Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 13 – Solutions

Exercise 1 (*Smoothness criteria for covector fields*)**:** Let M be a smooth manifold and let $\omega\colon M \to T^*M$ be a rough covector field on M. Prove that the following assertions are equivalent:

(a) ω is smooth.

(b) In every smooth coordinate chart the component functions of ω are smooth.

(c) Every point of M is contained in some smooth coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field X on M, the function $\omega(X)\colon M \to \mathbb{R}$ is smooth on M.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on U.

[Hint: Try proving $(a) \implies (b) \implies (c) \implies (a)$ and $(c) \implies (d) \implies (e) \implies (b)$.]

Solution:

(a) $\implies$ (b): Suppose that ω is smooth. Let $\big(U, (x^i)\big)$ be a smooth chart for M. This gives a corresponding smooth chart $\big(\pi^{-1}(U), \big((x^i), (\xi_i)\big)\big)$ for T^*M. It is characterized by sending $\xi_i \lambda^i|_p$ to $\big((x^i(p)), (\xi_i)\big)$, where $p \in U$ and $(\lambda_i|_p)$ is the dual basis of $(\partial/\partial x^i|_p)$. By definition, the component functions of ω with respect to the smooth chart $\big(U, (x^i)\big)$ are the functions $\omega_i\colon U \to \mathbb{R}$ determined by

$$\omega_p = \sum_i \omega_i(p) \cdot \lambda^i|_p\,,\ p \in U.$$

Therefore, the coordinate representation $\hat{\omega}$ of ω with respect to these charts on U and $\pi^{-1}(U)$ is the map

$$\begin{aligned}\hat{\omega}\colon \hat{U} &\to \hat{U} \times \mathbb{R}^n \\ \hat{x} &\mapsto \big(\hat{x}, \big(\omega_i \circ \varphi^{-1}(\hat{x})\big)\big)\,.\end{aligned}$$

Since by hypothesis ω, and thus also $\hat{\omega}$, is smooth, we conclude that each $\omega_i \circ \varphi^{-1}$, and thus ω_i itself, is smooth.

(b) $\implies$ (c): Immediate.

(c) $\implies$ (a): By hypothesis, there exists an atlas $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ of M such that for all α, the covector field ω has smooth component functions in each chart $(U_\alpha, \varphi_\alpha)$. By the computation in $(a) \Rightarrow (b)$ we see that the coordinate representation of ω with respect to the smooth charts $\big(U_\alpha, \varphi_\alpha\big)$ and $\big(\pi^{-1}(U_\alpha), (\varphi_\alpha, (\xi_{\alpha,i})\big)$ is smooth. Hence, ω is smooth by [*Exercise Sheet* 3, *Exercise* 1(*b*)].

(c) $\implies$ (d): Let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be an atlas for which ω has smooth component functions $\omega_{\alpha,i}$, and write $\varphi_\alpha = (x^i_\alpha)$. Let $X_{\alpha,i}$ be the component functions of X on U_α, which are smooth by *Proposition 7.2*. Then, for any $p \in U_\alpha$, we have

$$\omega(X)(p) = \omega_p(X_p) = \sum_i \sum_j \omega_{\alpha,i}(p) X_{\alpha,j}(p) \underbrace{\lambda^i|_p \left(\frac{\partial}{\partial x^j_\alpha}\bigg|_p \right)}_{=\delta^i_j} = \sum_i \omega_{\alpha,i}(p) X_{\alpha,i}(p)\,,$$

as $(\lambda^i|_p)$ is the dual basis of $(\partial/\partial x^i_\alpha|_p)$, and since all functions $\omega_{\alpha,i}$ and $X_{\alpha,i}$ are smooth, we infer that $\omega(X)|_{U_\alpha}$ is smooth. As $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is an atlas for M, it follows from [*Exercise Sheet* 3, *Exercise* 2(*a*)] that $\omega(X)$ is smooth.

