Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 12 – Solutions

Exercise 1: Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma\colon J \to M$ be an integral curve of V. Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\widetilde{\gamma}\colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} := aV$ on M, where $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma}\colon \widehat{J} \to M,\ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where $\widehat{J} := \{t \in \mathbb{R} \mid t+b \in J\}$.

Solution:

(a) If $t \in \widetilde{J}$, then

$$\widetilde{\gamma}'(t) = a\gamma'(at) = aV_{\gamma(at)} = \widetilde{V}_{\widetilde{\gamma}(t)}.$$

(b) If $t \in \widehat{J}$, then

$$\widehat{\gamma}'(t) = \gamma'(t+b) = V_{\gamma(t+b)} = V_{\widehat{\gamma}(t)}.$$

Exercise 2 (*The Euler vector field*)**:** Consider the *Euler vector field* on $\mathbb{R}^n$, i.e., the vector field V on $\mathbb{R}^n$ whose value at a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x_1}\bigg|_x + \ldots + x^n \frac{\partial}{\partial x_n}\bigg|_x .$$

(a) Check that V is a smooth vector field on $\mathbb{R}^n$.

(b) Let $c \in \mathbb{R}$ and let $f\colon \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}$ be a smooth function which is *positively homogeneous of degree* c, i.e., $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$.

[Hint: Differentiate the above relation with respect to both x^i and λ.]

(c) Compute the integral curves of V.

Solution:

(a) Note that the component functions of V with respect to the standard coordinate frame for $\mathbb{R}^n$ are linear, hence smooth. Therefore, V is a smooth vector field on $\mathbb{R}^n$ by *Proposition 7.2*.

(b) Using the chain rule, we obtain

$$\lambda^c \frac{\partial f}{\partial x^i}(x) = \frac{\partial}{\partial x^i}\big(\lambda^c f(x)\big) = \frac{\partial}{\partial x^i}\big(f(\lambda x)\big) = \lambda \frac{\partial f}{\partial x^i}(\lambda x) \tag{1}$$

and

$$c\lambda^{c-1} f(x) = \frac{d}{d\lambda}\big(\lambda^c f(x)\big) = \frac{d}{d\lambda}\big(f(\lambda x)\big) = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(\lambda x). \tag{2}$$

Since

$$(Vf)(x) = V_x f = \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x),$$

we have

$$(Vf)(\lambda x) = V_{\lambda x} f = \sum_{i=1}^{n} (\lambda x^i) \frac{\partial f}{\partial x^i}(\lambda x) \overset{(1)}{=} \lambda^c \sum_{i=1}^{n} x^i \frac{\partial f}{\partial x^i}(x) = \lambda^c (Vf)(x) \tag{3}$$

but also

$$(Vf)(\lambda x) = V_{\lambda x} f = \sum_{i=1}^{n} (\lambda x^i) \frac{\partial f}{\partial x^i}(\lambda x) \overset{(2)}{=} c\lambda^c f(x). \tag{4}$$

It follows now from (3) and (4) that

$$(Vf)(x) = cf(x) \text{ for every } x \in \mathbb{R}^n \setminus \{(0,0)\}.$$

(c) Since at $p = (0, \dots, 0) \in \mathbb{R}^n$ we have $V_p = (0, \dots, 0)$, the unique maximal integral curve of V starting at p is the constant curve $\gamma_0 \colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto (0, \dots, 0)$.

Now, if $\gamma \colon J \to \mathbb{R}^n$ is a smooth curve, written in standard coordinates as

$$\gamma(t) = \big(\gamma^1(t), \dots, \gamma^n(t)\big),$$

then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\dot{\gamma}^j(t) = \gamma^j(t) \ \text{ for every } 1 \le j \le n,$$

which yields

$$\gamma^j \colon J = \mathbb{R} \to \mathbb{R},\ \gamma^j(t) = c_j e^t, \quad 1 \le j \le n,$$

for some constants $c_j \in \mathbb{R}$. Therefore, the unique maximal integral curve of V starting at $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ is the smooth curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto \big(p^1 e^t, \dots, p^n e^t\big).$$

Observation: The Euler vector field V is a *complete* vector field on $\mathbb{R}^n$.

