Homework 8
Introduction to Quantum Information Processing

Exercise 1 *Useful identity for the realization of CNOT gate in experiments*

We consider two qubits (for example: two spins 1/2, or in general two two-level systems) and the following operations:

- Rotations of angle $\frac{\pi}{2}$ around the z axis for each spin:

$$R_1 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_1^z}{2}\right) \text{ et } R_2 = \exp\left(-i\frac{\pi}{2}\frac{\sigma_2^z}{2}\right)$$

- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

- The evolution operator $U = \exp\left(-i\frac{t}{\hbar}\mathcal{H}\right)$ associated two the anisotropic Heisenberg interaction:

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z.$$

We let the system evolve for a time $t = \frac{\pi}{4J}$.

a) Compute the product

$$(I \otimes H)\, U\, (R_1 \otimes R_2)\, (I \otimes H)$$

where I the 2×2 identity matrix.

b) Show that this product is equivalent to a 4×4 CNOT gate defined by

$$\text{CNOT}|x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle$$

Here $x, y \in \{0, 1\}$ and $\oplus$ is addition mod 2. Bit $|x\rangle$ is called the control bit and qbit $|y\rangle$ is caled the target bit.

Exercise 2 *Refocusing technique*

Consider a Hamailtonian of Heisenberg type (anisotropic with only the z type term)

$$\mathcal{H} = \hbar J \sigma_1^z \otimes \sigma_2^z$$

for the interaction of two qubits. Let

$$R_1 = \exp\left(i\pi\frac{\sigma_1^x}{2}\right),$$

the π-pulse (or rotation around x-axis) acting on the first spin. This can be realized for example by RMN techniques as seen in class.

We consider the natural evolution (induced by the interaction) of the two spins during a time interval $\frac{t}{2}$, followed by a π-pulse, followed by the natural time evolution of the two spins during a time $\frac{t}{2}$, and followed again by a π-pulse. The totale evolution is

$$U_{tot} = (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-i\frac{t}{2}\frac{\mathcal{H}}{\hbar}}$$

a) Show the following identity valid for all times t:

$$(R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}}\, (R_1 \otimes \mathbb{I}_2)\, e^{-\frac{it}{\hbar}\mathcal{H}} = \mathbb{I}_1 \otimes \mathbb{I}_2$$

b) In practice $J << 1$. This implies a physical interpretation of this identity. Can you say a few words about it ?