Exercise 1 *Rotations on the Bloch sphere*

A general vector can be written in the form $\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|\downarrow\rangle$ in the Bloch sphere.

a) The eigenvectors for σ_z basis are $|\uparrow\rangle$ and $|\downarrow\rangle$, corresponding to $(\theta = 0, \phi = 0)$ and $(\theta = \pi, \phi = 0)$, respectively.

The eigenvectors for σ_y basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{i}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{i}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$ and $(\theta = \frac{\pi}{2}, \phi = -\frac{\pi}{2})$, respectively.

The eigenvectors for σ_x basis are $\frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, corresponding to $(\theta = \frac{\pi}{2}, \phi = 0)$ and $(\theta = \frac{\pi}{2}, \phi = \pi)$, respectively.

The corresponding representation over the Bloch sphere is shown in Figure 1.

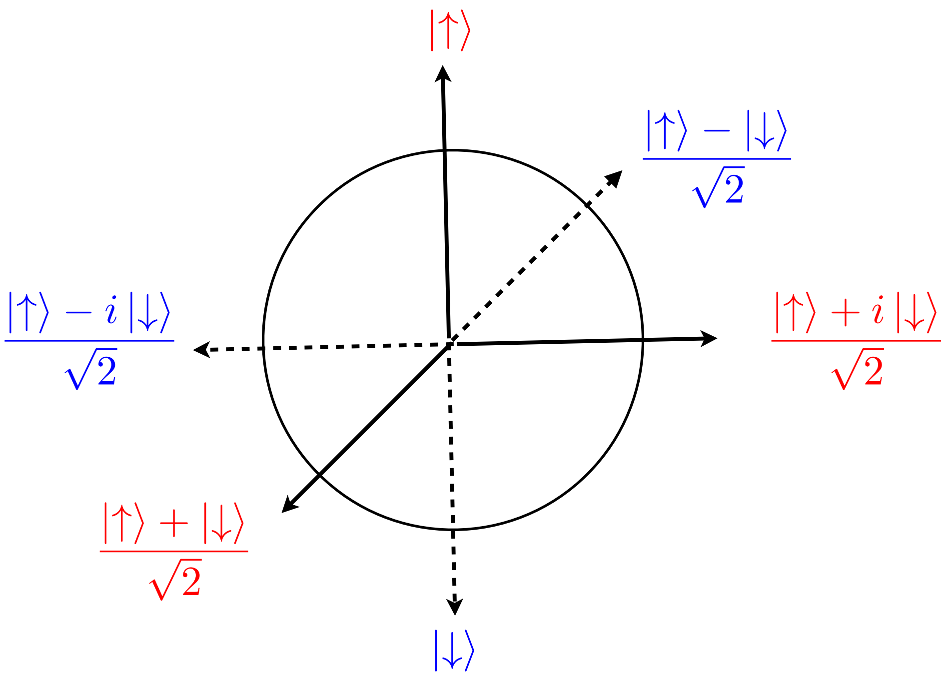

FIGURE 1 – Representation of basis vectors on Bloch Sphere

b) Using the general formula proved in homework 2 :

$$\exp\left(i\frac{\theta}{2}\vec{\sigma}\cdot\vec{n}\right) = \cos\left(\frac{\theta}{2}\right)I + i\vec{\sigma}\cdot\vec{n}\sin\left(\frac{\theta}{2}\right),$$

we obtain

$$\begin{aligned}\exp\Big(-i\frac{\alpha}{2}\sigma_x\Big) &= \cos\Big(\frac{\alpha}{2}\Big)I - i\sigma_x(\sin\Big(\frac{\alpha}{2}\Big)) \\ &= \begin{pmatrix} \cos\big(\frac{\alpha}{2}\big) & -i\sin\big(\frac{\alpha}{2}\big) \\ -i\sin\big(\frac{\alpha}{2}\big) & \cos\big(\frac{\alpha}{2}\big) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp\Big(-i\frac{\beta}{2}\sigma_y\Big) &= \cos\Big(\frac{\beta}{2}\Big)I - i\sigma_y(\sin\Big(\frac{\beta}{2}\Big)) \\ &= \begin{pmatrix} \cos\big(\frac{\beta}{2}\big) & -\sin\big(\frac{\beta}{2}\big) \\ \sin\big(\frac{\beta}{2}\big) & \cos\big(\frac{\beta}{2}\big) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp\Big(-i\frac{\gamma}{2}\sigma_z\Big) &= \cos\Big(\frac{\gamma}{2}\Big)I - i\sigma_z(\sin\Big(\frac{\gamma}{2}\Big)) \\ &= \begin{pmatrix} \cos\big(\frac{\gamma}{2}\big) - i\sin\big(\frac{\gamma}{2}\big) & 0 \\ 0 & \cos\big(\frac{\gamma}{2}\big) + i\sin\big(\frac{\gamma}{2}\big) \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}.\end{aligned}$$

