Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

---

# Exercise Sheet 11

---

**Exercise 1 (to be submitted by Thursday, 05.12.2024, 16:00):**
Show that there is a smooth vector field on $\mathbb{S}^2$ which vanishes at exactly one point.

[Hint: Use the stereographic projection [*Exercise Sheet* 2*, Exercise* 5] and consider one of the coordinate vector fields.]

**Exercise 2:**

(a) Let $F\colon M \to N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that $X$ and $Y$ are $F$-related if and only if for every smooth real-valued function $f$ defined on an open subset of $N$, we have
$$X(f \circ F) = (Yf) \circ F.$$

(b) Consider the smooth map
$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$
and the smooth vector fields
$$X = \frac{d}{dt} \in \mathfrak{X}(\mathbb{R}) \quad \text{and} \quad Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2).$$
Show that $X$ and $Y$ are $F$-related.

(c) Let $F\colon M \to N$ be a diffeomorphism and let $X \in \mathfrak{X}(M)$. Prove that there exists a unique smooth vector field $Y$ on $N$ that is $F$-related to $X$. The vector field $Y$ is denoted by $F_*X$ and is called the *pushforward of $X$ by $F$*.

(d) Consider the open submanifolds
$$M \coloneqq \big\{(x,y) \in \mathbb{R}^2 \mid y > 0 \ \text{ and } \ x + y > 0\big\} \subseteq \mathbb{R}^2$$
and
$$N \coloneqq \big\{(u,v) \in \mathbb{R}^2 \mid u > 0 \ \text{ and } \ v > 0\big\} \subseteq \mathbb{R}^2$$
and the map
$$F\colon M \to N,\ (x,y) \mapsto \big(x+y, \frac{x}{y} + 1\big).$$

(i) Show that $F$ is a diffeomorpism and compute its inverse $F^{-1}$.

(ii) Compute the pushforward $F_*X$ of the following smooth vector field $X$ on $M$:

$$X_{(x,y)} = y^2 \frac{\partial}{\partial x}\bigg|_{(x,y)}.$$

(e) *Naturality of integral curves*: Let $F\colon M \to N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are $F$-related if and only if $F$ takes integral curves of $X$ to integral curves of $Y$.

**Exercise 3:**
Let $M$ be a smooth manifold and let $X$ and $Y$ be two smooth vector fields on $M$. Show that the Lie bracket of $X$ and $Y$, defined by

$$[X,Y]\colon C^\infty(M) \to C^\infty(M),\ f \mapsto XYf - YXf,$$

is also a smooth vector field on $X$.

**Exercise 4:**
Let $M$ be a smooth $n$-manifold and let $X, Y \in \mathfrak{X}(M)$.

(a) *Coordinate formula for the Lie bracket*: Let

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$$

be the coordinate expressions for $X$ and $Y$, respectively, in terms of some smooth local coordinates $(x^i)$ for $M$. Show that the Lie bracket $[X,Y]$ has the following coordinate expression:

$$[X,Y] = \sum_{j=1}^{n}\sum_{i=1}^{n} \left( X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

(b) Compute the Lie brackets $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$ of the coordinate vector fields $\partial/\partial x^i$ in any smooth chart $\big(U, (x^i)\big)$ for $M$.

(c) Assume now that

$$M = \mathbb{R}^3, \quad X = x\,\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\,\frac{\partial}{\partial z} \quad \text{and} \quad Y = \frac{\partial}{\partial x} + y\,\frac{\partial}{\partial z},$$

and compute the Lie bracket $[X,Y]$.

**Exercise 5** (*Properties of the Lie bracket*)**:**
Let $M$ be a smooth manifold. Show that the Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(a) *Bilinearity*: For all $a, b \in \mathbb{R}$ we have

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

(b) *Antisymmetry*:

$$[X, Y] = -[Y, X].$$

(c) *Jacobi identity*:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0.$$

(d) For all $f, g \in C^\infty(M)$ we have

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

**Exercise 6:**
Let $F\colon M \to N$ be a smooth map.

(a) *Naturality of the Lie bracket*: Let $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ be smooth vector fields such that $X_i$ is $F$-related to $Y_i$ for $i \in \{1, 2\}$. Show that $[X_1, X_2]$ is $F$-related to $[Y_1, Y_2]$.

(b) *Pushforwards of Lie brackets*: Assume that $F$ is a diffeomorphism and consider $X_1, X_2 \in \mathfrak{X}(M)$. Show that $F_*[X_1, X_2] = [F_*X_1, F_*X_2]$.

**Exercise 7:**

(a) *Restricting smooth vector fields to submanifolds*: Let $M$ be a smooth manifold, let $S$ be an immersed submanifold of $M$, and let $\iota\colon S \hookrightarrow M$ be the inclusion map. Prove the following assertions:

   (i) If $Y \in \mathfrak{X}(M)$ and if there is $X \in \mathfrak{X}(S)$ that is $\iota$-related to $Y$, then $Y \in \mathfrak{X}(M)$ is tangent to $S$.

   (ii) If $Y \in \mathfrak{X}(M)$ is tangent to $S$, then there is a unique smooth vector field on $S$, denoted by $Y|_S$, which is $\iota$-related to $Y$.

   [Hint: Determine first the candidate vector field on $S$ and then use *Theorem 5.6* and *Proposition 5.17* to show that it is smooth.]

(b) *Lie brackets of smooth vector fields tangent to submanifolds*: Let $M$ be a smooth manifold and let $S$ be an immersed submanifold of $M$. If $Y_1$ and $Y_2$ are smooth vector fields on $M$ that are tangent to $S$, then show that their Lie bracket $[Y_1, Y_2]$ is also tangent to $S$.