Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 10 – Solutions

Exercise 1:

(a) Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M. Show that π is a surjective smooth submersion.

(b) Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose that $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Show that the transition function $\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R})$ between Φ and Ψ is smooth.

(c) Consider the tangent bundle $\pi\colon TM \to M$ of a smooth n-manifold M and let $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ and $\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^n$ be the smooth local trivializations of TM associated with two smooth charts (U, φ) and (V, ψ) for M. Determine the transition function $\tau\colon U \cap V \to \mathrm{GL}(n, \mathbb{R})$ between Φ and Ψ.

(d) Consider the tangent bundle $\pi\colon T\mathbb{S}^2 \to \mathbb{S}^2$ of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Compute the transition function associated with the two local trivializations determined by stereographic coordinates.

(e) Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose that $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, and that for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. For each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, let $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ be the transition function between the smooth local trivializations Φ_α and Φ_β. Show that the following identity is satisfied for all $\alpha, \beta, \gamma \in A$:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p) \quad \text{for all } p \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{$\star$}$$

Solution:

(a) By definition of a smooth vector bundle, π is smooth and surjective, so it remains to check that it is a smooth submersion. Let $q \in E$ and set $p := \pi(q) \in M$. Again by definition of a smooth vector bundle, there exists an open neighborhood U of p in M and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ (assuming that $\pi\colon E \to M$ is of rank k) such that

$\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)}$, where $\pi_U \colon U \times \mathbb{R}^k \to U$ is the projection to the first factor, which is a smooth submersion by [*Exercise Sheet* 6*, Exercise* 2(*a*)]. It follows from [*Exercise Sheet* 6*, Exercise* 1(*a*)(*i*)] and [*Exercise Sheet* 6*, Exercise* 5(*a*)] that $\pi|_{\pi^{-1}(U)}$ itself is a smooth submersion, that is, its differential is surjective at every point of $\pi^{-1}(U)$, which is an open neighborhood of q in E. Since $q \in E$ was arbitrary, we conclude that π is a smooth submersion.

(b) Consider the standard bases $\{e_i\}_{i=1}^k$ of $\mathbb{R}^k$ and $\{E_{ij}\}_{i,j=1}^k$ of $\mathrm{GL}(k,\mathbb{R})$. For each $p \in U \cap V$, denote by $\alpha_{ij}(p) \in \mathbb{R}$ the (i,j)-th element of the matrix $\tau(p) \in \mathrm{GL}(k,\mathbb{R})$ and observe that

$$\tau(p) = \sum_{i,j=1}^{k} \alpha_{ij}(p)\, E_{ij}.$$

For each $j \in \{1,\dots,k\}$ we have

$$\tau(p) \cdot e_j = \big(\alpha_{1j}(p),\dots,\alpha_{kj}(p)\big) = \sum_{i=1}^{k} \alpha_{ij}(p)\, e_i.$$

If now for each $i \in \{1,\dots,k\}$ we denote by π_{ij} the (projection) map

$$\pi_{ij} \colon (U \cap V) \times \mathbb{R}^k \to \mathbb{R},\ \big(q,(v_{1j},\dots,v_{kj})\big) \mapsto v_{ij},$$

which is smooth by [*Exercise Sheet* 3*, Exercise* 4(*a*)], then we obtain

$$\big(\pi_{ij} \circ \Phi \circ \Psi^{-1}\big)(p,e_j) = \pi_{ij}\Big(p, \big(\alpha_{1j}(p),\dots,\alpha_{kj}(p)\big)\Big) = \alpha_{ij}(p).$$

Therefore, each map $\alpha_{ij} \colon U \cap V \to \mathbb{R},\ p \mapsto \alpha_{ij}(p)$ is smooth as a composite of smooth maps. In view of [*Exercise Sheet* 2*, Exercise* 2], which gives the smooth chart

$$\psi\colon\ \mathrm{GL}(k^2,\mathbb{R}) \to \mathbb{R}^{k^2},\ \sum_{i,j=1}^{k} m_{ij}\, E_{ij} \mapsto \sum_{i,j=1}^{k} m_{ij}\, \epsilon_{ij}$$

for $\mathrm{GL}(k,\mathbb{R})$, we now deduce readily that the transition function $\tau \colon U \cap V \to \mathrm{GL}(k,\mathbb{R})$ between Φ and Ψ is smooth; indeed, it has a smooth coordinate representation $\psi \circ \tau \circ \varphi^{-1}$ with respect to ψ and any (fixed) smooth chart φ for M around (an arbitrary point) $p \in U \cap V$, since its component functions α_{ij} are smooth.