c) The state vector $\cos\big(\frac{\theta}{2}\big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\big(\frac{\theta}{2}\big)\left|\downarrow\right\rangle$ is transformed to the vector $\cos\big(\frac{\theta-\alpha}{2}\big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\big(\frac{\theta-\alpha}{2}\big)\left|\downarrow\right\rangle$. Indeed :

$$\exp\Big(-i\frac{\alpha}{2}\sigma_x\Big)\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta}{2}\Big)\right)\left|\downarrow\right\rangle = \cos\Big(\frac{\theta-\alpha}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta-\alpha}{2}\Big)\left|\downarrow\right\rangle.$$

We see that the matrix $\exp\big(-i\frac{\alpha}{2}\sigma_x\big)$ acts as a rotation matrix of angle α around the X-axis.

Similarly we have,

$$\exp\Big(-i\frac{\gamma}{2}\sigma_z\Big)\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i\frac{\pi}{2}}\sin\Big(\frac{\theta}{2}\Big)\right)\left|\downarrow\right\rangle = e^{-i\frac{\gamma}{2}}\left(\cos\Big(\frac{\theta}{2}\Big)\left|\uparrow\right\rangle + e^{i(\frac{\pi}{2}+\gamma)}\sin\Big(\frac{\theta}{2}\Big)\left|\downarrow\right\rangle\right).$$

Therefore we see that $\exp\big(i\frac{\gamma}{2}\sigma_z\big)$ acts as a rotation of angle γ around the Z-axis.

d) The matrix $\exp\big(i\frac{\beta}{2}\sigma_y\big)$ is simply a rotation matrix around the Y axis on teh Blocj sphere. We can check this by calculation although the resulting formula is less easy to visualise.

Remark : The complex exponentials of Pauli matrices considered here are said to be representations of rotations (around the X, Y, Z axis) in the vector space $\mathbb{C}^2$. The Pauli matrices σ_x, σ_y, σ_z are said to be te generators of these rotations. This exercise is an illustration of the general theory of "group representations". One can define the rotation group on Euclidean vectors in $\mathbb{R}^3$ and ask the question : how does this rotation group act on two dimensional vectors of $\mathcal{C}^2$. The answer is given in this exercise!

Exercise 2 *Dynamics of spin 1/2*

(a) The evolution matrix is given by (here notation $U(t,0)$ means evolution from 0 to t)

$$U(t,0) = \exp\Big(- i\frac{t\delta}{2}\sigma_z + i\frac{t\omega_1}{2}\sigma_x\Big) = \exp\Big(\frac{a}{2}(n_x\sigma_x + n_z\sigma_z\Big)$$

with $a = t(\delta^2+\omega_1^2)^{1/2}$ et $n_x = \frac{\omega_1}{(\delta^2+\omega_1^2)^{1/2}}$, $n_z = \frac{\delta}{(\delta^2+\omega_1^2)^{1/2}}$. So applying the "generalized Euler formula"

$$U(t,0) = \cos\bigg(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\bigg)I + i\sin\bigg(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\bigg)\Big(\frac{\omega_1}{(\delta^2+\omega_1^2)^{1/2}}\sigma_x + \frac{\delta}{(\delta^2+\omega_1^2)^{1/2}}\sigma_z\Big)$$

This gives the final matrix :

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big) + i\delta\frac{\sin\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big)}{(\delta^2+\omega_1^2)^{1/2}} & i\omega_1\frac{\sin\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big)}{(\delta^2+\omega_1^2)^{1/2}} \\ i\omega_1\frac{\sin\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big)}{(\delta^2+\omega_1^2)^{1/2}} & \cos\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big) - i\delta\frac{\sin\big(\frac{t}{2}(\delta^2+\omega_1^2)^{1/2}\big)}{(\delta^2+\omega_1^2)^{1/2}} \end{bmatrix}$$

(b) In the limit $\omega_1 << \delta$ we obtain

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{t\delta}{2}\big) + i\sin\big(\frac{t\delta}{2}\big) & 0 \\ 0 & \cos\big(\frac{t\delta}{2}\big) - i\sin\big(\frac{t\delta}{2}\big) \end{bmatrix}$$

If the initial state is $\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)$ then the final state is

$$\frac{1}{\sqrt{2}}(e^{it\delta/2}|\uparrow\rangle + e^{-it\delta/2}|\downarrow\rangle = \frac{e^{-it\delta/2}}{\sqrt{2}}(|\uparrow\rangle + e^{-it\delta}|\downarrow\rangle$$

Trajectory : On the Bloch sphere the trajectory is along the equator (since $\theta = \pi/2$ et $\varphi = -t\delta$). It is periodic with period $T = \frac{2\pi}{\delta}$.

