VON NEUMANN ENTROPY & ENTANGLEMENT ENTROPY.

A fundamental quantity of interest in quantum information & communication is the quantum notion of entropy.

The density matrix is an analog of the probability distribution in usual prob theory an similarly von Neumann entropy is a sort of analog of Shannon's entropy of classical information theory.

However as we will see the phenomenon of entanglement between two systems allows to define an "entanglement entropy" with no real classical analog. This is a new kind of entropy not associated with statistical ensembles (or mixtures).

I. SHANNON ENTROPY.

Consider a discrete R.V X taking values $x = a_1, a_2, \ldots, a_K$ with probabilities $p_1, p_2, \ldots, p_K$. The Shannon entropy is by definition

$$H(X) = -\sum_{i=1}^{K} p_i \underbrace{\log_2 p_i}_{(\text{or } \ln p_i)}$$

This measures the amount of information in a stream (or source) of symbols a_i emitted with probabilities p_i. Essentially it gives the best possible rate of compression for this source. Intuition: $H(X)$ = uncertainty contained in the source.

Essential properties.

a) $H(X) \geq 0$.

b) $H(X)$ is maximal for $p(x = a_i) = \frac{1}{K}$

= uniform distribution

$$H_{max} = - K \cdot \frac{1}{K} \log_2 K = \log_2 K.$$

c) $H(X)$ is a concave functional of the probability distribution p_X. i.e

$$H[\alpha p_X + (1-\alpha) q_X] \geq \alpha H[p_X] + (1-\alpha) H[q_X]$$

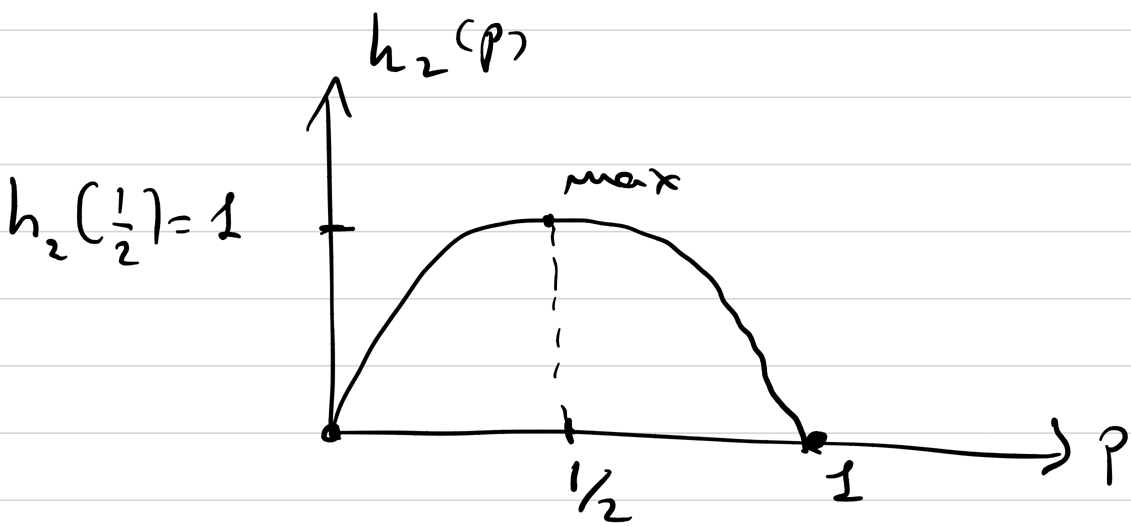
for $0 \leq \alpha \leq 1$.

(Notation here $H(X) = H[p_X]$).

Entropy of a binary source.

alphabet $\mathcal{A} = \{0, 1\}$. $\begin{cases} p(0) = p \\ p(1) = 1-p \end{cases}$

$$H(X) = h_2(p) \equiv -p \log_2 p - (1-p) \log_2 (1-p)$$



II. VON NEUMANN ENTROPY

$$\boxed{\begin{array}{l} \rho = \text{density matrix} \quad (\text{analog of } p_X) \\ S(\rho) \equiv -\mathrm{Tr}(\rho \ln \rho) \end{array}}$$

$$\left(\text{analog of } H(X) = H[p_X] = -\sum_{i=1}^{K} p_i \log_2 p_i\right)$$

Practical mathematical meaning of def:

$\rho^\dagger = \rho$; $\rho \geq 0$; $\mathrm{Tr}\,\rho = 1$

Thus ρ has eigenvalues $0 \leq \lambda_i \leq 1$ and is diagonal in an orthonormal eigenvector basis.

