Magnetic Moments, Spin 1/2. Phenomenological introduction.

1. Introduction

"Spin 1/2" is physical realization of qubits which arises as the magnetic moment carried by numerous particles (such as electrons, protons, some nuclei, some atoms, ect...). As such it can be manipulated by magnetic fields and thus plays an important role in quantum information processing.

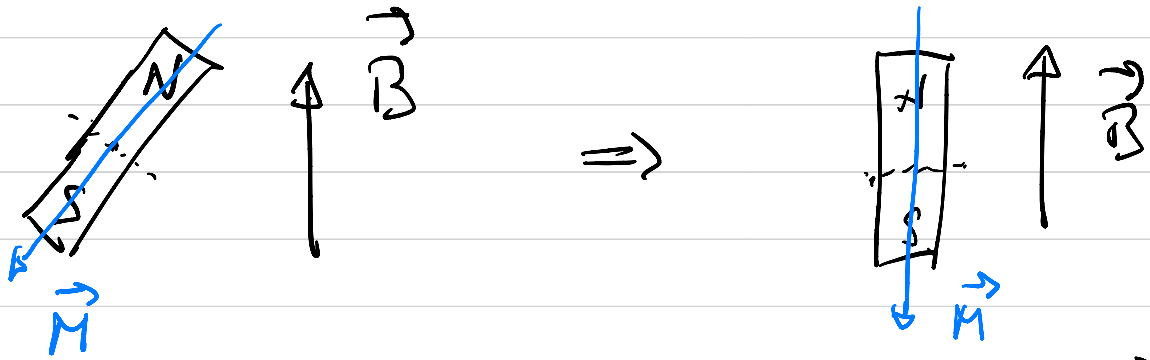
In the next few lectures we introduce the concept of spin 1/2 and its dynamical behavior in magnetic fields. As an application we will see how one can proceed to realize logical quantum gates such as H (Hadamard)

NOT, CNOT (control not) by Nuclear Magnetic Resonance.

The mathematical formalism developped in these next few lectures is essentially the same for various "two level systems" which are other realisations (natural or man-made) of quantum bits (although the physical interpretation is not in terms of magnetic moments & magnetic fields the math is basically similar).

2. Magnetic Moments: phenomelogy.

Think of a magnet, for example the needle of a compass. It has a North and a South pôle and the needle orients itself parallel to the magnetic field (of the earth say).



The magnet is described by a vector $\vec{M}$, called the magnetic moment and the energy function that the system minimizes when it orients itself favorably in the direction of $\vec{B}$ is

$$E \propto -\vec{M} \cdot \vec{B}$$

(we do not discuss proportionality constants here

There are various sources of magnetism that eventually constitute the magnetic moment $\vec{M}$ describing magnetic dipoles with a North pole and a South pole.

These magnetic dipoles come from orbital currents $\underbrace{\text{}}_{}$ loops (in atoms, solids, ...) which generate small magnetic fields, which add up to eventually contribute to the total Magnetic Moment $\vec{M}$.

But also there is another source of magnetism which <u>does not</u> come from orbital current loops but is <u>intrinsic</u> to particles such as electrons, protons, neutrons, nuclei, ect...
These particles are themselves "microscopic magnets" with a North and a South pole. The total $\vec{M}$ is eventually given by an addition of these

microscopic intrinsic contribution.

In ferromagnets (e.g fridge magnets, magnet of needle of compass):

- at "low" temperatures ($\sim < 1000\,°K$)

$$\vec{M} \quad = \quad \begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}$$

Microscopic contributions add up.