(d) $\implies$ (e): Let U be an open subset of M and let X be a smooth vector field on U. Let $p \in U$ and let (U_p, φ_p) be a smooth chart for M containing p. Let $\overline{V_p} \subseteq U_p$ be the preimage of a compact ball centered at $\varphi_p(p)$, and let V_p be its interior. Let $\psi_p \colon M \to \mathbb{R}$ be a smooth bump function with support in U_p such that $\psi_p|_{\overline{V_p}} \equiv 1$. Then the map $\psi_p X \colon M \to TM$ defined by

$$(\psi_p X)_q = \begin{cases} \psi_p(q) X_q & \text{if } q \in U, \\ 0 & \text{otherwise}, \end{cases}$$

is a smooth global vector field; indeed, it is smooth on U and on $M \setminus \operatorname{supp}(\psi_p)$ (as it is 0 on this set), which is an open cover of M by construction. Hence, $\omega(\psi_p X)$ is smooth by assumption. But then $\omega(X)|_{V_p} = \omega(\psi_p X)|_{V_p}$ is smooth as well by [*Exercise Sheet* 3, *Exercise* 2(*b*)]. We conclude that there is an open cover $\{V_p\}_{p \in U}$ of U such that $\omega(X)|_{V_p}$ is smooth for all $p \in U$, and thus $\omega(X) \colon U \to \mathbb{R}$ is smooth on U by [*Exercise Sheet* 3, *Exercise* 2(*a*)].

(e) $\implies$ (b): Let $\big(U, (x^i)\big)$ be a smooth chart for M and let ω_i be the component functions of ω with respect to this chart. By applying (e) to the smooth vector field $\partial/\partial x^i \colon U \to \mathbb{R}$, we infer that $\omega(\partial/\partial x^i)$ is smooth. But since for any $p \in U$ we have

$$\omega\left(\frac{\partial}{\partial x^i}\right)(p) = \sum_j \omega_j(p) \cdot \underbrace{\lambda^j|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right)}_{=\delta^j_i} = \omega_i(p),$$

and hence $\omega(\partial/\partial x^i) = \omega_i$, we deduce that the component functions ω^i of ω on $\big(U, (x^i)\big)$ are smooth.

Remark. The above arguments for $(d) \implies (e)$ and $(e) \implies (b)$ yield in particular the following: two (potentially rough) covector fields $\omega, \omega' \colon M \to T^*M$ are equal if and only if $\omega(X) = \omega'(X)$ for all smooth global vector fields X on M.

Exercise 2 (*Properties of the differential*)**:** Let M be a smooth manifold and let $f, g \in C^\infty(M)$. Prove the following assertions:

(a) If $a, b \in \mathbb{R}$, then $d(af + bg) = a\,df + b\,dg$.

(b) $d(fg) = f\,dg + g\,df$.

(c) $d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.

(d) If $J \subseteq \mathbb{R}$ is an interval containing the image of f and if $h \colon J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\,df$.

(e) If f is constant, then $df = 0$. Conversely, if $df = 0$, then f is constant on each connected component of M.

Solution:

(a) Fix $a, b \in \mathbb{R}$ and $p \in M$. For any $v \in T_pM$ we have

$$\begin{aligned} d(af + bg)_p(v) &= v(af + bg) = a\,v(f) + b\,v(g) \\ &= a\,df_p(v) + b\,dg_p(v) \\ &= \big(a\,df_p + b\,dg_p\big)(v). \end{aligned}$$

Therefore,

$$d(af + bg)_p = a\,df_p + b\,dg_p,$$

which yields the statement, since $p \in M$ was arbitrary.

(b) Fix $p \in M$. For any $v \in T_pM$ we have

$$\begin{aligned} d(fg)_p(v) &= v(fg) = f(p)\,vg + g(p)\,vf \\ &= f(p)\,dg_p(v) + g(p)\,df_p(v) \\ &= \big(f(p)\,dg_p + g(p)\,df_p\big)(v). \end{aligned}$$

Therefore,

$$d(fg)_p = f(p)\,dg_p + g(p)\,df_p,$$

which yields the statement, since $p \in M$ was arbitrary.

