Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 11 – Solutions

Exercise 1: Show that there is a smooth vector field on $\mathbb{S}^2$ which vanishes at exactly one point.

[Hint: Use the stereographic projection [*Exercise Sheet* 2, *Exercise* 5] and consider one of the coordinate vector fields.]

Solution: We view (u, v), resp. $(\widetilde{u}, \widetilde{v})$, as the component functions of σ, resp. $\widetilde{\sigma}$, where

$$\sigma(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = (u, v) \quad \text{and} \quad \widetilde{\sigma}(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) = (\widetilde{u}, \widetilde{v}),$$

so that

$$u = u(\widetilde{u}, \widetilde{v}) = \frac{\widetilde{u}}{\widetilde{u}^2 + \widetilde{v}^2} \quad \text{and} \quad v = v(\widetilde{u}, \widetilde{v}) = \frac{\widetilde{v}}{\widetilde{u}^2 + \widetilde{v}^2},$$

resp.

$$\widetilde{u} = \widetilde{u}(u, v) = \frac{u}{u^2 + v^2} \quad \text{and} \quad \widetilde{v} = \widetilde{v}(u, v) = \frac{v}{u^2 + v^2}.$$

Note that $\widetilde{\sigma} \circ \sigma^{-1}$ is given essentially by the same formula as $\sigma \circ \widetilde{\sigma}^{-1}$ (with the roles of (u, v) and $(\widetilde{u}, \widetilde{v})$ reversed), and thus its Jacobian is essentially the same matrix as the one in [*Exercise Sheet* 10, *Exercise* 1(*d*)] (where everything is now expressed in terms of (u, v) instead of $(\widetilde{u}, \widetilde{v})$); see [*Exercise Sheet* 2, *Exercise* 5].

We now consider the first coordinate vector field $X := \frac{\partial}{\partial u}$ associated with the chart $\left(\mathbb{S}^2 \setminus \{N\}, \sigma\right)$ for $\mathbb{S}^2$. It follows from *Proposition 7.2* that $X = 1\frac{\partial}{\partial u} + 0\frac{\partial}{\partial v}$ is a smooth vector field on $\mathbb{S}^2 \setminus \{N\}$, since its component functions with respect to the smooth coordinate frame $\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}$ are constant, and it is obvious that X does not vanish on $\mathbb{S}^2 \setminus \{N\}$. We claim that X extends to a smooth vector field on the whole $\mathbb{S}^2$ and that it vanishes precisely at the north pole $N \in \mathbb{S}^2$. Indeed, on $\mathbb{S}^2 \setminus \{N, S\}$ we have

$$\begin{aligned} X &= \frac{\partial \widetilde{u}}{\partial u} \frac{\partial}{\partial \widetilde{u}} + \frac{\partial \widetilde{v}}{\partial u} \frac{\partial}{\partial \widetilde{v}} = \frac{v^2 - u^2}{(u^2 + v^2)^2} \frac{\partial}{\partial \widetilde{u}} + \frac{-2uv}{(u^2 + v^2)^2} \frac{\partial}{\partial \widetilde{v}} \\ &= \left(\widetilde{v}^2 - \widetilde{u}^2\right) \frac{\partial}{\partial \widetilde{u}} + \left(-2\widetilde{u}\widetilde{v}\right) \frac{\partial}{\partial \widetilde{v}}. \end{aligned}$$

Since $N=(0,0,1)\in\mathbb{S}^2$ corresponds under $\widetilde{\sigma}$ to $(\widetilde{u},\widetilde{v})=(0,0)\in\mathbb{R}^2$, we infer that X can be extended to a vector field on $\mathbb{S}^2$ by defining its value at N to be zero; namely,

$$X\colon \mathbb{S}^2\to T\mathbb{S}^2,\ p\mapsto \begin{cases} \frac{\partial}{\partial u}\big|_p, & \text{if } p\neq N,\\ 0, & \text{if } p=N. \end{cases}$$

The above expression for X also shows that its component functions with respect to the smooth coordinate frame $\left\{\frac{\partial}{\partial \widetilde{u}},\frac{\partial}{\partial \widetilde{v}}\right\}$ associated with the chart $\left(\mathbb{S}^2\setminus\{S\},\widetilde{\sigma}\right)$ are smooth, and hence X is smooth (also) on $\mathbb{S}^2\setminus\{S\}$ by *Proposition 7.2*. Therefore, X is a smooth vector field on $\mathbb{S}^2$ which vanishes only at the north pole N of $\mathbb{S}^2$, as claimed.