- at very high temperature ($\sim > 1000\,°K$)

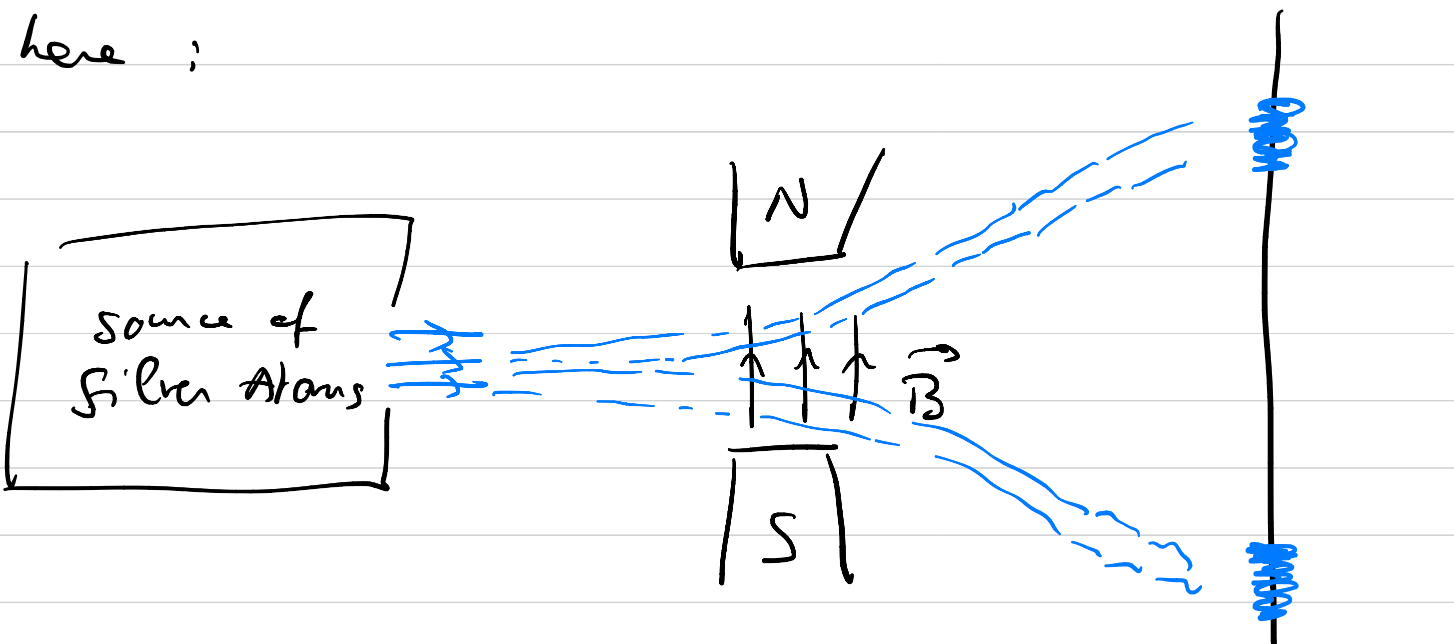
$$\vec{M} \approx 0 \quad = \quad \begin{array}{ccccccc} \uparrow & \downarrow & \uparrow & \uparrow & \downarrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\ \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow \end{array}$$

temperature introduces disorder and microscopic contributions cancel out

3) STERN-GERLACH experiment.

The Stern-Gerlach experiments (1922) put in evidence that particles (in this experiment electrons of silver atoms, ...) have an intrinsic magnetic moment which has quantum behavior and can be thought as an analog of polarization of photon.

We give a rough description of the experiment here:



Inhomogeneous Magnet producing

$$\vec{B}(z) = (0, 0, B_z(z))$$

We observe beam of atoms are deviated and split in two beams that fall in two distinct location called here "up" and "down".

Explanation: Ag atoms have an odd nb of electrons (thirteen) and each electron carries an intrinsic magnetic moment. These add up to produce $\vec{M}$ for Ag atoms. We have the energy function:

$$E \propto -\vec{M} \cdot \vec{B}(z) = -M_z B_z(z)$$

$$\Rightarrow \quad \text{Force} = -\frac{dE}{dz} = M_z \underbrace{\frac{d B_z(z)}{dz}}_{\geq 0 \text{ say}} .$$

Observing the two blobs up & down on the screen leads us to conclude that M_z takes two values $\propto (+1)$ and $\propto (-1)$.

Quantum theory.

