Dynamics of spin in magnetic field (case of constant field and Larmor Precession).

1) Recap about the Bloch sphere.

We saw that a good parametrization for qubit state vectors in $\mathbb{C}^2$ is

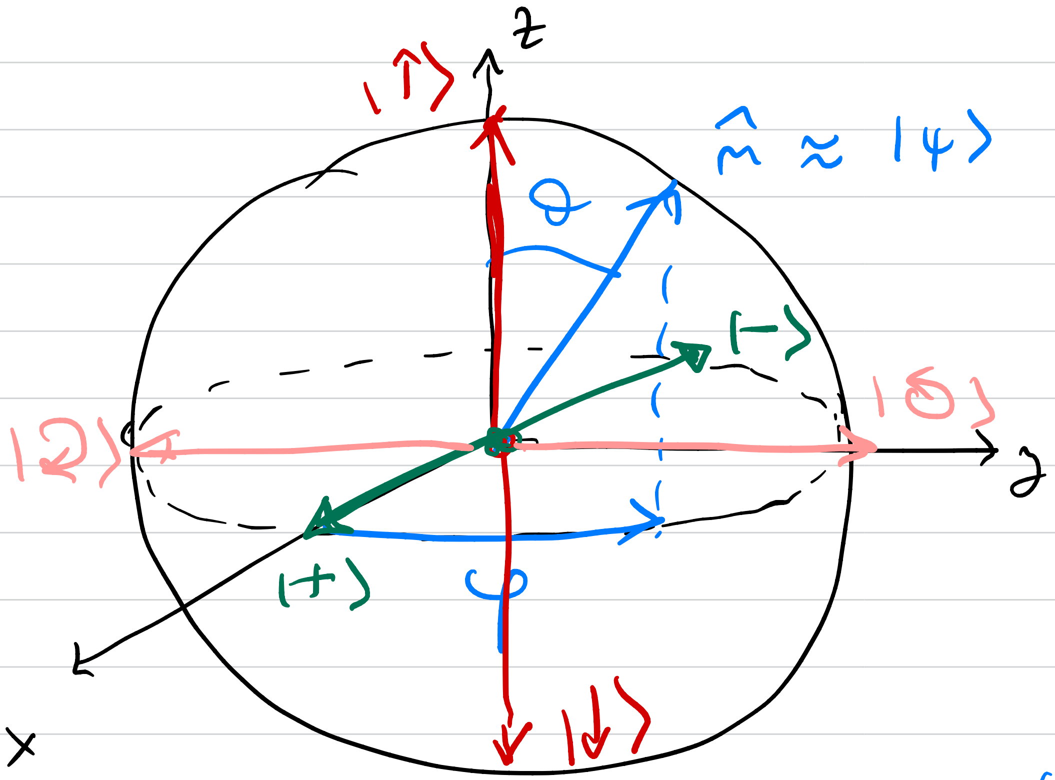
$$|\psi\rangle = \left(\cos\frac{\theta}{2}\right)|\uparrow\rangle + \left(\sin\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle$$

In this chapter $|\uparrow\rangle = |0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$$|\downarrow\rangle = |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

we use a notation that is intuitive when talking about spin.

Geometrical representation:



$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \left(\sin\frac{\theta}{2}\right) e^{i\varphi}|\downarrow\rangle \approx \hat{n} = \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix}$$

$0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$

Remarks:

$|\uparrow\rangle$, $|\downarrow\rangle$ are orthogonal (Z basis)

$|+\rangle$, $|-\rangle$ are orthogonal (X basis)

$= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

$|\otimes\rangle$, $|\odot\rangle$ are orthogonal (Y basis).

$= \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle) \qquad = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$.

Z basis = eigenvectors of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

X basis = eigenvectors of $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Y basis = eigenvectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Eigenvalues are always ± 1 for these matrices.

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Recall also the above parametrization follows from:

$$|\psi\rangle = \alpha |\uparrow\rangle + \beta |\downarrow\rangle$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$

normalisation removes one parameter.

Moreover a global phase is unobservable

so one can always adjust $\alpha \in \mathbb{R}$.

$\Rightarrow$ all in all take $\alpha \in \mathbb{R}$, $\beta \in \mathbb{C}$ with

$$\alpha = \cos\frac{\theta}{2} \quad , \quad \beta = \left(\sin\frac{\theta}{2}\right) e^{i\varphi}.$$

2) Hamiltonian.