The eigenvalues of $\rho \ln \rho$ are $(\lambda_i \ln \lambda_i)$

Thus

$$\boxed{S(\rho) = -\operatorname{Tr} \rho \ln \rho = -\sum_{i=1}^{d} \lambda_i \ln \lambda_i}$$

where $d = \dim \mathcal{H}$.

von Neumann entropy = Shannon entropy of prob distr defined by eigenvalues of ρ.

Basic Properties.

1) $S(\rho) \geq 0$

2) $S(\rho)$ maximal for $\lambda_i = \frac{1}{d}$, $i = 1 \ldots d$

$\Leftrightarrow \quad \rho = \frac{1}{d} \mathbb{1}$, $d = \dim \mathcal{H}$.

3) $S(\rho) = 0 \Leftrightarrow \rho = |\psi\rangle\langle\psi|$ is a "pure state"

<u>Proof of 3)</u> :

$\rho = |\psi\rangle\langle\psi| \Rightarrow \lambda_1 = 1 \; ; \; \lambda_j = 0 \, , \, j \neq 1$

$\Rightarrow \quad S(\rho) = -1 \ln 1 - 0 \ln 0 \; \ldots$
$\quad = 0 .$

converse also true.

<u>Remark</u>: The entropy of a pure state vector is always zero. Thus pure state vectors do carry any uncertainty about the state of the system.

$\uparrow$

(Not to be confused with the uncertainty of the outcome of the measurement process.)

4) <u>Concavity</u> (No proof here)

$$S(\alpha \rho_1 + (1-\alpha)\rho_2) \geq \alpha S(\rho_1) + (1-\alpha) S(\rho_2)$$

$\forall \; 0 \leq \alpha \leq 1 .$

III. ENTROPY OF A QUBIT.

Recall for one qubit $\rho = \frac{1}{2}(\mathbb{1} + \vec{a} \cdot \vec{\sigma})$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\|\vec{a}\| \leq 1$

($\vec{a}$ in the Bloch Ball).

To compute $S(\rho)$ we must compute the eigenvalues of ρ. In fact here we use a slightly heuristic argument (could be made rigorous). There exist a change of basis such that "z-axis is aligned with $\vec{a}$" so in the new basis

$$\rho \to \tilde{\rho} = \frac{1}{2}(\mathbb{1} + \|\vec{a}\|\sigma_z)$$

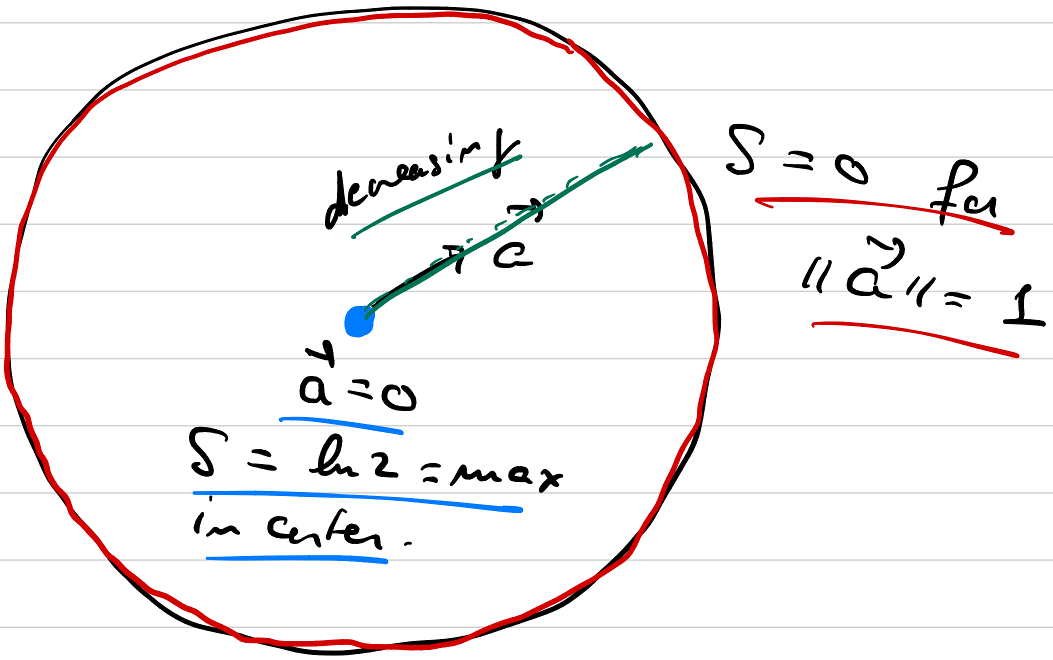
$$= \frac{1}{2}\begin{pmatrix} 1 + \|\vec{a}\| & 0 \\ 0 & 1 - \|\vec{a}\| \end{pmatrix}$$