It turns out that M_z is an observable described by a 2×2 matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

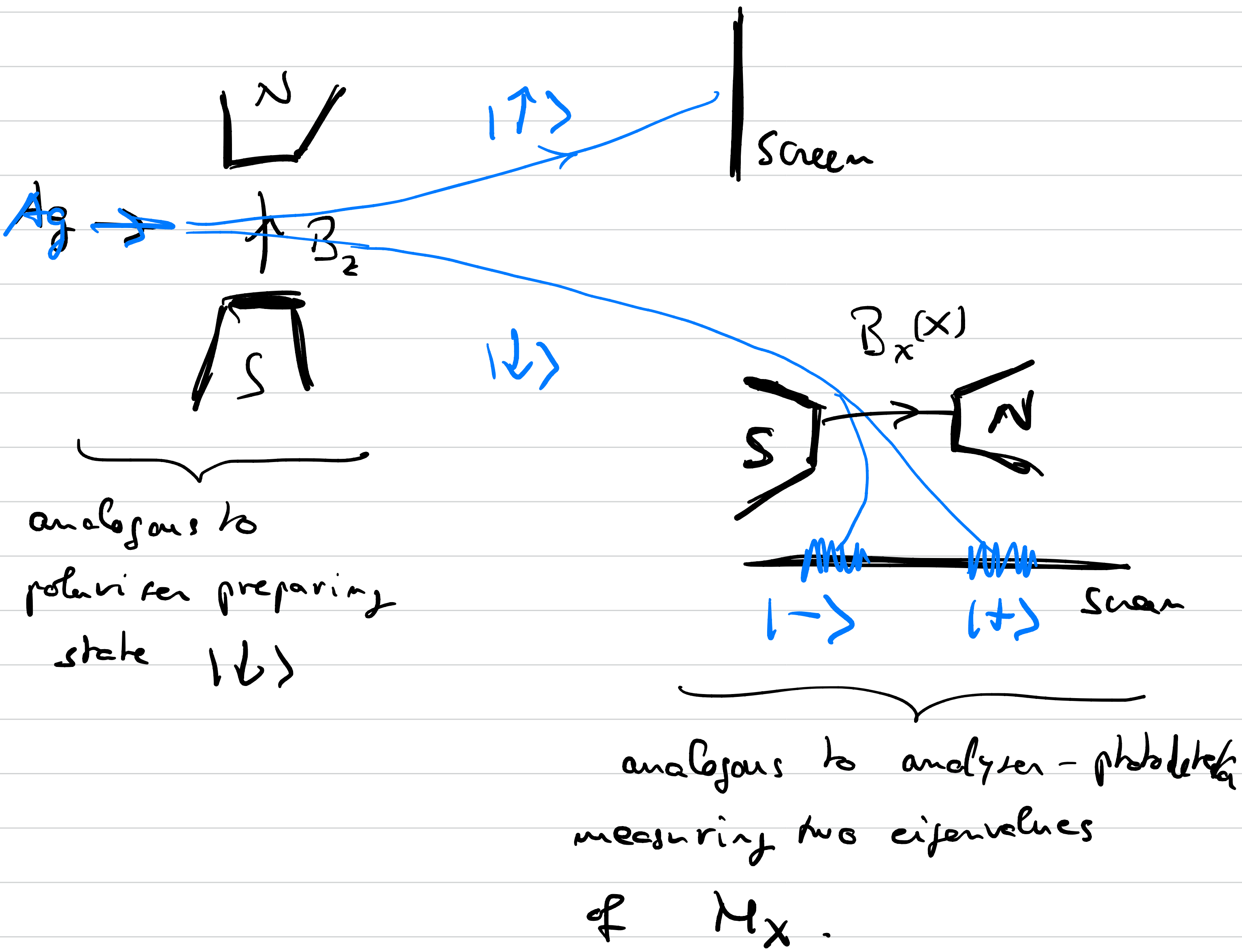
with two eigenvalues $(+1)$ & (-1) obtained in the measurement process. The eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle$

(in Dirac notation).

$$\sigma_z = (+1)|\uparrow\rangle\langle\uparrow| + (-1)|\downarrow\rangle\langle\downarrow| .$$

Remark: the Stern-Gerlach apparatus is analogous to the analyser-photodector apparatus

One can go further and proceed with the following experiment (analogous to the polarizer – analyzer – photodetector setting):



This leads us to introduce an observable M_x with again two eigenvalues $(+1)$ & (-1)

but eigenvectors $|+\rangle$ & $|-\rangle$ corresponding to a decomposition of $|\downarrow\rangle$.