Exercise 2:

(a) Let $F\colon M\to N$ be a smooth map. Let $X\in\mathfrak{X}(M)$ and $Y\in\mathfrak{X}(N)$. Show that X and Y are F-related if and only if for every smooth real-valued function f defined on an open subset of N, we have
$$X(f\circ F)=(Yf)\circ F.$$

(b) Consider the smooth map
$$F\colon \mathbb{R}\to\mathbb{R}^2,\ t\mapsto(\cos t,\sin t)$$
and the smooth vector fields
$$X=\frac{d}{dt}\in\mathfrak{X}(\mathbb{R}) \quad\text{and}\quad Y=x\,\frac{\partial}{\partial y}-y\,\frac{\partial}{\partial x}\in\mathfrak{X}(\mathbb{R}^2).$$
Show that X and Y are F-related.

(c) Let $F\colon M\to N$ be a diffeomorphism and let $X\in\mathfrak{X}(M)$. Prove that there exists a unique smooth vector field Y on N that is F-related to X. The vector field Y is denoted by F_*X and is called the *pushforward of X by F*.

(d) Consider the open submanifolds
$$M\coloneqq\left\{(x,y)\in\mathbb{R}^2\mid y>0\ \text{ and }\ x+y>0\right\}\subseteq\mathbb{R}^2$$
and
$$N\coloneqq\left\{(u,v)\in\mathbb{R}^2\mid u>0\ \text{ and }\ v>0\right\}\subseteq\mathbb{R}^2$$
and the map
$$F\colon M\to N,\ (x,y)\mapsto\left(x+y,\frac{x}{y}+1\right).$$

(i) Show that F is a diffeomorpism and compute its inverse F^{-1}.

(ii) Compute the pushforward F_*X of the following smooth vector field X on M:
$$X_{(x,y)}=y^2\frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

(e) *Naturality of integral curves*: Let $F\colon M \to N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if F takes integral curves of X to integral curves of Y.

Solution:

(a) For any point $p \in M$ and any smooth real-valued function f defined on an open neighborhood of $F(p)$ we have

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)(f)$$

and

$$\big((Yf) \circ F\big)(p) = (Yf)\big(F(p)\big) = Y_{F(p)}f.$$

Therefore, X and Y are F-related (i.e., $dF_p(X_p) = Y_{F(p)}$ for every $p \in M$) if and only if for every smooth real-valued function f defined on an open subset of N it holds that $X(f \circ F) = (Yf) \circ F$.

(b) *1st way*: We prove the claim using the definition of F-related vector fields. To this end, recall that the differential of F at an arbitrary point $t \in \mathbb{R}$ is represented (with respect to the bases $\{d/dt\big|_t\}$ for $T_t\mathbb{R} \cong \mathbb{R}$ and $\{\partial/\partial x\big|_{F(t)}, \partial/\partial y\big|_{F(t)}\}$ for $T_{F(t)}\mathbb{R}^2 \cong \mathbb{R}^2$) by the Jacobian of F at t, which is the 2×1-matrix

$$\begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}.$$

Hence,

$$dF_t(X_t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \cdot (1) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} = -\sin(t)\frac{\partial}{\partial x}\bigg|_{F(t)} + \cos(t)\frac{\partial}{\partial y}\bigg|_{F(t)} = Y_{F(t)}$$

for any $t \in \mathbb{R}$, which shows that X and Y are F-related.

2nd way: We may alternatively prove the assertion using (a) as follows: For every smooth real-valued function $f = f(x, y)$ defined on an open subset of $\mathbb{R}^2$ and for any $t \in \mathbb{R}$ we have

$$\begin{aligned} X(f \circ F)(t) = X_t(f \circ F) &= \frac{d}{dt}\bigg|_t (f \circ F) \\ &= \left(\frac{\partial f}{\partial x}\big(F(t)\big),\ \frac{\partial f}{\partial y}\big(F(t)\big)\right) \cdot \big(F_1'(t),\ F_2'(t)\big)^T \\ &= -\sin(t)\,\frac{\partial f}{\partial x}\big(F(t)\big) + \cos(t)\,\frac{\partial f}{\partial y}\big(F(t)\big) \end{aligned}$$

and

$$\begin{aligned} \big((Yf) \circ F\big)(t) = (Yf)\big(F(t)\big) &= Y_{F(t)}f \\ &= \left(\cos(t)\,\frac{\partial}{\partial y}\bigg|_{F(t)} - \sin(t)\,\frac{\partial}{\partial x}\bigg|_{F(t)}\right) f \\ &= \cos(t)\,\frac{\partial f}{\partial y}\big(F(t)\big) - \sin(t)\,\frac{\partial f}{\partial x}\big(F(t)\big). \end{aligned}$$

It follows from part (a) that X and Y are F-related.