Note: We may also argue somewhat differently as follows (the same also applies for (a) above, and this method will be used in (c) below as well): Let X be a smooth global vector field on M. For any $p \in M$ we have

$$d(fg)(X)(p) = X_p(fg) = f(p)\,X_p(g) + g(p)\,X_p(f) = (f\,dg)(X)(p) + (g\,df)(X)(p).$$

Therefore,

$$d(fg)(X) = (f\,dg)(X) + (g\,df)(X)$$

for any smooth global vector field X, which yields the statement.

(c) Let $U := M \setminus g^{-1}(0)$. Let X be a smooth vector field on U. Given $p \in U$, note that

$$0 = X_p(1) = X_p\big(g \cdot (1/g)\big) = g(p) X_p(1/g) + \big(1/g(p)\big) X_p(g),$$

which yields

$$X_p(1/g) = -X_p(g)/\big(g(p)^2\big).$$

Therefore,

$$d(1/g)(X)(p) = X_p(1/g) = -X_p(g)/\big(g(p)^2\big) = \big(-(dg)/g^2\big)(X)(p)$$

for all X and p, which implies that $d(1/g) = -(dg)/g^2$. It follows that

$$d(f/g) \overset{(b)}{=} (1/g)\, df + f\, d(1/g) = (1/g)\, df - (f/g^2)\, dg = (g\, df - f\, dg)/g^2,$$

as desired.

(d) Fix $p \in M$ and $v \in T_pM$. Write $v = v^i \frac{\partial}{\partial x^i}\big|_p$ and note that

$$\left.\frac{\partial}{\partial x^i}\right|_p (h \circ f) = \frac{\partial (h \circ f)}{\partial x^i}(p) = h'\big(f(p)\big) \frac{\partial f}{\partial x^i}(p) = h'\big(f(p)\big) \left.\frac{\partial}{\partial x^i}\right|_p f$$

by the chain rule. Therefore,

$$\begin{aligned} d(h \circ f)_p(v) = v(h \circ f) &= \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right)(h \circ f) \\ &= v^i h'\big(f(p)\big) \left.\frac{\partial}{\partial x^i}\right|_p f = h'\big(f(p)\big)\, vf \\ &= (h' \circ f)(p)\, df_p(v). \end{aligned}$$

Since $v \in T_pM$ was arbitrary, we infer that $d(h \circ f)_p = (h' \circ f)(p)\, df_p$, and since $p \in M$ was arbitrary, we conclude that $d(h \circ f) = (h' \circ f)\, df$.

Note: We may alternatively argue as follows: Let X be a smooth global vector field on M and let $p \in M$ be arbitrary. To avoid confusion, denote by $df_p \colon T_pM \to T_{f(p)}\mathbb{R}$ the differential of f at $p \in M$ as a linear map between tangent spaces, and by $d^{\text{cov}} f$ the covector field determined by f. They are related as follows: for every $p \in M$ and $v \in T_pM$, we have

$$d^{\text{cov}} f_p(v) = [df_p(v)](\mathrm{Id}_{\mathbb{R}}).$$

This follows from the fact that the natural identifaction of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ is provided by evaluation at $\mathrm{Id}_{\mathbb{R}}$. Therefore, if $p \in M$ and $v \in T_pM$ are arbitrary, then we have

$$\begin{aligned} d^{\text{cov}}(h \circ f)_p(v) &= [d(h \circ f)_p(v)](\mathrm{Id}_{\mathbb{R}}) = \big[dh_{f(p)}\big(df_p(v)\big)\big]\,(\mathrm{Id}_{\mathbb{R}}) \\ &= h'\big(f(p)\big) \cdot [df_p(v)](\mathrm{Id}_{\mathbb{R}}) = h'\big(f(p)\big) \cdot d^{\text{cov}} f_p(v), \end{aligned}$$

where we used that for any $t \in J$, the differential $dh_t\colon T_tJ \to T_{h(t)}\mathbb{R}$ is the map given by scalar multiplication with $h'(t)$. As $p \in M$ and $v \in T_pM$ were arbitrary, we conclude that $d^{\text{cov}}(h \circ f) = (h' \circ f)\, d^{\text{cov}} f$.