The energy observable (Hamiltonian) of a spin 1/2 in a magnetic field is the 2x2 matrix

$$H = -\gamma\frac{\hbar}{2}\vec{B}\cdot\vec{\sigma}$$

$$= -\gamma\frac{\hbar}{2}\left(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z\right)$$

$$= -\gamma\frac{\hbar}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$$

[the value of γ depends on the particle or system and represents a coupling strength].

Special easy case:

$$\vec{B} = (0,\ 0,\ B_0) \quad /\!/ \ z \text{ direction}$$

$$H = -\gamma \frac{\hbar}{2} B_0 \sigma_z$$

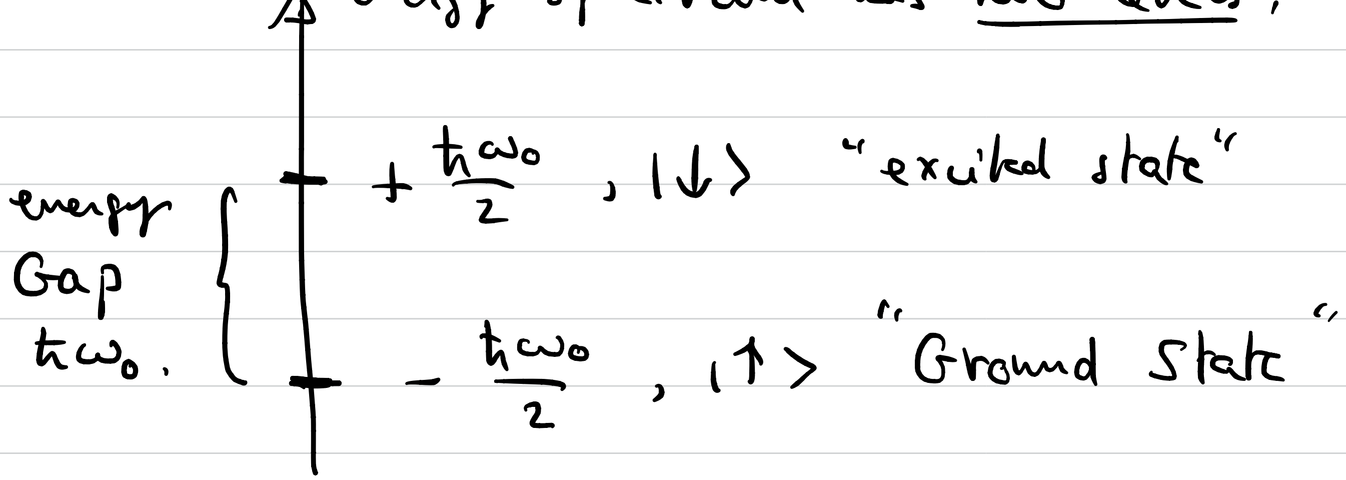
<u>Notation</u> $\gamma B_0 = \omega_0 \Rightarrow \boxed{H = -\frac{\hbar\omega_0}{2}\sigma_z}$

Here ω_0 has unit $[s^{-1}]$
$\hbar$ " " $[J.s] \Rightarrow$ Planck constant.
H " " $[J]$.

Eigenvalues & eigenvectors:

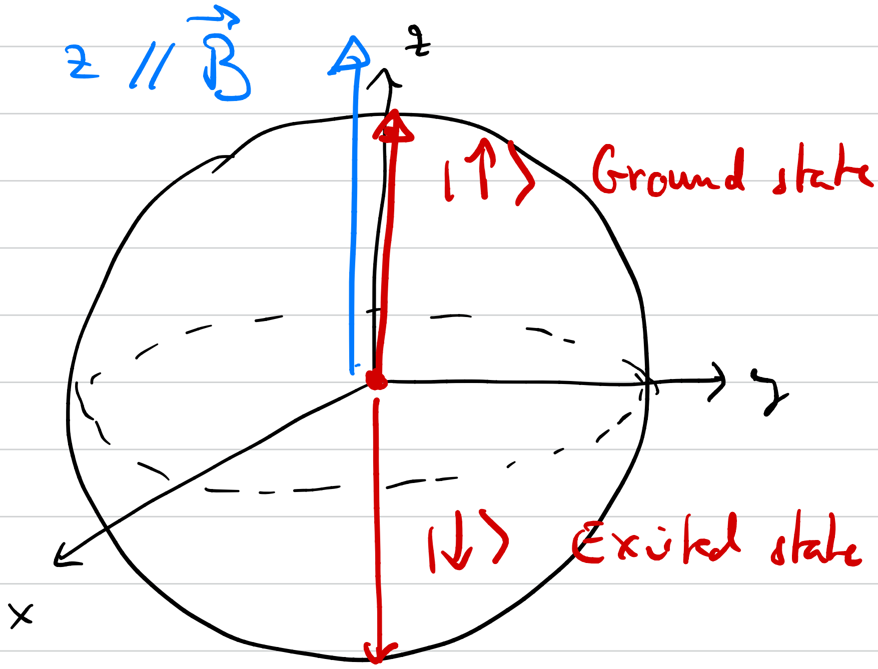
$$H = -\frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} H|\uparrow\rangle = -\frac{\hbar\omega_0}{2}|\uparrow\rangle \\ H|\downarrow\rangle = +\frac{\hbar\omega_0}{2}|\downarrow\rangle \end{cases}$$

energy spectrum has <u>two levels</u>;