(c) Denote by $(x^1,\dots,x^n)$ and $(\widetilde{x}^1,\dots,\widetilde{x}^n)$ the coordinate functions of the smooth coordinate charts (U,φ) and (V,ψ), respectively, and recall that the associated smooth local trivializations Φ and Ψ, respectively, are defined as follows:

$$\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k,\ v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \big(p,(v^1,\dots,v^n)\big)$$

and

$$\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k,\ \widetilde{v}^i \frac{\partial}{\partial \widetilde{x}^i}\bigg|_p \mapsto \big(p,(\widetilde{v}^1,\dots,\widetilde{v}^n)\big).$$

Since

$$\frac{\partial}{\partial \widetilde{x}^i}\bigg|_p = \frac{\partial x^j}{\partial \widetilde{x}^i}(\widehat{p})\frac{\partial}{\partial x^j}\bigg|_p,$$

we have

$$
\begin{aligned}
\left(\Phi \circ \Psi^{-1}\right)\left(p,(v^1,\ldots,v^n)\right) &= \Phi\left(v^i \frac{\partial}{\partial \widetilde{x}^i}\bigg|_p\right) = \Phi\left(\left(v^i \frac{\partial x^j}{\partial \widetilde{x}^i}(\widehat{p})\right)\frac{\partial}{\partial x^j}\bigg|_p\right) \\
&= \left(p, \left(v^i \frac{\partial x^1}{\partial \widetilde{x}^i}(\widehat{p}),\ldots, v^i \frac{\partial x^n}{\partial \widetilde{x}^i}(\widehat{p})\right)\right) \\
&= \left(p, A_p \cdot (v^1,\ldots,v^n)^T\right),
\end{aligned}
$$

where

$$
A_p := \left(\frac{\partial x^j}{\partial \widetilde{x}^i}(\widehat{p})\right)_{i,j=1,\ldots,n} \in \mathrm{GL}(n,\mathbb{R})
$$

is the Jacobian matrix at $\widehat{p} = \varphi(p) = \psi(p)$ of the transition map

$$
\varphi \circ \psi^{-1} \colon \psi(U \cap V) \to \varphi(U \cap V).
$$

(Recall also that the matrix A_p represents the differential $d\left(\varphi \circ \psi^{-1}\right)_{\widehat{p}}$ with respect to coordinate bases.) Therefore, the transition function τ between Φ and Ψ is the map

$$
\tau \colon U \cap V \to \mathrm{GL}(n,\mathbb{R}),\ p \mapsto A_p = \left(\frac{\partial x^j}{\partial \widetilde{x}^i}(\widehat{p})\right)_{i,j}.
$$

(d) We use the same notation as the one used in [*Exercise Sheet* 2, *Exercise* 5]. According to the solution of part (c), the transition function $\tau \colon \mathbb{S}^2 \setminus \{N,S\} \to \mathrm{GL}(2,\mathbb{R})$ between the two smooth local trivializations of $T\mathbb{S}^2$ determined by the stereographic coordinates $\left(\mathbb{S}^2 \setminus \{N\}, \sigma\right)$ and $\left(\mathbb{S}^2 \setminus \{S\}, \widetilde{\sigma}\right)$ is given at every point $p \in \mathbb{S}^2 \setminus \{N,S\}$ by the Jacobian matrix at $\widehat{p} = \sigma(p) = \widetilde{\sigma}(p)$ of the transition map $\sigma \circ \widetilde{\sigma}^{-1}$. We saw in [*Exercise Sheet* 2, *Exercise* 5] that $\sigma \circ \widetilde{\sigma}^{-1}$ is given by the formula

$$
\left(\sigma \circ \widetilde{\sigma}^{-1}\right)(\widetilde{u},\widetilde{v}) = \left(\frac{\widetilde{u}}{\widetilde{u}^2+\widetilde{v}^2}, \frac{\widetilde{v}}{\widetilde{u}^2+\widetilde{v}^2}\right) = (u,v),\ \ (\widetilde{u},\widetilde{v}) \in \mathbb{R}^2 \setminus \{(0,0)\}.
$$

