Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 13

Exercise 1 (*Smoothness criteria for covector fields*)**:**
Let M be a smooth manifold and let $\omega\colon M \to T^*M$ be a rough covector field on M. Prove that the following assertions are equivalent:

(a) ω is smooth.

(b) In every smooth coordinate chart the component functions of ω are smooth.

(c) Every point of M is contained in some smooth coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field X on M, the function $\omega(X)\colon M \to \mathbb{R}$ is smooth on M.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X)\colon U \to \mathbb{R}$ is smooth on U.

[Hint: Try proving $(a) \implies (b) \implies (c) \implies (a)$ and $(c) \implies (d) \implies (e) \implies (b)$.]

Exercise 2 (*Properties of the differential*)**:**
Let M be a smooth manifold and let $f, g \in C^\infty(M)$. Prove the following assertions:

(a) If $a, b \in \mathbb{R}$, then $d(af + bg) = a\, df + b\, dg$.

(b) $d(fg) = f\, dg + g\, df$.

(c) $d(f/g) = (g\, df - f\, dg)/g^2$ on the set where $g \neq 0$.

(d) If $J \subseteq \mathbb{R}$ is an interval containing the image of f and if $h\colon J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\, df$.

(e) If f is constant, then $df = 0$. Conversely, if $df = 0$, then f is constant on each connected component of M.

Exercise 3:

(a) *Derivative of a function along a curve*: Let M be a smooth manifold, $\gamma\colon J \to M$ be a smooth curve, and $f\colon M \to \mathbb{R}$ be a smooth function. Show that the derivative of $f \circ \gamma\colon J \to \mathbb{R}$ is given by

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\big(\gamma'(t)\big).$$

(b) Let M be a smooth manifold and let $f \in C^\infty(M)$. Show that $p \in M$ is a critical point of f if and only if $df_p = 0$.

Exercise 4:

Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. For any $f \in C^\infty(M)$, show that $d\big(f|_S\big) = \iota^*(df)$. Conclude that the pullback of df to S is zero if and only if f is constant on each connected component of S.

Exercise 5 (to be submitted by Thursday, 19.12.2024, 16:00):

(a) Consider the smooth map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (s,t) \mapsto (st, e^t)$$

and the smooth covector field

$$\omega = x dy - y dx \in \mathfrak{X}^*(\mathbb{R}^2).$$

Compute $F^*\omega$.

(b) Consider the function

$$f\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 + z^2$$

and the map

$$F\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$ [1]

Compute $F^*(df)$ and $d(f \circ F)$ separately, and verify that they are equal.

(c) Consider the smooth manifold

$$M := \big\{(x,y) \in \mathbb{R}^2 \mid x > 0\big\}$$

and the smooth function

$$f\colon M \to \mathbb{R},\ (x,y) \mapsto \frac{x}{x^2+y^2}.$$

Compute the coordinate representation for df and determine the set of all points $p \in M$ at which $df_p = 0$.

(d) Let M be a compact, connected, smooth manifold of dimension $n > 0$. Show that every exact smooth covector field on M vanishes at least at two points of M.

[1] Note that F is the inverse of the stereographic projection from the north pole $N \in \mathbb{S}^2$; see [*Exercise Sheet* 2, *Exercise* 5].