(e) In view of the fact that the differential of f as defined in *Chapter 3* (i.e., as a linear map $df_p\colon T_pM \to T_p\mathbb{R}$) and as defined in *Chapter 8* (i.e., as a linear map $df_p\colon T_pM \to \mathbb{R}$) is the same object (due to the canonical identification between $\mathbb{R}$ and $T_p\mathbb{R}$), the assertion is simply a special case of [*Exercise Sheet* 5, *Exercise* 5(*b*)].

Exercise 3:

(a) *Derivative of a function along a curve*: Let M be a smooth manifold, $\gamma\colon J \to M$ be a smooth curve, and $f\colon M \to \mathbb{R}$ be a smooth function. Show that the derivative of $f \circ \gamma\colon J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\big(\gamma'(t)\big).$$

(b) Let M be a smooth manifold and let $f \in C^\infty(M)$. Show that $p \in M$ is a critical point of f if and only if $df_p = 0$.

Solution:

(a) Using the definitions, for any $t \in J$ we have

$$df_{\gamma(t)}\big(\gamma'(t)\big) = \gamma'(t)\, f = d\gamma\left(\left.\frac{d}{dt}\right|_t\right)(f) = \left.\frac{d}{dt}\right|_t (f \circ \gamma) = (f \circ \gamma)'(t).$$

(b) Since the differential df_p is a linear map with codomain the 1-dimensional $\mathbb{R}$-vector space $T_p\mathbb{R} \cong \mathbb{R}$, it is surjective if and only if there exists $v \in T_pM \setminus \{0\}$ such that $df_p(v) \in \mathbb{R} \setminus \{0\} \cong T_p\mathbb{R} \setminus \{0\}$. Therefore, $p \in M$ is a critical point of f if and only if df_p is not surjective if and only if $df_p = 0$ (i.e., the zero linear map).

Remark. Let M be a smooth manifold and let $f \in C^\infty(M)$. If γ is a smooth curve in M, then we have two different meanings for the expression $(f \circ \gamma)'(t)$. On the one hand, $f \circ \gamma$ can be interpreted as a smooth curve in $\mathbb{R}$, and thus $(f \circ \gamma)'(t)$ is its velocity (vector) at the point $(f \circ \gamma)(t)$, which is an element of the tangent space $T_{(f\circ\gamma)(t)}\mathbb{R}$. [*Exercise Sheet* 4, *Exercise* 5] shows that this tangent vector is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as an element of $T_{(f\circ\gamma)(t)}\mathbb{R}$. On the other hand, $f \circ \gamma$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ordinary derivative. *Exercise* 4(a) above shows that this derivative is equal to $df_{\gamma(t)}\big(\gamma'(t)\big)$, thought of as a real number.

Exercise 4: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. For any $f \in C^\infty(M)$, show that $d\big(f|_S\big) = \iota^*(df)$. Conclude that the pullback of df to S is zero if and only if f is constant on each connected component of S.

Solution: Since $f|_S = f \circ \iota$, by *Proposition 8.13* we obtain

$$\iota^*(df) = d(f \circ \iota) = d\big(f|_S\big).$$

It follows from the above relation and from *Exercise* 2(e) that $\iota^*(df) = 0$ if and only if f is constant on each connected component of S.

Exercise 5:

(a) Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (st, e^t)$$

and the smooth covector field

$$\omega = xdy - ydx \in \mathfrak{X}^*(\mathbb{R}^2).$$

Compute $F^*\omega$.

(b) Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 + z^2$$

and the map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$ [1]

Compute $F^*(df)$ and $d(f \circ F)$ separately, and verify that they are equal.

(c) Consider the smooth manifold

$$M := \big\{(x,y) \in \mathbb{R}^2 \mid x > 0\big\}$$

and the smooth function

$$f\colon M \to \mathbb{R},\ (x,y) \mapsto \frac{x}{x^2+y^2}.$$

Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$.

(d) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M.

Solution:

(a) We have

$$\begin{aligned} F^*\omega &= (x \circ F)\, d\,(y \circ F) - (y \circ F)\, d\,(x \circ F) \\ &= (st)\, d\,(e^t) - \,(e^t)\, d\,(st) \\ &= ste^t\, dt - e^t(s\, dt + t\, ds) \\ &= (-te^t)\, ds + se^t(t-1)\, dt. \end{aligned}$$

[1] Note that F is the inverse of the stereographic projection from the north pole $N \in \mathbb{S}^2$; see [*Exercise Sheet* 2, *Exercise* 6].