It turns out that

$$M_x \propto \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= (+1)\,|+\rangle\langle+| + (-1)\,|-\rangle\langle-|$$

with $|+\rangle = \dfrac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$|-\rangle = \frac{|\uparrow\rangle - |\downarrow\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and $\sigma_x\,|+\rangle = (+1)\,|+\rangle$, $\sigma_x\,|-\rangle = (-1)\,|-\rangle$.

Similarly we can proceed also with a Stern-Gerlach apparatus oriented along the y-axis which leads to introduce the observable

$$M_y \propto \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

#.

Note: There are types of particles with magnetic moments that can take more than two values. All integers are possible 2, 3, 4, 5 ....
(there are more than two spots on the screen).

These are described by matrices (analogous) to Pauli matrices (2 x 2, 3 x 3, 4 x 4, ...).

Notation integer = 2S + 1 with S = half-integer

S = 1/2 (spin 1/2) → 2 x 2 matrices

S = 1 (spin 1) → 3 x 3 matrix

S = 3/2 (spin 3/2) → 4 x 4 matrix

4) Spin 1/2 : formalism.

- In summary "spin 1/2" particles carry an "intrinsic magnetic moment" described as a quantum observable as :

$$\vec{M} = (M_x, M_y, M_z) \propto (\sigma_x, \sigma_y, \sigma_z)$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

[we do not discuss proportionality factor which depends on the system/particle].

- The "energy observable" of a magnetic moment in a magnetic field or "Hamiltonian" is given by

$$H = -\gamma \vec{B} \cdot \vec{\sigma}$$
$$= -\gamma (B_x \sigma_x + B_y \sigma_y + B_z \sigma_z).$$

(units for H are Joule [J], for B-field Tesla [T],

and $\gamma \sim [J][T^{-1}]$.).

- <u>The Hamiltonian is the generator of the dynamics in the following sense</u>:

$$|\psi_t\rangle = U_t \, |\psi_0\rangle$$

state at time t ↑ ; ↑ initial state at time $t=0$

and the unitary U_t is found from the Schroedinger equation:

$$i\hbar \frac{d}{dt} U_t = H U_t$$

Remark: we also have $\underbrace{i\hbar \frac{d}{dt} |\psi_t\rangle = H |\psi_t\rangle}_{\text{Schroedinger equation.}}$

5) Rough intuition for Schroedinger's equation.

State of a photon of energy $\hbar\omega$ and momentum $\hbar k$ $\Big(k = \underset{\substack{\uparrow \\ \text{wave length}}}{\frac{2\pi}{\lambda}}, \quad \omega = 2\pi \underset{\substack{\uparrow \\ \text{frequency}}}{\nu} \Big)$

in Dirac notation:

$$|\psi_t\rangle = e^{-i\omega t} \underbrace{|k\rangle}_{\text{orbital state.}} \otimes \underbrace{|\vartheta, \varphi\rangle}_{\text{polarization state}}$$

in component notation this corresponds to

$$e^{-i\omega t} e^{-ikz} \begin{pmatrix} \cos\vartheta \\ (\sin\vartheta) e^{i\varphi} \end{pmatrix}$$

<u>Now</u> $\frac{d}{dt}|\psi_t\rangle = -i\omega|\psi_t\rangle = -\frac{i}{\hbar}\hbar\omega|\psi_t\rangle$

$$\Rightarrow \boxed{i\hbar\frac{d}{dt}|\psi_t\rangle = \underbrace{\hbar\omega}_{\substack{\text{energy observable here } 1\times 1 \\ \text{matrix.}}} |\psi_t\rangle}$$