$$\Rightarrow S(\rho) = S(\tilde{\rho})$$

$$= -\frac{1+\|\vec{a}\|}{2}\ln\left(\frac{1+\|\vec{a}\|}{2}\right) - \frac{1-\|\vec{a}\|}{2}\ln\left(\frac{1-\|\vec{a}\|}{2}\right)$$

= binary entropy function of a distr

$$\begin{cases} p(0) = \dfrac{1+\|\vec{a}\|}{2} \\ p(1) = \dfrac{1-\|\vec{a}\|}{2} \end{cases}$$



IV. ENTANGLEMENT ENTROPY.

Recall for $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ pure states can be classified into

<u>product states.</u> & <u>entangled states.</u>

$|\psi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$ possible to factorize.

$|\psi\rangle$ impossible to factorise.

(For density matrices (or mixed states) this distinction is a bit more subtle and not discussed here).

<u>What we discuss here is a measure of the "quantity of entanglement" for a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.</u>

Definition. Entanglement entropy.

Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The associated DM is $\rho = |\psi\rangle\langle\psi|$ (a pure state),

Let

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}(|\psi\rangle\langle\psi|)$$

$$\rho_B = \mathrm{Tr}_{\mathcal{H}_A}(|\psi\rangle\langle\psi|)$$

the partial or reduced DMs.

We define the entanglement entropy as:

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \ln \rho_A)$$

$$S(\rho_B) = -\mathrm{Tr}(\rho_B \ln \rho_B)$$

Theorem: If ρ_A & ρ_B come from a pure state as above, we have $S(\rho_A) = S(\rho_B)$.

This is a consequence of the important Schmidt Thm.

Examples.

1) **Product states.** $|\psi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$

$$\Rightarrow \quad \rho = |\varphi_A\rangle\langle\varphi_A| \otimes |\varphi_B\rangle\langle\varphi_B| .$$
$$= \rho_A \otimes \rho_B$$

with
$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\, \rho = |\varphi_A\rangle\langle\varphi_A|$$
$$\rho_B = \mathrm{Tr}_{\mathcal{H}_A}\, \rho = |\varphi_B\rangle\langle\varphi_B|$$

$$\Rightarrow \quad \boxed{S(\rho_A) = S(\rho_B) = 0}$$

so the entanglement entropy of product states is zero. "there is no entanglement"

2) **Bell state.**

Let $|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$

$$\rho = |\mathrm{Bell}\rangle\langle \mathrm{Bell}|$$

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B} |\mathrm{Bell}\rangle\langle\mathrm{Bell}|$$

$$= \frac{1}{2}\left(|0\rangle_A \langle 0|_A + |1\rangle_B \langle 1|_B\right)$$

$$= \frac{1}{2}\mathbb{1}_A \qquad \text{maximally random!}$$

$$\rho_B = \frac{1}{2}\mathbb{1}_B \qquad \text{also. maximally random!}$$

$$\Rightarrow \boxed{S(\rho_A) = S(\rho_B) = \ln 2 = \text{max entropy of one qubit}}$$

The entanglement entropy of a Bell state is maximal.

$$\boxed{\begin{array}{l}\text{Bell states are not globally random since} \\ \qquad S(|\mathrm{Bell}\rangle\langle\mathrm{Bell}|) = 0 \\ \text{but they are locally maximally random since} \\ \qquad S(\rho_A) = S(\rho_B) = \ln 2.\end{array}}$$

Quite astonishing fact.

Entanglement entropy behaves in a very non-classical way. It has no classical analog. This plays a very important role as you will see in more advanced classes on QIT.

For example if (X, Y) is a pair of classical R.V then it is always true that

$$H(X, Y) \geq H(X) \;\&\; H(Y)$$

entropies of marginal distr.

But for $\rho = |\psi\rangle\langle\psi|$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ we <u>may</u> have

$$S(\rho) \leq S(\rho_A) \;\&\; S(\rho_B)$$

for example with $\rho = |\text{Bell}\rangle\langle\text{Bell}|$ indeed

$$S(\rho) = 0 \leq \ln 2 = S(\rho_A) = S(\rho_B).$$