(b) On the one hand, by *Exercise* 2 we obtain

$$df = d(x^2 + y^2 + z^2) = 2x\,dx + 2y\,dy + 2z\,dz,$$

and since

$$d\,(x \circ F) = d\left(\frac{2u}{u^2+v^2+1}\right) = \frac{2(u^2+v^2+1) - 4u^2}{(u^2+v^2+1)^2}\,du + \frac{-4uv}{(u^2+v^2+1)^2}\,dv,$$

$$d\,(y \circ F) = d\left(\frac{2v}{u^2+v^2+1}\right) = \frac{-4uv}{(u^2+v^2+1)^2}\,du + \frac{2(u^2+v^2+1) - 4v^2}{(u^2+v^2+1)^2}\,dv,$$

$$d\,(z \circ F) = d\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right) = \frac{4u}{(u^2+v^2+1)^2}\,du + \frac{4v}{(u^2+v^2+1)^2}\,dv,$$

we compute that

$$\begin{aligned}
F^*(df) &= 2(x \circ F)\,d\,(x \circ F) + 2(y \circ F)\,d\,(y \circ F) + 2(z \circ F)\,d\,(z \circ F)\\
&= 2\,\frac{2u}{u^2+v^2+1}\,d\left(\frac{2u}{u^2+v^2+1}\right) + 2\,\frac{2v}{u^2+v^2+1}\,d\left(\frac{2v}{u^2+v^2+1}\right) +\\
&\quad + 2\,\frac{u^2+v^2-1}{u^2+v^2+1}\,d\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)\\
&= \frac{\big(8u(u^2+v^2+1) - 16u^3\big) - 16uv^2 + 8u(u^2+v^2-1)}{(u^2+v^2+1)^3}\,du +\\
&\quad + \frac{-16u^2v + \big(8v(u^2+v^2+1) - 16v^3\big) + 8v(u^2+v^2-1)}{(u^2+v^2+1)^3}\,dv\\
&= 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(f \circ F)(u,v) &= \left(\frac{2u}{u^2+v^2+1}\right)^2 + \left(\frac{2v}{u^2+v^2+1}\right)^2 + \left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)^2\\
&= \frac{(u^2+v^2+1)^2}{(u^2+v^2+1)^2}\\
&= 1,
\end{aligned}$$

whence $d(f \circ F) = 0$ according to *Exercise* 2(e).

(c) Given a point $p = (x_0, y_0) \in M$, the differential df_p of f at p is represented in coordinates (x, y) by the row matrix D_p whose components are the partial derivatives of f at $p = (x_0, y_0)$; namely,

$$D_p = \left(\frac{\partial f}{\partial x}(x_0, y_0),\, \frac{\partial f}{\partial y}(x_0, y_0)\right) = \left(\frac{y_0^2 - x_0^2}{(x_0^2 + y_0^2)^2},\, \frac{-2x_0y_0}{(x_0^2 + y_0^2)^2}\right).$$

In view of *Exercise* 3(b), to find the points $p \in M$ at which $df_p = 0$, we have to solve the system

$$(\Sigma) : \begin{cases} y^2 - x^2 = 0\\ -2xy = 0 \end{cases}$$

under the restriction that $x > 0$. It is straightforward to see that (Σ) has no solutions $(x, y) \in M$; in other words,

$$\{p \in M \mid df_p = 0\} = \emptyset.$$

(d) Let $\omega \in \mathfrak{X}^*(M)$ be exact and let $f \in C^\infty(M)$ such that $\omega = df$. Since M is compact, f attains its minimum at a point $p \in M$ and its maximum at a point $q \in M$, and since df is represented in coordinates by the gradient of (the coordinate representation of) f, we have $df_p = 0 = df_q$. Note also that if $p = q$, then f is constant, and thus $df = 0$ by *Exercise* 2(e).