Bloch sphere illustration:

With a constant $\vec{B} = (0, 0, B_0)$ field in the z direction



What if $\vec{B}$ has any general orientation?

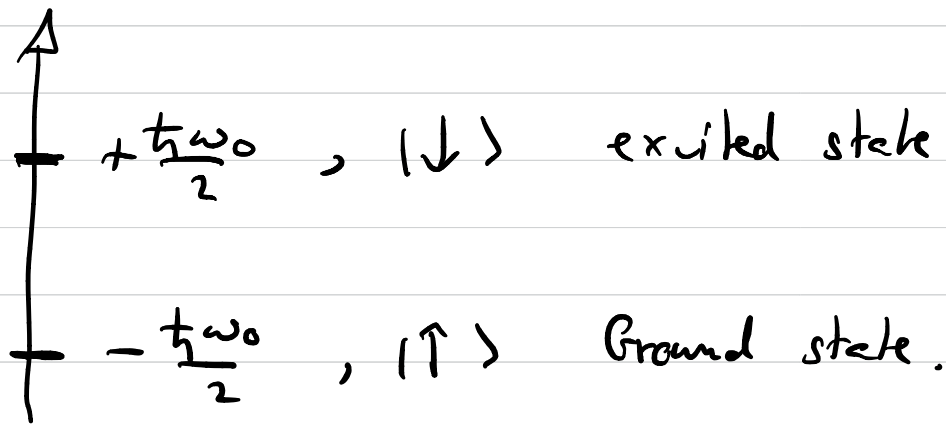
Obviously the energy spectrum will be the same with eigenvalues:

$$\pm \hbar\omega_0 \quad \text{and} \quad \omega_0 = \gamma \|\vec{B}\| = \gamma\sqrt{B_x^2 + B_y^2 + B_z^2}.$$

The eigenvectors (or eigenstates) will be qubit states with (θ, φ) pointing in the direction of $\frac{\vec{B}}{\|\vec{B}\|} = \hat{n}$.

Interaction of this two level system with photons.

Qualitative discussion.



- If initialy system is in state $|\uparrow\rangle$ and we send a photon (a grain of energy) of frequency $\omega = \omega_0$ tuned to the gap, so we give an energy exactly $\hbar\omega = \hbar\omega_0$, the state will flip to $|\downarrow\rangle$ the excited state. This is an absorption process.
- Conversely if initially the state is $|\downarrow\rangle$ excited, it can release an energy $\hbar\omega = \hbar\omega_0$ in the form of a photon and the system falls in the GS. This is an emission process.

3) Dynamics in constant field: Larmor precession.

We want to solve the equation

$$\begin{cases} i\hbar \frac{d}{dt} U_t = H\, U_t \\ U_{t=0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

and then compute $|\psi_t\rangle$ given that

$$|\psi_0\rangle = \left(\cos\frac{\theta}{2}\right)|\uparrow\rangle + \left(\sin\frac{\theta}{2}\right)e^{i\varphi}\,|\downarrow\rangle .$$

We also want to visualize the solution as a function of time on the Bloch sphere.

For $H = -\gamma\frac{\hbar}{2}\vec{B}\cdot\vec{\sigma}$ with $\vec{B}$ time-independent

this is very easy (time-dependent case see Next week.)

Because H is independent of time the following can be explicitly checked by plugging in ODE :

$$U_t = \exp\left(-i \frac{t}{\hbar} H\right)$$

Notice that $U_{t=0} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ initial cond is satisfied.

Now we address the exponential.