Remark. The statement from *Exercise* 2(b) is referred to as *the Euler's homogeneous function theorem* in the literature. In fact, it can also be shown that the converse to Euler's homogeneous function theorem holds: *if* $f \in C^\infty\big(\mathbb{R}^n \setminus \{(0,0)\}\big)$ *satisfies* $Vf = cf$, *where* V *is the Euler vector field on* $\mathbb{R}^n$ *and* $c \in \mathbb{R}$, *then* f *is positively homogeneous of degree* c.

Exercise 3:

(a) Consider the open submanifold

$$M := \big\{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\big\} \subseteq \mathbb{R}^2,$$

the map

$$F\colon M \to M,\ (x, y) \mapsto \left(xy,\ \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \quad \text{and} \quad Y := y\,\frac{\partial}{\partial x}$$

on M.

(i) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1}.

(ii) Compute the pushforwards F_*X and F_*Y of X and Y, respectively.

(iii) Compute the Lie brackets $[X, Y]$ and $[F_*X, F_*Y]$.

(iv) Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

(b) Compute the flow of each of the following smooth vector fields on $\mathbb{R}^2$:

(i) $U = y\,\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(ii) $V = x\,\dfrac{\partial}{\partial x} + 2y\,\dfrac{\partial}{\partial y}$.

(iii) $W = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

Solution:

(a)(i) Define the map

$$G\colon M \to M,\ (x, y) \mapsto \left(\sqrt{x/y},\ \sqrt{xy}\right).$$

Note that G is smooth, as the functions

$$(x, y) \in M \mapsto x/y \in \mathbb{R}_{>0} \quad \text{and} \quad (x, y) \in M \mapsto xy \in \mathbb{R}_{>0}$$

are smooth (they are rational polynomials with non-vanishing denominator), and the function $u \in \mathbb{R}_{>0} \mapsto \sqrt{u} \in \mathbb{R}_{>0}$ is smooth as well. Observe also that

$$(G \circ F)(x,y) = \left(\sqrt{\frac{xy}{y/x}}, \sqrt{xy \cdot \frac{y}{x}} \right) = (x,y)$$

and

$$(F \circ G)(x,y) = \left(\sqrt{\frac{x}{y}} \cdot \sqrt{xy}, \frac{\sqrt{xy}}{\sqrt{x/y}} \right) = (x,y)$$

for all $(x,y) \in M$, so F and G are mutually inverse. Hence, F is a diffeomorphism with inverse $F^{-1} = G$. Furthermore, the Jacobian of F is given by

$$DF(x,y) = \begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \quad \text{for all } (x,y) \in M.$$

(a)(ii) The push-forward F_*X of X is the unique vector field on M that is F-related to X, i.e.,

$$dF_{(x,y)}\left(X_{(x,y)}\right) = (F_*X)_{F(x,y)} \quad \text{for all } (x,y) \in M.$$

The matrix of $dF_{(x,y)}$ with respect to the bases provided by $\partial/\partial x$ and $\partial/\partial y$ is precisely $DF(x,y)$. Since X is given with respect to this basis by $(x,y)^T$ and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2xy \\ 0 \end{pmatrix},$$

we obtain that

$$(F_*X)_{F(x,y)} = 2xy \left.\frac{\partial}{\partial x}\right|_{F(x,y)}.$$

Replacing (x,y) by $G(x,y)$ yields

$$(F_*X)_{(x,y)} = 2x \left.\frac{\partial}{\partial x}\right|_{(x,y)},$$

and thus $F_*X = 2x \frac{\partial}{\partial x}$.

Similarly, since Y is given by $(y,0)^T$ with respect to the standard basis and since

$$\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} \cdot \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y^2 \\ -y^2/x^2 \end{pmatrix},$$

we obtain

$$(F_*Y)_{F(x,y)} = y^2 \left.\frac{\partial}{\partial x}\right|_{F(x,y)} - \frac{y^2}{x^2} \left.\frac{\partial}{\partial y}\right|_{F(x,y)}.$$

Replacing (x,y) by $G(x,y)$ yields

$$(F_*Y)_{(x,y)} = xy \left.\frac{\partial}{\partial x}\right|_{(x,y)} - y^2 \left.\frac{\partial}{\partial y}\right|_{(x,y)},$$

and thus $F_*Y = xy \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y}$.

