Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 12

Exercise 1:
Let V be a smooth vector field on a smooth manifold M, let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma \colon J \to M$ be an integral curve of V. Prove the following assertions:

(a) *Rescaling lemma*: For any $a \in \mathbb{R}$, the curve

$$\widetilde{\gamma} \colon \widetilde{J} \to M,\ t \mapsto \gamma(at)$$

is an integral curve of the vector field $\widetilde{V} := aV$ on M, where $\widetilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma*: For any $b \in \mathbb{R}$, the curve

$$\widehat{\gamma} \colon \widehat{J} \to M,\ t \mapsto \gamma(t+b)$$

is also an integral curve of V on M, where $\widehat{J} := \{t \in \mathbb{R} \mid t + b \in J\}$.

Exercise 2 (*The Euler vector field*)**:**
Consider the *Euler vector field* on $\mathbb{R}^n$, i.e., the vector field V on $\mathbb{R}^n$ whose value at a point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x_1}\bigg|_x + \ldots + x^n \frac{\partial}{\partial x_n}\bigg|_x .$$

(a) Check that V is a smooth vector field on $\mathbb{R}^n$.

(b) Let $c \in \mathbb{R}$ and let $f \colon \mathbb{R}^n \smallsetminus \{0\} \to \mathbb{R}$ be a smooth function which is *positively homogeneous of degree* c, i.e., $f(\lambda x) = \lambda^c f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Prove that $Vf = cf$.

[Hint: Differentiate the above relation with respect to both x^i and λ.]

(c) Compute the integral curves of V.

Exercise 3 (to be submitted by Thursday, 12.12.2024, 16:00):

(a) Consider the open submanifold

$$M := \left\{(x, y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\right\} \subseteq \mathbb{R}^2,$$

the map

$$F\colon M \to M,\ (x, y) \mapsto \left(xy, \frac{y}{x}\right),$$

and the smooth vector fields

$$X := x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} \quad \text{and} \quad Y := y\,\frac{\partial}{\partial x}$$

on M.

(i) Show that F is a diffeomorphism, compute its Jacobian matrix $DF(x, y)$ at an arbitrary point $(x, y) \in M$, and determine its inverse F^{-1}.

(ii) Compute the pushforwards F_*X and F_*Y of X and Y, respectively.

(iii) Compute the Lie brackets $[X, Y]$ and $[F_*X, F_*Y]$.

(iv) Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

(b) Compute the flow of each of the following smooth vector fields on $\mathbb{R}^2$:

(i) $U = y\,\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$.

(ii) $V = x\,\dfrac{\partial}{\partial x} + 2y\,\dfrac{\partial}{\partial y}$.

(iii) $W = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

Exercise 4:
Let $\theta\colon \mathbb{R} \times M \to M$ be a smooth global flow on a smooth manifold M. Show that the infinitesimal generator V of θ is a smooth vector field on M, and that each curve $\theta^{(p)}\colon \mathbb{R} \to M$ is an integral curve of V.

Exercise 5:

(a) *Naturality of flows*: Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y. Show that if X and Y are F-related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t:

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ {\scriptstyle\theta_t}\big\downarrow & & \big\downarrow{\scriptstyle\eta_t} \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

(b) *Diffeomorphism invariance of flows*: Let $F\colon M \to N$ be a diffeomorphism. Show that if $X \in \mathfrak{X}(M)$ and if θ is the flow of X, then the flow of $F_*X \in \mathfrak{X}(N)$ is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Definition. Let V be a (rough) vector field on a smooth manifold M. A point $p \in M$ is called a *singular point* of V if $V_p = 0 \in T_pM$; otherwise, it is called a *regular point* of V.

Exercise 6:
Let V be a smooth vector field on a smooth manifold M and let $\theta\colon \mathfrak{D} \to M$ be the flow generated by V. Prove the following assertions:

(a) If $p \in M$ is a singular point of V, then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.

(b) If $p \in M$ is a regular point of V, then $\theta^{(p)}\colon \mathfrak{D}^{(p)} \to M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]