Thus, its Jacobian at an arbitrary point $(\widetilde{u},\widetilde{v}) \in \mathbb{R}^2 \setminus \{(0,0)\}$ is the matrix

$$
J\left(\sigma \circ \widetilde{\sigma}^{-1}\right)(\widetilde{u},\widetilde{v}) = \begin{pmatrix} \dfrac{\widetilde{v}^2-\widetilde{u}^2}{(\widetilde{u}^2+\widetilde{v}^2)^2} & \dfrac{-2\widetilde{u}\widetilde{v}}{(\widetilde{u}^2+\widetilde{v}^2)^2} \\ \dfrac{-2\widetilde{u}\widetilde{v}}{(\widetilde{u}^2+\widetilde{v}^2)^2} & \dfrac{\widetilde{u}^2-\widetilde{v}^2}{(\widetilde{u}^2+\widetilde{v}^2)^2} \end{pmatrix}
$$

(whose determinant equal to $-\dfrac{1}{(\widetilde{u}^2+\widetilde{v}^2)^2}$, which is clearly non-zero).

(e) Fix indices $\alpha,\beta,\gamma \in A$ and a point $p \in U_\alpha \cap U_\beta \cap U_\gamma$. According to *Lemma 6.5*, for any $v \in \mathbb{R}^k$ we have

$$
\begin{aligned}
\left(p, \tau_{\alpha\gamma}(p)v\right) &= \left(\Phi_\alpha \circ \Phi_\gamma^{-1}\right)(p,v) \\
&= \left(\Phi_\alpha \circ \Phi_\beta^{-1}\right) \circ \left(\Phi_\beta \circ \Phi_\gamma^{-1}\right)(p,v) \\
&= \left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)\left(p, \tau_{\beta\gamma}(p)v\right) \\
&= \left(p, \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v\right),
\end{aligned}
$$

which implies that

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p).$$

Since α, β, γ and p were arbitrary, we obtain $(\star)$.

Exercise 2 (*Smooth vector bundle construction lemma*)**:** Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose that for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that $(\star)$ is satisfied for all $\alpha, \beta, \gamma \in A$. Show that there is a smooth vector bundle $E \to M$ of rank k with smooth local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose transitions functions are the given maps $\tau_{\alpha\beta}$.

[Hint: Define an appropriate equivalence relation on $\coprod \left(U_\alpha \times \mathbb{R}^k\right)$ and use the vector bundle chart lemma.]

Solution: We first fix some notation: Set

$$\mathcal{E} \coloneqq \coprod \left(U_\alpha \times \mathbb{R}^k\right);$$

and for $(p, v) \in U_\alpha \times \mathbb{R}^k$, denote by $(p, v)_\alpha$ its image in $\mathcal{E}$.

As suggested by the hint, consider now the following relation $\sim$ on $\mathcal{E}$: two points $(p, v)_\alpha, (p', v')_\beta \in \mathcal{E}$ are equivalent if and only if

$$p = p' \quad \text{and} \quad v = \tau_{\alpha\beta}(p) \cdot v',$$

in which case we write $(p, v)_\alpha \sim (p', v')_\beta$. Let us check that $\sim$ indeed an equivalence relation on $\mathcal{E}$:

- *Reflexivity*: By applying $(\star)$ to $\alpha = \beta = \gamma$ we obtain $\tau_{\alpha\alpha} \equiv \mathrm{Id}_{k\times k}$. It follows that $v = \tau_{\alpha\alpha}(p)v$ for all $(p, v) \in U_\alpha \times \mathbb{R}^k$, and thus $(p, v)_\alpha \sim (p, v)_\alpha$.
- *Symmetry*: Suppose that $(p, v)_\alpha \sim (p', v')_\beta$, i.e., $p = p'$ and $v = \tau_{\alpha\beta}(p)v'$. By applying $(\star)$ to α, β and $\gamma = \alpha$ we obtain $\tau_{\beta\alpha}(p) = \big(\tau_{\alpha\beta}(p)\big)^{-1}$. Thus,

  $$v' = (\tau_{\alpha\beta}(p))^{-1} \cdot v = \tau_{\beta\alpha}(p') \cdot v,$$

  whence $(p', v')_\beta \sim (p, v)_\alpha$.
- *Transitivity*: Suppose that $(p, v)_\alpha \sim (p', v')_\beta$ and $(p', v')_\beta \sim (p'', v'')_\gamma$. Then

  $$p = p' = p'' \quad \text{and} \quad v = \tau_{\alpha\beta}(p) \cdot v', \;\; v' = \tau_{\beta\gamma}(p') \cdot v''.$$

  In particular, we obtain

  $$v = \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p') \cdot v'' \overset{(\star)}{=} \tau_{\alpha\gamma}(p) \cdot v'',$$

  which shows that $(p, v)_\alpha \sim (p'', v'')_\beta$.

Next, set

$$E \coloneqq \mathcal{E}/\sim$$

and denote by $[(p,v)_\alpha] \in E$ the equivalence class of $(p,v)_\alpha \in \mathcal{E}$. Note that the map $\mathcal{E} \to M$ sending $(p,v)_\alpha$ to p factors through E, because if $(p,v)_\alpha \sim (p',v')_\beta$, then in particular $p = p'$. So, consider the map

$$\pi\colon E \to M,\ [(p,v)_\alpha] \mapsto p.$$

Now, let α be arbitrary and let us verify that

$$\begin{aligned} \Psi_\alpha\colon U_\alpha \times \mathbb{R}^k &\to \pi^{-1}(U_\alpha) \\ (p,v) &\mapsto [(p,v)_\alpha] \end{aligned}$$

is a bijection. For injectivity, suppose that $\Psi_\alpha(p,v) = \Psi_\alpha(p',v')$. In particular, we obtain $p = p'$ and $v = \tau_{\alpha\alpha}(p)v' = v'$. For surjectivity, let $[(p',v')_\beta] \in \pi^{-1}(U_\alpha)$ be arbitrary. Notice that $p := \pi([p',v']) \in U_\alpha$, and set $v = \tau_{\alpha\beta}(p)v'$. Then we have $(p,v)_\alpha \sim (p',v')_\beta$, and thus $[(p',v')_\beta] = \Psi_\alpha(p,v)$. Hence, Ψ_α is bijective, as claimed. Finally, with a similar argument it is straightforward to check that $\Psi_\alpha(\{p\} \times \mathbb{R}^k) = \pi^{-1}(p)$.

By bijectivity we may write $\Phi_\alpha = (\Psi_\alpha)^{-1}$. To endow the fibers $\pi^{-1}(p)$ with a vector space structure, let α_p be such that $p \in U_{\alpha_p}$. We endow $\pi^{-1}(p)$ with the structure of a k-dimensional real vector space via the bijection $\pi^{-1}(p) \cong \{p\} \times \mathbb{R}^k$ provided by Φ_{α_p}. We denote the resulting real vector space by $E_p = \pi^{-1}(p)$. Since we chose α_p at random, we have to check that the choice does not matter. To this end, let α be arbitrary and take $p \in U_\alpha$. We have to check that $\Phi_\alpha|_{E_p}$ is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k$. So, pick $(p,v)_{\alpha_p} \in E_p$, and set $v' = \tau_{\alpha,\alpha_p}(p) \cdot v$, so that $(p,v')_\alpha \sim (p,v)_{\alpha_p}$. Then

$$\Phi_\alpha\big([(p,v)_{\alpha_p}]\big) = \Phi_\alpha\big([(p,v')_\alpha]\big) = (p,v') = \big(p, \tau_{\alpha,\alpha_p}(p) \cdot v\big).$$

As $\tau_{\alpha,\alpha_p}(p) \in \mathrm{GL}(k,\mathbb{R})$, we infer that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is an isomorphism of real vector spaces.