(c) Since F is a diffeomorphism, we may define the following rough vector field on N:

$$Y\colon N \to TN,\ q \mapsto dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big).$$

It is clear that this is the unique (rough) vector field on N that is F-related to X. We now observe that Y is the composition of the following smooth maps (see also [*Exercise Sheet* 5, *Exercise* 4(*a*)]):

$$N \xrightarrow{F^{-1}} M \xrightarrow{X} TM \xrightarrow{dF} TN,$$

so it is smooth by [*Exercise Sheet* 3, *Exercise* 3(*e*)].

Remark. Given a diffeomorphism $F\colon M \to N$, the pushforward of any $X \in \mathfrak{X}(M)$ by F is defined explicitly by the formula

$$(F_*X)_q = dF_{F^{-1}(q)}\big(X_{F^{-1}(q)}\big),$$

as already demonstrated in the proof of (c) above. As long as the inverse map F^{-1} of F can be computed explicitly, the pushforward of a smooth vector field can be computed directly from this formula. This observation will be applied in (d) below.

(d) It is straightforward to check that the inverse of F is given by the formula

$$F^{-1}(u,v) = \left(u - \frac{u}{v}, \frac{u}{v}\right).$$

The differential of F at an arbitrary point $(x,y) \in M$ is represented by the Jacobian of F at (x,y), given by

$$DF(x,y) = \begin{pmatrix} 1 & 1 \\ \dfrac{1}{y} & -\dfrac{x}{y^2} \end{pmatrix},$$

and thus $dF_{F^{-1}(u,v)}$ is represented by the matrix

$$DF\left(u - \frac{u}{v}, \frac{u}{v}\right) = \begin{pmatrix} 1 & 1 \\ \dfrac{v}{u} & \dfrac{v - v^2}{u} \end{pmatrix}.$$

For any $(u,v) \in N$ we have

$$X_{F^{-1}(u,v)} = X_{\left(u-\frac{u}{v}, \frac{u}{v}\right)} = \frac{u^2}{v^2} \frac{\partial}{\partial x}\bigg|_{\left(u-\frac{u}{v}, \frac{u}{v}\right)}.$$

Therefore, we obtain

$$(F_*X)_{(u,v)} = \frac{u^2}{v^2} \frac{\partial}{\partial u}\bigg|_{(u,v)} + \frac{u}{v} \frac{\partial}{\partial v}\bigg|_{(u,v)}.$$

(e) Assume first that X and Y are F-related. Let γ be an integral curve of X. By definition and by [*Exercise Sheet* 4, *Exercise* 5(*b*)] we obtain

$$(F \circ \gamma)'(t) = dF_{\gamma(t)}\big(\gamma'(t)\big) = dF_{\gamma(t)}\big(X_{\gamma(t)}\big) = Y_{F(\gamma(t))} = Y_{(F\circ\gamma)(t)},$$

which shows that $F \circ \gamma$ is an integral curve of Y.

Assume now that F takes integral curves of X to integral curves of Y. Let $p \in M$ and let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be an integral curve of X starting at p. Then $\gamma(0) = p$ and $\gamma'(0) = X_p$. Moreover, by assumption, $F \circ \gamma\colon (-\varepsilon, \varepsilon) \to N$ is an integral curve of Y starting at $F(p)$, so $Y_{(F\circ\gamma)(0)} = (F \circ \gamma)'(0)$. Therefore, by [*Exercise Sheet* 4*, Exercise* 5(*b*)] we obtain

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p\big(\gamma'(0)\big) = dF_p\big(X_p\big).$$

Since $p \in M$ was arbitrary, we conclude that X and Y are F-related.

Exercise 3: Let M be a smooth manifold and let X and Y be two smooth vector fields on M. Show that the Lie bracket $[X, Y]$ of X and Y, defined by

$$[X, Y]\colon C^\infty(M) \to C^\infty(M),\ f \mapsto XYf - YXf,$$

is also a smooth vector field on X.

