The Schmidt Theorem.

Theorem Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a pure state of a bipartite system. Let $\rho_{AB} = |\psi\rangle\langle\psi|$ the corresponding density matrix.

Then $\rho_A = Tr_B\, \rho_{AB}$ and $\rho_B = Tr_A\, \rho_{AB}$ have the same non-zero eigenvalues.

Moreover $S(\rho_A) = S(\rho_B)$ since of $\{\lambda_\alpha^A\}$ & $\{\lambda_\beta^B\}$ are the set of non zero eigenvalues of ρ_A & ρ_B we have $S(\rho_A) = -\sum_\alpha \lambda_\alpha^A \ln \lambda_\alpha^A$; $S(\rho_B) = -\sum_\beta \lambda_\beta^B \ln \lambda_\beta^B$.

<u>Proof.</u> Based on SVD decomposition.

Let $$|\psi\rangle = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \psi_{ij} \, |i\rangle_A \otimes |j\rangle_B$$

where $d_A = \dim \mathcal{H}_A$, $d_B = \dim \mathcal{H}_B$.

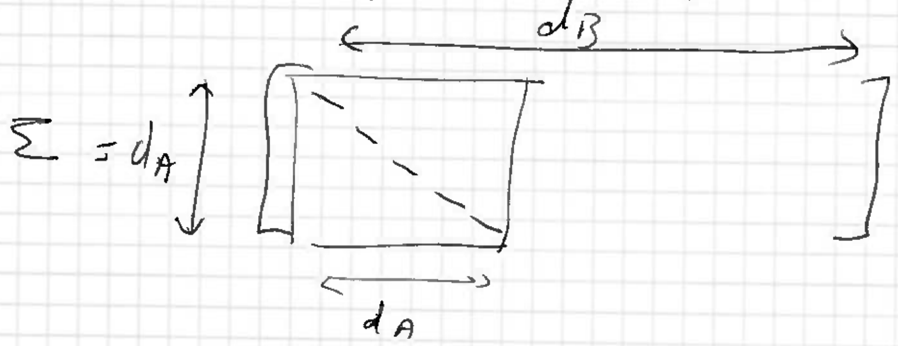
Assume the case $d_A \leq d_B$ (other one is similar).

By SVD the "matrix" ψ_{ij} which is $d_A \times d_B$;

$$\psi_{ij} = \sum_{k=1}^{d_A} \sum_{l=1}^{d_B} U_{ik} \, \Sigma_{kl} \, V^*_{jl}$$

with U : $d_A \times d_A$ unitary, V : $d_B \times d_B$ unitary

Σ : $d_A \times d_B$ a fat (diagonal) matrix of sing value $\sigma_\alpha \geq 0$.



We have

$(|i\rangle \otimes |j\rangle)(\langle l| \otimes \langle m|)$

$$|\psi\rangle\langle\psi| = \cancel{\sum_{k=1}^{d_A} \sum_{l=1}^{d_B} \sum_{k'=1}^{d_A} \sum_{l'=1}^{d_B}}$$

$$= \sum_{i,j,l,m} \psi_{ij} \, \psi^*_{lm} \, |i,j\rangle_{AB} \langle l,m|_{AB}$$

The density matrix $\rho_{AB} = |\psi\rangle\langle\psi|$ has matrix elements

$$\rho_{ij;lm} = \psi_{ij}\,\psi^*_{lm}.$$

Thus $\rho_A = \mathrm{Tr}_B\,\rho_{AB}$ has matrix elements

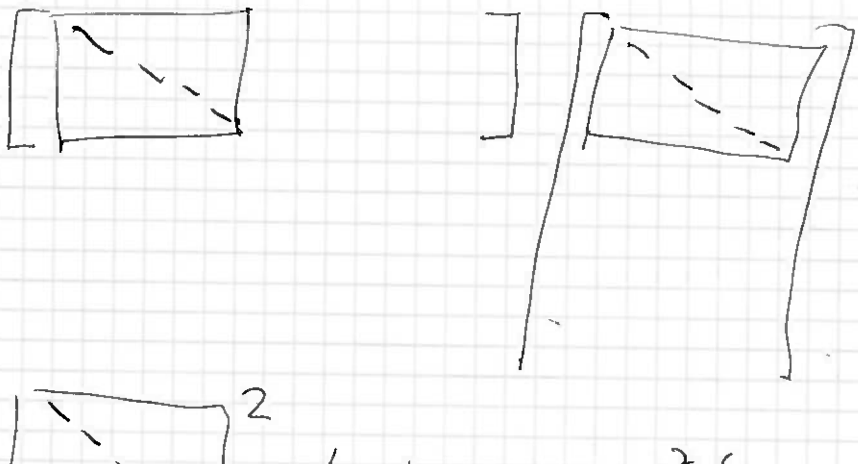
$$(\rho_A)_{il} = \sum_{j=1}^{d_B} \psi_{ij}\,\psi^*_{lj}$$