Finally, to apply the Vector Bundle Chart Lemma, we have to verify that the Φ_α's are compatible. Let α, β be such that $U_\alpha \cap U_\beta \neq \emptyset$. Take $(p,v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$. We want to compute $\big(\Phi_\alpha \circ \Phi_\beta^{-1}\big)(p,v)$. By construction, we see that $\Phi_\beta^{-1}(p,v) = [(p,v)_\beta]$. Now, let $v' = \tau_{\alpha\beta}(p) \cdot v$, so that $(p,v')_\alpha \sim (p,v)_\beta$. Then we have

$$\big(\Phi_\alpha \circ \Phi_\beta^{-1}\big)(p,v) = \Phi_\alpha\big([(p,v')_\alpha]\big) = (p,v') = \big(p, \tau_{\alpha\beta}(p) \cdot v\big).$$

Since by hypothesis the maps $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k,\mathbb{R})$ are smooth, the Vector Bundle Chart Lemma implies that E has a unique topology and smooth structure such that $\pi\colon E \to M$ is a smooth vector bundle of rank k, and the Φ_α's are its local trivializations, with transition functions the $\tau_{\alpha\beta}$'s.

Exercise 3:

(a) Show that the zero section of every smooth vector bundle is smooth.

[Hint: Consider $\Phi \circ \zeta$, where Φ is a local trivialization.]

(b) Let $\pi\colon E \to M$ be a smooth vector bundle. Show that if $f, g \in C^\infty(M)$ and if $\sigma, \tau \in \Gamma(E)$, then $f\sigma + g\tau \in \Gamma(E)$.

[Hint: Consider $\Phi \circ (f\sigma + g\tau)$, where Φ is a local trivialization of E.]

(c) Let $E := M \times \mathbb{R}^k$ be a product bundle over a topological manifold M. Show that there is a natural one-to-one correspondence between (continuous) sections of E and continuous functions from M to $\mathbb{R}^k$.

Moreover, if M is a smooth manifold, show that this is a one-to-one correspondence between smooth sections of E and smooth functions from M to $\mathbb{R}^k$. Deduce that there is a natural identification between the space $C^\infty(M)$ and the space of smooth sections of the trivial line bundle $M \times \mathbb{R} \to M$.

(d) Let $\pi\colon E \to M$ be a smooth vector bundle. Show that each element of E is in the image of a smooth global section of E.

[Hint: Use *Lemma 6.10*.]

Solution:

(a) Let $p \in M$ and let $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a smooth local trivialization, where U is a neighborhood of p. Given $q \in U$, write 0_q for the zero element of $E_q = \pi^{-1}(q)$. By definition we have $\zeta(q) = 0_q \in E_q$. Since $\Phi|_{E_q}\colon E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism, we obtain

$$\Phi\big(\zeta(q)\big) = \Phi|_{E_q}(0_q) = 0_{\{q\}\times\mathbb{R}^k} = (q, 0).$$

Hence, $\Phi \circ \zeta|_U = \mathrm{Id}_U \times \mathbb{O}_{\mathbb{R}^k}$ is smooth by [*Exercise Sheet* 3*, Exercise* 4]. As Φ is a diffeomorphism, we infer that $\zeta|_U$ is smooth, and as p was arbitrary, we conclude that ζ is smooth by [*Exercise Sheet* 3*, Exercise* 2].

Remark. By arguing as above (essentially replacing the words "smooth" with "continuous" and "diffeomorphism" with "homeomorphism"), we can also show that, more generally, the zero section of a (topological) vector bundle is continuous.

