Homework 9 graded
Introduction to Quantum Information Processing

Exercise 1 *Dynamics of 1-qubit density matrix*

In class we showed that the general form of a 1-qubit density matrix is

$$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$$

where $\vec{a} = a_x, a_y, a_z)$ is a vector in the unit three dimensional ball (the Bloch *ball*) $\|\vec{a}\| \leq 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the three usual Pauli matrices. Consider the dynamics of this mixed state generated by the Hamiltonian of the qubit in a static plus rotating magnetic field in the rotating frame (as seen in class, $\omega_1 \propto$ the strength of the rotating field and and $\delta = \omega - \omega_0$ the detuning between the Larmor and rotating field frequencies)

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

a) Show that the density matrix at time t is of the form

$$\rho_t = \frac{1}{2}(I + \vec{a}(t) \cdot \vec{\sigma})$$

and compute the vector $\vec{a}(t)$. *Hint*: From the definition of the density matrix you can infer that

$$\rho_t = U_t \rho U_t^\dagger$$

with U_t the evolution operator.

b) Check that $\|\vec{a}(t)\| = \|\vec{a}\|$. So the vector $\vec{a}(t)$ evolves on a sphere (inside the Bloch ball) of radius given by the initial vector.

c) Find a simple proof of the last statement without ever computing $\vec{a}(t)$.

Exercise 2 *The difference between a Bell state and a statistical mixture of* $|00\rangle$ *and* $|11\rangle$

We consider a source that distributes to A and B either an EPR pair in the perfect Bell state $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, or distributes a pair of qubits in a statistical mixture of states $|00\rangle$, $|11\rangle$ with uniform probabilities 1/2. This exercise illustrates in many ways that the two kind of situations are completely different.

a) Write down the density matrix ρ_{Bell} associated to the Bell state in Dirac notation as well as in matrix array form (in the computational basis).

b) Write down the density matrix ρ_{stat} associated to the statistical mixture above in Dirac notation as well as in matrix array form (in the computational basis).

c) In a Bell/CSHS experiment one measures the observable

$$\mathcal{B} = A \otimes B + A \otimes B' - A' \otimes B + A' \otimes B'$$

What is the theoretical average if the state when the state is ρ_{Bell} ? (Use results proven in class and no need to reproduce calculations). And now compute the theoretical average if the state is ρ_{stat}. What are the values of the of these two averages for the optimal CSHS-angles $\alpha = 0$, $\alpha' = -\frac{\pi}{4}$, $\beta = \frac{\pi}{8}$, $\beta' = -\frac{\pi}{8}$?

Exercise 3 *Density matrix: a decoherence model*

In the following, we will study a model of decoherence of one qubit interacting with the environment. The whole system is defined in the hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{E}} \otimes \mathcal{H}_b$ where $\mathcal{H}_{\mathcal{E}}$ is the Hilbert space describing the possible states of the environment and $\mathcal{H}_b = \mathbb{C}^2$ is the Hilbert space describing the possible states of the qubit.

Let $|\phi_0\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathcal{H}_b$ be the initial state of the qubit and $|\mathcal{E}\rangle \in \mathcal{H}_b$ that of the environment (or sometimes called *heat-bath*). Let $(|i\rangle)_{i\geq 1} \in \mathcal{H}_{\mathcal{E}}$ be an "infinite" orthonormal basis of the environment $\mathcal{H}_{\mathcal{E}}$. We define the evolution operator $U = \sum_{i=1}^{+\infty} |i\rangle\langle i| \otimes \mathcal{D}(\theta_i)$ for some distinct angles $\theta_i \in \mathbb{R}$, and the dephasing operator: $\mathcal{D}(\theta_i) = |0\rangle\langle 0| + e^{i\theta_i}|1\rangle\langle 1|$.

If the environment makes a transition from state $|\mathcal{E}\rangle$ to $|i\rangle$, we let $\mu(\theta_i) = P(|\mathcal{E}\rangle \to |i\rangle)$ the probability of such a transition. Note that $\mu(\theta_i) = e^{i\arg\langle i|\mathcal{E}\rangle}\sqrt{\mu(\theta_i)}$.

a) What is the initial global state $|\psi_0\rangle$ of the whole system?

b) Show that U is a unitary operator (describe your steps).

c) The state of the system evolves (in discrete time steps say) with a power $n \in \mathbb{N}$ of the operator U as $|\psi_n\rangle = U^n|\psi_0\rangle$. Show that $\mathcal{D}(\theta_i)^n = \mathcal{D}(n\theta_i)$ and deduce that

$$|\psi_n\rangle = \sum_{i=1}^{+\infty} e^{i\arg\langle i|\mathcal{E}\rangle}\sqrt{\mu(\theta_i)}\,|i\rangle \otimes (\mathcal{D}(n\theta_i)|\phi_0\rangle)$$

d) Now let's consider the density matrix of the qubit itself: $\rho_n = \mathrm{tr}_{\mathcal{H}_{\mathcal{E}}}[|\psi_n\rangle\langle\psi_n|]$. First, using only the result of question (a), show that we have initially:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

e) For any angle $\theta \in \mathbb{R}$, show that we have:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{pmatrix}$$

f) Now let's consider $\hat{\theta}$ a random variable taking values $\theta_i in \mathbb{R}$ with probability partial $\mu(\theta_i)$. Use the result of question (c) and (e) to show that the density matrix of the qubit coincide with the following expression:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\mathbb{E}[e^{-in\hat{\theta}}] \\ \alpha^*\beta\mathbb{E}[e^{in\hat{\theta}}] & |\beta|^2 \end{pmatrix}$$

g) Now say that the values θ_i form a quasicontinuum and that μ is the PDF of a gaussian distribution of mean 0 and variance σ^2. Show that the density matrix of the qubit evolves as:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-\frac{1}{2}\sigma^2 n^2} \\ \alpha^*\beta e^{-\frac{1}{2}\sigma^2 n^2} & |\beta|^2 \end{pmatrix}$$

Calculate $\rho_\infty = \lim_{n\to\infty} \rho_n$.

h) (not graded) How does the von Neumann entropy of the qubit evolve from initial time $n = 0$ to final time $n \to +\infty$?