Alternatively, working as in the solution of [*Exercise Sheet* 11*, Exercise* $2(d)(ii)$] and using coordinates (u,v) in the codomain, we compute that

$$DF\big(G(u,v)\big) = \begin{pmatrix} \sqrt{uv} & \sqrt{\dfrac{u}{v}} \\ -\dfrac{\sqrt{uv}}{\frac{u}{v}} & \sqrt{\dfrac{v}{u}} \end{pmatrix},$$

as well as

$$X_{G(u,v)} = \sqrt{\frac{u}{v}}\,\frac{\partial}{\partial x}\bigg|_{G(u,v)} + \sqrt{uv}\,\frac{\partial}{\partial y}\bigg|_{G(u,v)} \quad \text{and} \quad Y_{G(u,v)} = \sqrt{uv}\,\frac{\partial}{\partial x}\bigg|_{G(u,v)},$$

whence

$$(F_*X)_{(u,v)} = 2u\,\frac{\partial}{\partial u}\bigg|_{(u,v)} \quad \text{and} \quad (F_*Y)_{(u,v)} = uv\,\frac{\partial}{\partial u}\bigg|_{(u,v)} - v^2\,\frac{\partial}{\partial v}\bigg|_{(u,v)}.$$

(a)(iii) By [*Exercise Sheet* 11*, Exercise* $4(a)$] we have

$$\begin{aligned}[X,Y] &= \Bigg(\underbrace{\left(x\frac{\partial y}{\partial x} - y\frac{\partial x}{\partial x}\right)}_{=-y} + \underbrace{\left(y\frac{\partial y}{\partial y} - 0\cdot\frac{\partial x}{\partial y}\right)}_{=y}\Bigg)\frac{\partial}{\partial x} + \\ &\quad + \Bigg(\underbrace{\left(x\frac{\partial 0}{\partial x} - y\frac{\partial y}{\partial x}\right)}_{=0} + \underbrace{\left(y\frac{\partial 0}{\partial y} - 0\cdot\frac{\partial y}{\partial y}\right)}_{=0}\Bigg)\frac{\partial}{\partial y} \\ &= 0.\end{aligned}$$

Now, as $[F_*X, F_*Y] = F_*[X,Y]$ by [*Exercise Sheet* 11*, Exercise* $6(b)$], we conclude that

$$[F_*X, F_*Y] = 0.$$

One can also see this with a direct calculation, using part (b) and part (a) of [*Exercise Sheet* 11*, Exercise* 4]; namely, we have

$$\begin{aligned}[F_*X, F_*Y] &= \Bigg(\underbrace{\left(2x\,\frac{\partial(xy)}{\partial x} - xy\,\frac{\partial(2x)}{\partial x}\right)}_{=0} + \underbrace{\left(0\cdot\frac{\partial(xy)}{\partial y} + y^2\,\frac{\partial(2x)}{\partial y}\right)}_{=0}\Bigg)\frac{\partial}{\partial x} \\ &\quad + \Bigg(\underbrace{\left(2x\,\frac{\partial(-y^2)}{\partial x} - xy\,\frac{\partial 0}{\partial x}\right)}_{=0} + \underbrace{\left(0\cdot\frac{\partial(-y^2)}{\partial y} + y^2\cdot\frac{\partial 0}{\partial y}\right)}_{=0}\Bigg)\frac{\partial}{\partial y} \\ &= 0.\end{aligned}$$

(a)(iv) If $\gamma\colon J \to \mathbb{R}^2$ is a smooth curve in M, written in coordinates as $\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big)$, then the condition $\gamma'(t) = Y_{\gamma(t)}$ for γ to be an integral curve of Y translates to the system

$$\frac{d\gamma^1}{dt}(t) = \gamma^2(t) \quad \text{and} \quad \frac{d\gamma^2}{dt}(t) = 0 \quad \text{for } t \in J,$$

whence γ^2 is a constant function and γ^1 is an affine function. Since we also require that $\gamma(0) = (1,1)$, we infer that the unique maximal integral curve of Y starting at $(1,1) \in M$ is the smooth curve

$$\gamma\colon J = (-1, +\infty) \to M \subseteq \mathbb{R}^2,\ t \mapsto \big(t+1, 1\big)$$

whose image is the straight line segment $\{(x,1) \in \mathbb{R}^2 \mid x > 0\}$.