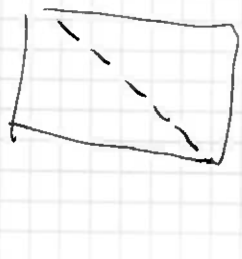
$\Rightarrow$ $\boxed{\rho_A = \psi\psi^\dagger}$ where "ψ is matrix ψ_{ij}."

Now the SVD is $\psi = U\Sigma V^\dagger$

$$\Rightarrow \quad \rho_A = U\Sigma V^\dagger\, V\Sigma^T U^\dagger$$

$$= U \underbrace{\Sigma\Sigma^T}_{d_A \times d_A \text{ matrix}} U^\dagger \qquad \text{since } V^\dagger V = \mathbb{1}_{d_B\times d_B}$$

$\Sigma\Sigma^T =$ 

$=$  ^{2} {matrix of σ_α^2}.

On the other hand:

$\rho_B = Tr_A\, \rho_{AB}$ has matrix elements:

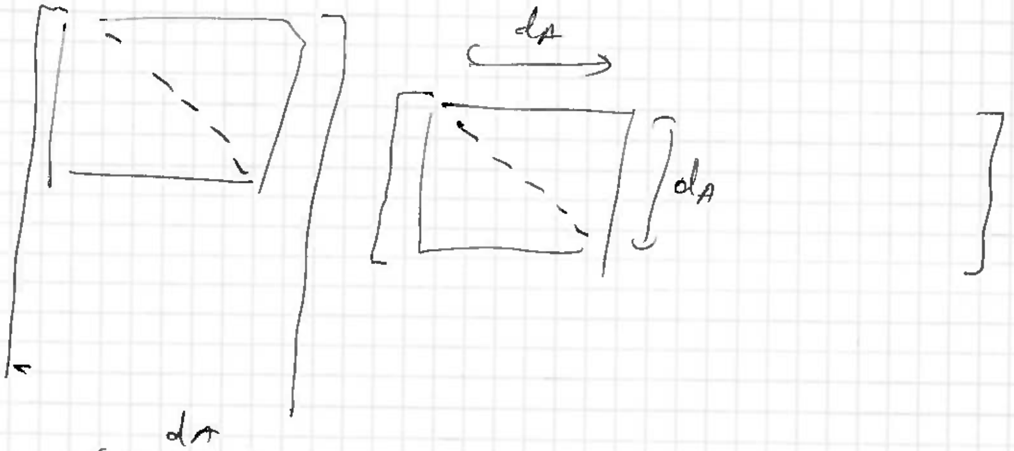
$$(\rho_B)_{jm} = \sum_{i=1}^{d_A} \psi_{ij}\, \psi_{im}^* = \sum_{i=1}^{d_A} (\psi^+)_{mi}\, \psi_{ij}$$

$$\Rightarrow \quad \boxed{\rho_B = \psi^+ \psi}$$

Now from $\psi = U \Sigma V^+$ we get

$$\rho_B = V \Sigma^+ \underbrace{U^+ U}_{\mathbb{1}_{d_A \times d_A}} \Sigma V^+ = V \underbrace{\Sigma^T \Sigma}_{d_B \times d_B} V^+$$

$\Sigma^T \Sigma =$



$$= \begin{bmatrix} \begin{bmatrix} \ddots \end{bmatrix}^2_{d_A \times d_A} & 0 \\ 0 & 0 \end{bmatrix} \quad d_B \times d_B$$

has non zero eigenvalues given by $\begin{bmatrix} \ddots \end{bmatrix}^2$ same as ρ_A

CQFD ∎