By choosing the reference frame appropriately we align $\vec{B}$ with z. So we here look at the case: $\vec{B} = (0, 0, B_0)$

$$U_t = \exp\left(-i \frac{t}{\hbar} \left(-\frac{\gamma B_0 \hbar}{2} \sigma_z\right)\right)$$

$$= \exp\left(i \frac{t}{2} \omega_0 \sigma_z\right)$$

$$= \begin{pmatrix} e^{i\frac{t\omega_0}{2}} & 0 \\ 0 & e^{-i\frac{t\omega_0}{2}} \end{pmatrix}.$$

(Remark: the exp of a diagonal matrix is just the diag matrix with exp on the diagonal.)

$$\Rightarrow \quad |\psi_t\rangle = U_t |\psi_0\rangle$$

$$= \left(\cos\frac{\vartheta}{2}\right) U_t |\uparrow\rangle + \left(\sin\frac{\vartheta}{2}\right) e^{i\varphi} U_t |\downarrow\rangle$$

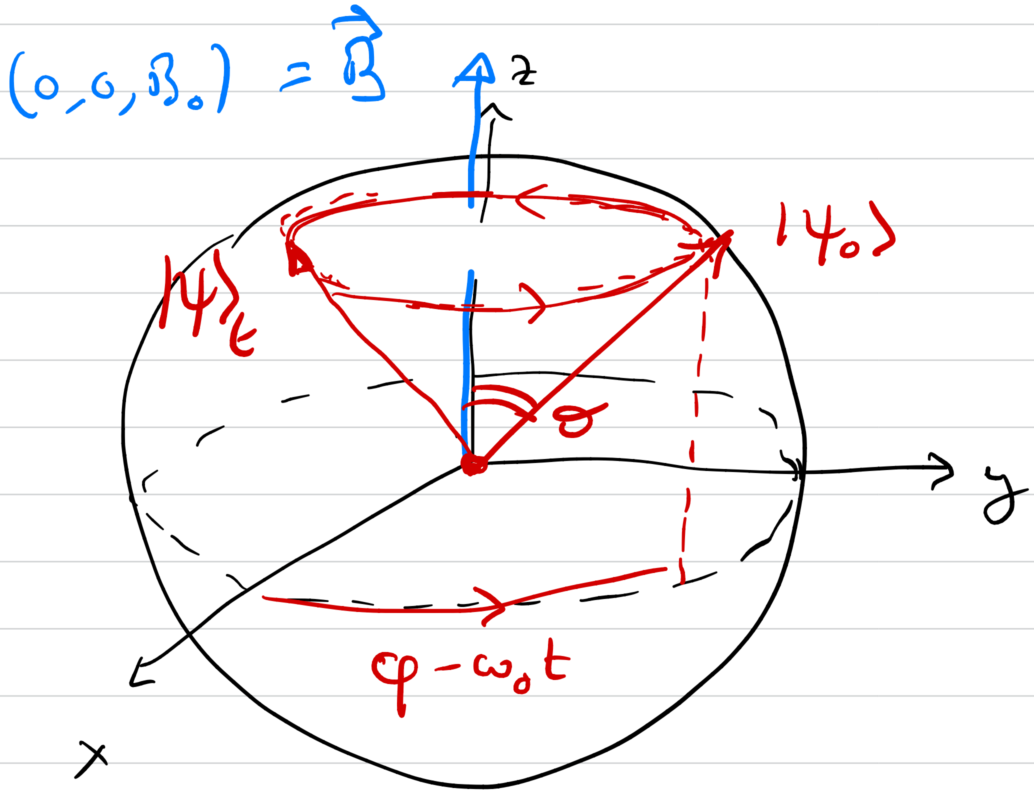
$$= \left(\cos\frac{\vartheta}{2}\right) e^{i\frac{t\omega_0}{2}} |\uparrow\rangle + \left(\sin\frac{\vartheta}{2}\right) e^{i\varphi} e^{-i\frac{t\omega_0}{2}} |\downarrow\rangle$$

$$= \underbrace{e^{i\frac{t\omega_0}{2}}}_{\uparrow} \left\{ \left(\cos\frac{\vartheta}{2}\right) |\uparrow\rangle + \left(\sin\frac{\vartheta}{2}\right) e^{i(\varphi - t\omega_0)} |\downarrow\rangle \right\}$$

Global unobservable phase.

<u>This is the solution of the Schroedinger equation.</u>

Evolution on Bloch sphere (constant magnfield).



- The state precesses around the z-axis with frequency ω_0. Period of rotation:

$$\varphi - \omega_0 T = \varphi - 2\pi \Rightarrow \boxed{T = \frac{2\pi}{\omega_0}}$$

- This is called the <u>Larmor precession</u>.

- Of course if $\vec{B}$ has general orientation the Larmor precession is around $\vec{B}$-axis with frequency $\omega_0 = \gamma \|\vec{B}\|$ and period $T = \frac{2\pi}{\gamma \|\vec{B}\|}$.