(b) To deal with all the cases we argue exactly as in the solution to *Exercise* 2(c). Thus, we only provide the details for the solution of (ii).

(b)(i) The unique maximal integral curve of U starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(\frac{1}{2}t^2 + p^2 t + p^1,\ t + p^2\big)$, which is a smooth immersion. Hence, the flow of the complete vector field U is the map

$$\theta_U\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t, (x,y)\big) \mapsto \left(\frac{1}{2}t^2 + yt + x,\ t + y\right).$$

(b)(ii) Observe first that the unique maximal integral curve of V starting at $p = (0,0)$ is the constant curve $\gamma_0\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (0,0)$; see *Exercise* 6(a). Now, if $\gamma\colon J \to \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = \big(\gamma^1(t), \gamma^2(t)\big)$, then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\begin{aligned} \dot{\gamma}^1(t) &= \gamma^1(t), \\ \dot{\gamma}^2(t) &= 2\gamma^2(t). \end{aligned}$$

Therefore, there are constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} \gamma^1 &\colon J = \mathbb{R} \to \mathbb{R},\ \gamma^1(t) = c_1 e^t, \\ \gamma^2 &\colon J = \mathbb{R} \to \mathbb{R},\ \gamma^2(t) = c_1 e^{2t}, \end{aligned}$$

so the unique maximal integral curve of V starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(p^1 e^t, p^2 e^{2t}\big)$, which is in passing a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0,0)\}$; see *Exercise* 6(b).

In conclusion, V is a complete vector field on $\mathbb{R}^2$ whose flow is the map

$$\theta_V\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t, (x,y)\big) \mapsto \big(xe^t, ye^{2t}\big).$$

(b)(iii) The unique maximal integral curve of W starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \big(p^1 e^t, p^2 e^{-t}\big)$, which is a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0,0)\}$. Hence, the flow of the complete vector field W is the map

$$\theta_W\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ \big(t, (x,y)\big) \mapsto \big(xe^t, ye^{-t}\big).$$

Exercise 4: Let $\theta\colon \mathbb{R}\times M \to M$ be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of θ is a smooth vector field on M, and that each curve $\theta^{(p)}\colon \mathbb{R} \to M$ is an integral curve of V.

Solution: By definition of the infinitesimal generator, we have

$$V_p = \left.\frac{d}{dt}\right|_{t=0} \theta(t,p) \quad \text{for all } p \in M. \tag{$\star$}$$

First, to show that V is smooth, we apply *Proposition 7.4*: Given an open subset U of M, a smooth real-valued function f on U, and a point $p \in U$, we have

$$\begin{aligned} Vf(p) = V_p f &= \left(\left.\frac{d}{dt}\right|_{t=0} \theta(t,p) \right) f \\ &= \left.\frac{d}{dt}\right|_{t=0} (f\circ\theta)(t,p) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} (f\circ\theta)(t,p). \end{aligned}$$

Since the composite map $f\circ\theta$ is smooth, its partial derivative with respect to t is smooth as well. Thus, $Vf(p)$ depends smoothly on p, which implies that V is smooth.

Next, fix $p \in M$ and $s \in \mathbb{R}$. We have to show that

$$\left.\frac{d}{dt}\right|_{t=s} \theta(t,p) = V_{\theta(s,p)} \overset{(\star)}{=} \left.\frac{d}{dt}\right|_{t=0} \theta\big(t,\theta(s,p)\big).$$

By definition of a flow, we have

$$\theta(t+s,p) = \theta\big(t,\theta(s,p)\big),$$

and by first differentiating the above relation with respect to t and then evaluating at $t=0$ we obtain the required identity.