Solution: The $\mathbb{R}$-linearity of $[X, Y]$ follows immediately from the $\mathbb{R}$-linearity of both X and Y. Let us now verify the product rule:

$$\begin{aligned}
[X, Y](fg) &= XY(fg) - YX(fg) \\
&= X\big(fYg + gYf\big) - Y\big(fXg + gXf\big) \\
&= \big(\cancel{Xf\,Yg} + f\,XYg + Xg\,Yf + g\,XYf\big) - \\
&\qquad \big(\cancel{Yf\,Xg} + f\,YXg + Yg\,Xf + g\,YXf\big) \\
&= f\big(XYg - YXg\big) + g\big(XYf - YXf\big) \\
&= f\,[X, Y]g + g\,[X, Y]f.
\end{aligned}$$

In conclusion, $[X, Y]$ is a smooth vector field on M by *Proposition 7.5*.

Exercise 4: Let M be a smooth n-manifold and let $X, Y \in \mathfrak{X}(M)$.

(a) *Coordinate formula for the Lie bracket*: Let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for X and Y, respectively, in terms of some smooth local coordinates (x^i) for M. Show that the Lie bracket $[X, Y]$ has the following coordinate expression:

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $\big(U, (x^i)\big)$ for M.

(c) Assume now that $M = \mathbb{R}^3$,

$$X = x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\,\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial z},$$

and compute the Lie bracket $[X, Y]$.

Solution:

(a) Denote by $U \subseteq M$ the coordinate domain. For any $f \in C^\infty(U)$ we have

$$\begin{aligned}
[X,Y](f) = XY(f) - YX(f) &= X\left(\sum_j Y^j \frac{\partial f}{\partial x^j}\right) - Y\left(\sum_i X^i \frac{\partial f}{\partial x^i}\right) \\
&= \sum_j \left[X(Y^j)\frac{\partial f}{\partial x^j} + Y^j X\left(\frac{\partial f}{\partial x^j}\right)\right] - \sum_i \left[Y(X^i)\frac{\partial f}{\partial x^i} + X^i Y\left(\frac{\partial f}{\partial x^i}\right)\right] \\
&= \sum_{j,i} \left[X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + Y^j X^i \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right)\right] - \sum_{i,j} \left[Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} + X^i Y^j \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)\right] \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j} - \sum_{i,j}(X^i Y^j) \underbrace{\left[\frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right)\right]}_{=0} \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}.
\end{aligned}$$

(b) Recall that the component functions of each coordinate vector field $\partial/\partial x^j$ in the coordinate frame $(\partial/\partial x^i)$ associated with the smooth chart $\big(U, (x^i)\big)$ are constant, so it follows immediately from (a) that

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

(c) By part (a) we obtain

$$\begin{aligned}
[X,Y] &= \big((x\cdot 0 - 1\cdot 1) + (1\cdot 0 - 0\cdot 0) + (x(y+1)\cdot 0 - 1\cdot 0)\big)\,\frac{\partial}{\partial x} \\
&\quad + \big((x\cdot 0 - 1\cdot 0) + (x\cdot 0 - 0\cdot 0) + (x(y+1)\cdot 0 - y\cdot 0)\big)\,\frac{\partial}{\partial y} \\
&\quad + \big((x\cdot 0 - 1\cdot (y+1)) + (1\cdot 1 - 0\cdot x) + (x(y+1)\cdot 0 - y\cdot 0)\big)\,\frac{\partial}{\partial z} \\
&= -\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.
\end{aligned}$$

Exercise 5 (*Properties of the Lie bracket*)**:** Let M be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
[aX + bY, Z] &= a[X,Z] + b[Y,Z], \\
[Z, aX + bY] &= a[Z,X] + b[Z,Y].
\end{aligned}$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Solution:

(a) We first make the following observation: given $\lambda, \mu \in \mathbb{R}$ and $U, V, W \in \mathfrak{X}(M)$, for any $f \in C^\infty(M)$ it holds that

$$(\lambda V + \mu W)\, Uf = \lambda VUf + \mu WUf \quad \text{and} \quad U(\lambda V + \mu W)f = \lambda UVf + \mu UWf.$$

Indeed, for any $p \in M$ we have

$$\begin{aligned}\big((\lambda V + \mu W)\, Uf\big)(p) &= (\lambda V + \mu W)_p(Uf) = (\lambda V_p + \mu W_p)(Uf) \\ &= \lambda V_p(Uf) + \mu W_p(Uf) = \lambda V(Uf)(p) + \mu W(Uf)(p) \\ &= \big(\lambda VUf + \mu WUf\big)(p),\end{aligned}$$

which yields the first equality above, while the second one is obtained analogously.