(b) Let $p \in M$ and let $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ be a smooth local trivialization, where U is an open neighborhood of $p \in M$. For $q \in U$, denote by $+_q$ the addition and by $\cdot_q$ the scalar multiplication of E_q. By definition we have

$$(f\sigma + g\tau)(q) = f(q) \cdot_q \sigma(q) +_q g(q) \cdot_q \tau(q) \in E_q.$$

Since $\Phi|_{E_q}\colon E_q \to \{q\} \times \mathbb{R}^k$ is a vector space isomorphism, we obtain

$$\Phi \circ (f\sigma + g\tau)(q) = f(q)(\Phi|_{E_q})\big(\sigma(q)\big) + g(q)(\Phi|_{E_q})\big(\tau(q)\big) \in \{q\} \times \mathbb{R}^k.$$

According to [*Exercise Sheet* 3*, Exercise* 4], showing that the map $\Phi \circ (f\sigma + g\tau)$ is smooth is equivalent to checking that its post-composition with both projections $\mathrm{pr}_1\colon U \times \mathbb{R}^k \to U$ and $\mathrm{pr}_2\colon U \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth. By the above formula we obtain

$$\mathrm{pr}_1 \circ \Phi \circ (f\sigma + g\tau) = \mathrm{Id}_U,$$

so it remains to check post-composition with pr_2. To this end, set $\hat{\sigma} := \mathrm{pr}_2 \circ \Phi \circ \sigma$ and $\hat{\tau} := \mathrm{pr}_2 \circ \Phi \circ \tau$, and note that both of them are smooth functions from U to $\mathbb{R}^k$. The above formula gives

$$\mathrm{pr}_2 \circ \Phi \circ (f\sigma + g\tau)(q) = f(q)\hat{\sigma}(q) + g(q)\hat{\tau}(q).$$

Due to the smoothness of the maps involved, this is also smooth. Thus, $\Phi \circ (f\sigma + g\tau)|_U$ is smooth, and as Φ is a diffeomorphism, we infer that $(f\sigma + g\tau)|_U$ is smooth. Since $p \in M$ was arbitrary, we conclude that $f\sigma + g\tau$ is a smooth global section of E by [*Exercise Sheet* 3*, Exercise* 2(*a*)].

(c) Consider the projection maps of the given product bundle:

$$\pi = \pi_M \colon E = M \times \mathbb{R}^k \to M,\ (p, v) \mapsto p,$$

and

$$\pi_{\mathbb{R}^k} \colon E = M \times \mathbb{R}^k \to \mathbb{R}^k,\ (p, v) \mapsto v.$$

Note that they are both continuous.

Now, let $f \colon M \to \mathbb{R}^k$ be a continuous function. Consider the continuous map

$$\sigma_f \colon M \to E,\ \sigma_f(p) = \big(p, f(p)\big)$$

and observe that

$$\big(\pi \circ \sigma_f\big)(p) = p = \mathrm{Id}_M(p),$$

so σ_f is a global section of E. Conversely, if $\sigma \colon M \to E = M \times \mathbb{R}^k$ is a global section of E, then $f_\sigma := \pi_{\mathbb{R}^k} \circ \sigma \colon M \to \mathbb{R}^k$ is a continuous map. Finally, it is easy to check that the assignments $f \mapsto \sigma_f$ and $\sigma \mapsto f_\sigma$ just described are inverse to each other; in other words, we have $\sigma = \sigma_{f_\sigma}$ and $f = f_{\sigma_f}$.

If M is a smooth manifold, and hence $\pi \colon E = M \times \mathbb{R}^k \to M$ is a smooth product bundle of rank k over M, then the above construction yields a one-to-one correspondence between smooth sections of E and smooth functions from M to $\mathbb{R}^k$, taking into account *Exercise 3*(e) and *Exercise 4* from *Exercise Sheet 3*. In particular, if $k = 1$, then there is a natural identification between the space $C^\infty(M)$ of smooth functions on M and the space of smooth sections of the trivial smooth line bundle $M \times \mathbb{R} \to M$.

