\varepsilon^i = \sum_{j<k} b^i_{jk}\, \varepsilon^j \wedge \varepsilon^k \quad \text{and} \quad [E_j, E_k] = c^i_{jk}\, E_i.$$

Show that $b^i_{jk} = -c^i_{jk}$.