Midterm exam

Exercise 1. Quiz. (18 points) Answer each yes/no question below (1 pt) and provide a short justification (proof or counter-example) for your answer (2 pts).

a) Let $(\Omega, \mathcal{F})$ be a measurable space and let $g\colon \Omega \to \Omega'$ be some function onto the set Ω'. We construct a collection $\mathcal{F}'$ of subsets of Ω' in the following way:

$$\mathcal{F}' = \{F' \subseteq \Omega' : g(F) = F', \text{ for some } F \in \mathcal{F}\}.$$

where $g(F) = F'$ means that $F' \subset \Omega'$ is the image of set the F. Does $(\Omega', \mathcal{F}')$ always form a measurable space?

b) Let $\{A_1, A_2, A_3\}$ be a collection of pairwise independent events on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $B = A_1 \cup A_2$. Are the events B and A_3 also independent?

c) Let X, Y be two square integrable random variables. For two constants $a, b \in \mathbb{R}$, is it true that $\text{Cov}(aX + b, Y) = a\text{Cov}(X, Y) + b$?

d) Let X be a standard Gaussian distribution i.e., $X \sim \mathcal{N}(0, 1)$. Let Y be a random variable defined as follows:

$$Y = \begin{cases} X, & |X| < a \\ -X, & |X| \geq a \end{cases}$$

where $a > 0$ is a constant. Then, is Y a standard Gaussian random variable?

e) Let X, Y be the same as in part **d)**. Is $Z = (X, Y)$ a Gaussian random vector?

f) Let $U \sim \text{Uniform}[0, 1]$ and define

$$X_n = n 1_{\left[0, \frac{1}{n^2}\right]}(U), \quad n = 1, 2, \ldots$$

Does X_n converge in quadratic mean to some random variable X?

Exercise 2. (12 points) Let X, Y be two independent random variable with $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$. Recall that the pmf of a $\text{Binomial}(n, p)$ random variable X is given by $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

a) Compute the probability mass function of $X + Y$ using convolution. *Hint: Recall Vandermonde's identity:* $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$

b) Compute the characteristic functions of X and Y. *Hint: Recall the Binomial identity* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

c) Check your result for part **a)**. Compute the probability mass function of $X + Y$ using the characteristic functions in part **b)**.

Exercise 3. (12 points) For this problem, we remind you of basic definitions related to vector spaces in the attached appendix.

a) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{V}$ be a space of all $\mathcal{F}$-measurable random variables on this space. Show that $\mathcal{V}$ forms a vector space over the reals.

In the interest of time you may skip axioms 1-2 and 5-8 (in the appendix) as these are trivial.

With regard to axioms 3 and 4 (in the appendix): Describe the zero-element of this vector space. If $X \in \mathcal{V}$ is an $\mathcal{F}$-measurable random variable, what is its additive inverse?

b) Suppose Ω is finite. Describe a basis for $\mathcal{V}$.

c) Suppose Ω is finite. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field and let $\mathcal{W}$ be the space of all $\mathcal{G}$-measurable random variables. Then, is $\mathcal{W}$ a linear subspace of $\mathcal{V}$? Why or why not?

Exercise 4. (16 points) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ be an $\mathcal{F}$-measurable random variable.

a) Assume the p^{th}-moment of X is finite i.e., $\mathbb{E}(|X|^p) < \infty$ for some $p \in \mathbb{N}$. Then, show that $\forall$ $p' \in [0, p]$, we have $\mathbb{E}(|X|^{p'}) < \infty$.

b) Assume that X is integrable, e.i. $\mathbb{E}(|X|) < \infty$. Show that $\exp(\mathbb{E}(\log |X|)) \leq \mathbb{E}(|X|)$. Conclude from this that for any $y_1, \ldots, y_k \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_k \geq 0$ such that $\sum_{i=1}^{k} \alpha_k = 1$, we have the following inequality:

$$|y_1|^{\alpha_1} |y_2|^{\alpha_2} \ldots |y_k|^{\alpha_k} \leq \alpha_1 |y_1| + \alpha_2 |y_2| + \cdots + \alpha_k |y_k|.$$

In class, we have seen the Cauchy-Schwarz inequality i.e., $\mathbb{E}(|XY|) \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$. In the next part, we will extend it to higher moments of the random variables X and Y by proving a result known as Holder's inequality. That is, let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We will show that

$$\mathbb{E}(|XY|) \leq (\mathbb{E}(|X|^p))^{1/p} (\mathbb{E}(|Y|^q))^{1/q}.$$

c) First, show that for any $x, y \in \mathbb{R}$ and $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have $|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}$. This result is well-known as Young's inequality for products.

d) Finally, let X, Y be $\mathcal{F}$-measurable random variables with $\mathbb{E}(|X|^{\max\{p,q\}}), \mathbb{E}(|Y|^{\max\{p,q\}}) < \infty$. Show that

$$\frac{\mathbb{E}(|XY|)}{(\mathbb{E}(|X|^p))^{1/p} (\mathbb{E}(|Y|^q))^{1/q}} \leq 1,$$

from which Holder's inequality follows immediately.

Appendix for Exercise 3:

$\mathcal{V}$ is a vector space over $\mathbb{R}$ if the eight following axioms must be satisfied for every $u, v, w \in \mathcal{V}$, and $a, b \in \mathbb{R}$.

1. **Associativity of vector addition:** $u + (v + w) = (u + v) + w$
2. **Commutativity of vector addition:** $u + v = v + u$
3. **Identity element of vector addition:** There exists an element $\mathbf{0} \in \mathcal{V}$, called the zero vector, such that $v + \mathbf{0} = v$ for all $v \in \mathcal{V}$.
4. **Inverse elements of vector addition:** For every $v \in \mathcal{V}$, there exists an element $-v \in \mathcal{V}$, called the additive inverse of v, such that $v + (-v) = \mathbf{0}$.
5. **Compatibility of scalar multiplication with field multiplication:** $a(bv) = (ab)v$.
6. **Identity element of scalar multiplication:** $1v = v$
7. **Distributivity of scalar multiplication with respect to vector addition:** $a(u + v) = au + av$
8. **Distributivity of scalar multiplication with respect to field addition:** $(a + b)v = av + bv$

Given $\{v_1, \dots, v_k\} \in \mathcal{V}$, a *linear combination* of elements $\{v_1, \dots, v_k\}$ is an element of the form

$$a_1 v_1 + \cdots + a_k v_k$$

for some $a_1, \dots, a_k \in \mathbb{R}$.

The elements of a subset $\{v_1, \dots, v_k\}$ are said to be *linearly independent* if no element v_i can be written as a linear combination of $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k\}$.

The *span* of $\{v_1, \dots, v_k\}$ is the set of all linear combinations of elements of $\{v_1, \dots, v_k\}$.

A subset of a vector space is a *basis* if its elements are linearly independent and span the vector space.

A *linear subspace* or vector subspace $\mathcal{W}$ of a vector space $\mathcal{V}$ is a non-empty subset of $\mathcal{V}$ that is closed under vector addition and scalar multiplication; that is, the sum of two elements of $\mathcal{W}$ and the product of an element of $\mathcal{W}$ by a scalar belong to $\mathcal{W}$.