Now, given $a, b \in \mathbb{R}$, using the previous observation, for any $f \in C^\infty(M)$ we have

$$\begin{aligned}[aX + bY, Z](f) &= \big(aX + bY\big)Zf - Z\big(aX + bY\big)f \\ &= aXZf + bYZf - aZXf - bZYf \\ &= a\big(XZf - ZXf\big) + b\big(YZf - ZYf\big) \\ &= a[X, Z](f) + b[Y, Z](f) \\ &= \big(a[X, Z] + b[Y, Z]\big)(f),\end{aligned}$$

which yields the first part of the statement, while the second one is obtained similarly.

(b) For any $f \in C^\infty(M)$ we have

$$[X, Y](f) = XYf - YXf = -\big(YXf - XYf\big) = -[Y, X](f),$$

which yields the statement.

(c) By expanding all the brackets and using linearity we obtain

$$\begin{aligned}&\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = \\ &\qquad = X[Y, Z] - [Y, Z]X + Y[Z, X] - [Z, X]Y + Z[X, Y] - [X, Y]Z \\ &\qquad = XYZ - XZY - YZX + ZYX + YZX - YXZ - ZXY + XZY + \\ &\qquad\quad + ZXY - ZYX - XYZ + YXZ \\ &\qquad = 0.\end{aligned}$$

(d) We first make the following observation: if $V \in \mathfrak{X}(M)$ and $s, t \in C^\infty(M)$, then

$$(sV)\,h = s\,(Vh) \quad \text{(as smooth functions on } M\text{)},$$

since for any $p \in M$ we have

$$\big((sV)\,h\big)(p) = (sV)_p\,h = \big(s(p)V_p\big)h = s(p)V_ph = s(p)(Vh)(p) = \big(s\,(Vh)\big)(p).$$

Now, fix $f, g \in C^\infty(M)$. Using the previous observation and the fact that smooth vector fields are derivations of $C^\infty(M)$ by *Proposition 7.5*, for any $h \in C^\infty(M)$ we have

$$\begin{aligned}
[fX, gY](h) &= (fX)(gY)(h) - (gY)(fX)(h) \\
&= (fX)\big(g(Yh)\big) - (gY)\big(f(Xh)\big) \\
&= g(fX)(Yh) + (Yh)(fX)(g) - f(gY)(Xh) - (Xh)(gY)(f) \\
&= gf\big(X(Yh)\big) + f(Xg)(Yh) - fg\big(Y(Xh)\big) - g(Yf)(Xh) \\
&= fg\big((XY - YX)(h)\big) + (fXg)Y(h) - (gYf)X(h) \\
&= \big(fg[X,Y] + (fXg)Y - (gYf)X\big)(h),
\end{aligned}$$

whence the desired relation.

Remark. A *Lie algebra (over* $\mathbb{R}$*)* is an $\mathbb{R}$-vector space $\mathfrak{g}$ endowed with a map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* and usually denoted by $(X, Y) \mapsto [X, Y]$, which satisfies the following properties for all $X, Y, Z \in \mathfrak{g}$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$\begin{aligned}
[aX + bY, Z] &= a[X, Z] + b[Y, Z], \\
[Z, aX + bY] &= a[Z, X] + b[Z, Y].
\end{aligned}$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

According to *Exercise* 4, the infinite-dimensional $\mathbb{R}$-vector space $\mathfrak{X}(M)$ of all smooth vector fields on a smooth manifold M is a Lie algebra under the Lie bracket. Here are two more examples of Lie algebras:

(1) The $\mathbb{R}$-vector space $M_n(\mathbb{R})$ of real $n \times n$ matrices equipped with the *commutator bracket* $[A, B] := AB - BA$ becomes an n^2-dimensional Lie algebra, which is denoted by $\mathfrak{gl}(n, \mathbb{R})$.

(2) If V is an $\mathbb{R}$-vector space, then the $\mathbb{R}$-vector space of all linear maps from V to itself equipped with the *commutator bracket* $[L_1, L_2] := L_1 \circ L_2 - L_2 \circ L_1$ becomes a Lie algebra, which is denoted by $\mathfrak{gl}(V)$.

Under our usual identification of $n \times n$ matrices with linear maps from $\mathbb{R}^n$ to itself, $\mathfrak{gl}(\mathbb{R}^n)$ is the same as $\mathfrak{gl}(n, \mathbb{R})$.