Remark :In the rotating frame the state is (approximately) unaffected by the magnetic field (situation of extreme detuning).

(c) In the limit $\delta << \omega_1$ we obtain

$$U(t,0) = \begin{bmatrix} \cos\big(\frac{\omega_1 t}{2}\big) & i\sin\big(\frac{\omega_1 t}{2}\big) \\ i\sin\big(\frac{\omega_1 t}{2}\big) & \cos\big(\frac{\omega_1 t}{2}\big) \end{bmatrix}$$

If the initial state is $|\uparrow\rangle$ then the final state is

$$U(t,0)|\uparrow\rangle = \cos\bigg(\frac{\omega_1 t}{2}\bigg)|\uparrow\rangle + i\sin\bigg(\frac{\omega_1 t}{2}\bigg)|\downarrow\rangle$$

Trajectory : On the Bloch sphere the trajectory is rotating around the x -axis and is in the plane (yz) (because $\theta = \omega_1 t$ and $\varphi = \pi/2$). It is periodic with period $T = \frac{2\pi}{\omega_1}$: in effect this period corresponds to the parametrisation $\theta = \omega_1 t$.

Remark 1 : Within one period the evolution matrix changes sign but this sign gives a global phase which is not obervable at the level of probabilities of measurement outcomes.

Remark 2 :In the rotating frame this corresponds to spin flips in a situation of tuning.

Exercise 3 *Entanglement creation by a magnetic interaction*

We work entirely in Dirac notation but you can also check out the complement on the connection with component notation.

The final state is (using that $|\uparrow\rangle, |\downarrow\rangle$ are eigenvectors of σ_z with eigenvalues $+1$ et -1).

$$\begin{aligned} e^{-\frac{it}{\hbar}\mathcal{H}}|\psi_0\rangle &= e^{-itJ\sigma_1^z\otimes\sigma_2^z}\cdot\frac{1}{2}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right) \\ &= \frac{1}{2}\left(e^{-itJ}\,|\uparrow\uparrow\rangle - e^{itJ}\,|\uparrow\downarrow\rangle + e^{itJ}\,|\downarrow\uparrow\rangle - e^{-itJ}\,|\downarrow\downarrow\rangle\right) \\ &= \frac{e^{-itJ}}{2}\left(|\uparrow\uparrow\rangle - e^{2itJ}\,|\uparrow\downarrow\rangle + e^{2itJ}\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right). \end{aligned}$$

(a) for $t = \frac{\pi}{4J}$ on a $e^{2itJ} = e^{\frac{i\pi}{2}} = i$

$\Rightarrow |\psi_t\rangle = \frac{e^{-\frac{i\pi}{4}}}{2}\left(|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle - |\downarrow\downarrow\rangle\right)$.

(b) suppose the state can be written

$(\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle)\otimes(\gamma\,|\uparrow\rangle + \delta\,|\downarrow\rangle) = \alpha\gamma\,|\uparrow\uparrow\rangle + \alpha\delta\,|\uparrow\downarrow\rangle + \beta\gamma\,|\downarrow\uparrow\rangle + \beta\delta\,|\downarrow\downarrow\rangle$,

then $\alpha\gamma = 1,\ \alpha\delta = -i,\ \beta\gamma = i,\ \beta\delta = -1$.

One can always set $\alpha = 1$ (global phase). Thus $\gamma = 1,\ \delta = -i,\ \beta = i$ et $\delta = i$ $\Rightarrow$ contradiction on δ. You can also take any value for α and show the contradiction appears.

(c) At $t = \frac{\pi}{2J}$ with $e^{\pm itJ} = e^{\pm i\frac{\pi}{2}} = \pm i$,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{2}\left(-i\,|\uparrow\uparrow\rangle - i\,|\uparrow\downarrow\rangle + i\,|\downarrow\uparrow\rangle + i\,|\downarrow\downarrow\rangle\right) \\ &= \frac{-i}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right) \end{aligned}$$

is a product state. So another $\frac{\pi}{4J}$ time of evolution cancels the entanglement.

(d) At $t = \frac{\pi}{J}$ with $e^{\pm itJ} = e^{\pm i\pi} = -1$,

$$\begin{aligned} |\psi_t\rangle &= \frac{1}{2}\left(-\,|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle\right) \\ &= \frac{-1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{aligned}$$

is also a product state.