Exercise 5:

(a) *Naturality of flows*: Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y. Show that if X and Y are F-related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{\;F\;} & N_t \\ {\scriptstyle\theta_t}\downarrow & & \downarrow{\scriptstyle\eta_t} \\ M_{-t} & \xrightarrow{\;F\;} & N_{-t} \end{array}$$

(b) *Diffeomorphism invariance of flows*: Let $F\colon M \to N$ be a diffeomorphism. Show that if $X \in \mathfrak{X}(M)$ and if θ is the flow of X, then the flow of $F_*X \in \mathfrak{X}(N)$ is $\eta_t = F\circ\theta_t\circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Solution:

(a) Denote by $\mathcal{D}_X$ resp. $\mathcal{D}_Y$ the flow domain of θ resp. η. Fix $t \in \mathbb{R}$ and let $p \in M_t$. Then $t \in \mathcal{D}_X^{(p)}$ and $\theta^{(p)}\colon \mathcal{D}_X^{(p)} \to M$ is the unique maximal integral curve of X starting at p.

By [*Exercise Sheet* 11, *Exercise* 2(e)], $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$. Hence, by maximality, we obtain that $\mathcal{D}_X^{(p)} \subseteq \mathcal{D}_Y^{(F(p))}$, and thus $t \in \mathcal{D}_Y^{(F(p))}$, which shows that $F(p) \in N_t$. In conclusion, $F(M_t) \subseteq N_t$.

Finally, we have

$$F \circ \theta_t(p) = F\big(\theta(t,p)\big) \overset{(*)}{=} \eta\big(t, F(p)\big) = \eta_t \circ F(p),$$

where in $(*)$ we again used that $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$ and thus it is equal to $\eta^{(F(p))}$ where its defined (this uses the uniqueness part in the theorem about solutions to a system of ODEs).

(b) Denote by η the flow of F_*X. Applying part (a) to both F and F^{-1} we infer that $F(M_t) \subseteq N_t$ and $F^{-1}(N_t) \subseteq M_t$, so that $F(M_t) = N_t$ for each $t \in \mathbb{R}$. Furthermore, the commutativity of the above diagram shows that $\eta_t = F \circ \theta_t \circ F^{-1}$ for all $t \in \mathbb{R}$.

Exercise 6: Let V be a smooth vector field on a smooth manifold M and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. Prove the following assertions:

(a) If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.

(b) If $p \in M$ is a regular point of V, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]

Solution:

(a) If $V_p = 0$, then the constant curve $\gamma\colon \mathbb{R} \to M,\ t \mapsto p$ is clearly an integral curve of V, so it must be equal to $\theta^{(p)}$ by uniqueness and maximality.

(b) Assume that $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is not a smooth immersion. Then $\theta^{(p)\prime}(s) = 0$ for some $s \in \mathfrak{D}^{(p)}$. Set $q := \theta^{(p)}(s)$ and note that $V_q = 0$, since $\theta^{(p)}$ is an integral curve of V. Thus, q is a singular point of V, and by part (a) we infer that $\mathfrak{D}^{(q)} = \mathbb{R}$ and that $\theta^{(q)}$ is the constant curve $\theta^{(q)}(t) \equiv q$. It follows from *Theorem 7.26*(b) that $\mathfrak{D}^{(p)} = \mathbb{R}$ as well, and for all $t \in \mathbb{R}$ the group law gives

$$\theta^{(p)}(t) = \theta_t(p) = \theta_{t-s}\big(\theta(s,p)\big) = \theta_{t-s}(q) = q.$$

For $t = 0$ we obtain $q = \theta^{(p)}(0) = p$, and hence $\theta^{(p)}(t) \equiv p$ and $V_p = \theta^{(p)\prime}(0) = 0$, which contradicts the assumption that p is a regular point of V. This finishes the proof of (b).

Remark. It can be shown that if V is a smooth vector field on a smooth manifold M and if $p \in M$ is a regular point of V, then there exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\frac{\partial}{\partial s^1}$. Therefore, a flow in a neighborhood of a regular point behaves, up to diffeomorphism, just like translation along parallel coordinate lines in $\mathbb{R}^n$; see *Example 7.23*(1).