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Exercise 6: Let $F\colon M \to N$ be a smooth map.

(a) *Naturality of the Lie bracket*: Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be smooth vector fields such that X_i is F-related to Y_i for $i \in \{1,2\}$. Show that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

(b) *Pushforwards of Lie brackets*: Assume that F is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

Solution:

(a) Since X_i is F-related to Y_i for $i \in \{1,2\}$, by *Exercise* 2(a) we infer that for every smooth real-valued function f defined on an open subset of N we have

$$X_1(f \circ F) = (Y_1 f) \circ F \quad \text{and} \quad X_2(f \circ F) = (Y_2 f) \circ F.$$

Therefore,

$$\begin{aligned} [X_1, X_2](f \circ F) &= X_1 X_2 (f \circ F) - X_2 X_1 (f \circ F) \\ &= X_1\big((Y_2 f) \circ F\big) - X_2\big((Y_1 f) \circ F\big) \\ &= \big(Y_1(Y_2 f)\big) \circ F - \big(Y_2(Y_1 f)\big) \circ F \\ &= \big([Y_1, Y_2](f)\big) \circ F, \end{aligned}$$

and thus *Exercise* 1(a) implies that $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

(b) Follows immediately from part (a) and from *Exercise* 2(c).

Exercise 7:

(a) *Restricting smooth vector fields to submanifolds*: Let M be a smooth manifold, let S be an immersed submanifold of M, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

  (i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is ι-related to Y, then $Y \in \mathfrak{X}(M)$ is tangent to S.

  (ii) If $Y \in \mathfrak{X}(M)$ is tangent to S, then there is a unique smooth vector field on S, denoted by $Y|_S$, which is ι-related to Y.

  [Hint: Determine first the candidate vector field on S and then use *Theorem 5.6* and *Proposition 5.17* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds*: Let M be a smooth manifold and let S be an immersed submanifold of M. If Y_1 and Y_2 are smooth vector fields on M that are tangent to S, then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to S.

Solution:

(a)(i) Since X is ι-related to Y, it holds that $Y_p = d\iota_p(X_p)$ for all $p \in S$, which means that $Y_p \in T_pS$ for all $p \in S$, i.e., Y is tangent to S.

(a)(ii) Since by hypothesis we have $Y_p \in T_pS \cong d\iota_p(T_pS)$ for all $p \in S$, we may define a rough vector field $X\colon S \to TS$ by requiring that, for any $p \in S$, its value $X_p \in T_pS$ is the unique vector such that $d\iota_p(X_p) = Y_p$. By the injectivity of $d\iota_p$, it is clear that X is unique, and that it is ι-related to Y, so it remains to show that X is smooth. To this end, let $p \in S$ be arbitrary. By *Proposition 5.17* there is an open neighborhood V of p in S such that V is embedded in M. By *Theorem 5.6* there exists a smooth chart $\big(U, (x^i)\big)$ for M such that $V \cap U$ is a k-slice in U – we may assume that $V \cap U$ is the k-slice given by $x^{k+1} = \ldots = x^n = 0$ – and $(x^1, \ldots, x^k)$ are local coordinates for S in $V \cap U$. Consider the coordinate representation

$$Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i}$$

of Y on U. Since Y is tangent to S, by *Proposition 7.8* (evaluating the above expression at the coordinate function x^i with $i > k$) we infer that $Y^{k+1} = \ldots = Y^n = 0$ on $V \cap U$. Therefore,

$$X = \sum_{1 \leq i \leq k} Y^i|_{U\cap V} \frac{\partial}{\partial x^i}\bigg|_{U\cap V}$$

is the coordinate representation of X on $V \cap U$, and each component $Y^i|_{U\cap V}$ is smooth by [*Exercise Sheet* 8*, Exercise* 5(*a*)], so X is smooth on $U \cap V$, and we are done.

(b) Since $Y_1, Y_2 \in \mathfrak{X}(M)$ are tangent to S, by part (a)(ii) there exist (unique) smooth vector fields W_1, W_2 on S such that W_j is ι-related to Y_j for each $j \in \{1, 2\}$, where $\iota\colon S \hookrightarrow M$ is the inclusion map. But now *Exercise* 6(a) implies that $[W_1, W_2] \in \mathfrak{X}(S)$ is ι-related to $[Y_1, Y_2] \in \mathfrak{X}(M)$, and thus $[Y_1, Y_2]$ is tangent to S by part (a)(i).