(d) Fix $q \in E$ and set $p := \pi(q) \in M$. Consider the closed subset $A := \{p\} \subseteq M$ and the section

$$\sigma \colon A \to E,\ p \mapsto q \in E_p$$

of $E|_A = E_p$. We claim that σ extends to a smooth local section of E over some open neighborhood of p. Granting this claim for a moment, by *Lemma 6.10* there exists a smooth global section $\widetilde{\sigma}$ of E such that $\widetilde{\sigma}|_A = \sigma$; in particular, we also have $\widetilde{\sigma}(p) = \sigma(p) = q$, which shows that $q \in E$ lies in the image of the smooth global section $\widetilde{\sigma} \in \Gamma(E)$.

We now prove the above claim. By definition of a smooth vector bundle, there exists an open neighborhood U of p in M and a diffeomorphism $\Phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that

$$\pi_U \circ \Phi = \pi|_{\pi^{-1}(U)},$$

where $\pi_U \colon U \times \mathbb{R}^k \to U$ is the projection to the first factor. Since $q \in \pi^{-1}(U)$, its image under Φ is a pair $(p, v_q) \in U \times \mathbb{R}^k$ for some vector $v_q \in \mathbb{R}^k$. Consider now the map

$$t \colon U \to U \times \mathbb{R}^k,\ x \mapsto (x, v_q),$$

which is smooth by[*Exercise Sheet* 3*, Exercise* 4(*b*)], as well as the composite map

$$s := \Phi^{-1} \circ t \colon U \to \pi^{-1}(U),\ x \mapsto \Phi^{-1}(x, v_q),$$

which is also smooth by [*Exercise Sheet* 3*, Exercise* 3(*e*)], and satisfies

$$s(p) = \Phi^{-1}(p, v_q) = q = \sigma(p).$$

Moreover, we have

$$(\pi \circ s)(x) = \big((\pi \circ \Phi^{-1}) \circ t\big)(x) = (\pi_U \circ t)(x) = x = \mathrm{Id}_U(x) \ \text{ for every } x \in U.$$

Therefore, $s\colon U \to E$ is a smooth section of E over U and may also be regarded as a smooth extension of $\sigma\colon A \to E$. This proves the claim and completes the proof of (d).

Remark.

(1) Let $\pi\colon E \to M$ be a smooth vector bundle. According to *Exercise* 3, the set $\Gamma(E)$ of smooth global sections of E is an infinite-dimensional $\mathbb{R}$-vector space and a module over the ring $C^\infty(M)$.

(2) Using *Exercise* 5(a) and *Proposition 6.14* we give below another, somewhat more direct, solution to *Exercise* 3(d):

Fix $q \in E$ and set $p := \pi(q) \in M$. Consider the closed subset $A := \{p\} \subseteq M$ and the section

$$\sigma\colon A \to E,\ p \mapsto q \in E_p$$

of $E|_A = E_p$. There exists a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over an open neighborhood U of p, and hence a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over U (associated with Φ) by *Exercise* 5(a). We may thus write

$$\sigma(p) = \sum_{i=1}^{k} v^i \sigma_i(p) \in E_p$$

for some uniquely determined constants $v^i \in \mathbb{R}$, $1 \le i \le k$. We now define the map

$$s\colon U \to E,\ x \mapsto \sum_{i=1}^{k} v^i \sigma_i(x) \in E_x.$$

Note that s is a rough section of π, since $\big(\pi \circ s\big)(x) = x = \mathrm{Id}_U(x)$, and it is actually smooth by *Proposition 6.14*, since its component functions with respect to the smooth local frame $(\sigma_1, \dots, \sigma_k)$ are constant (namely, the constants $v^i \in \mathbb{R}$). Since we clearly have $s(p) = \sigma(p)$, the section s is a smooth extension of $\sigma\colon A \to E$ over U. Thus, the statement follows readily from *Lemma 6.10* (as above).

Exercise 4 (*Completion of smooth local frames for smooth vector bundles*)**:** Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Prove the following assertions:

(a) If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, where $1 \le m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ of E defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(b) If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of the fiber E_p for some $p \in M$, where $1 \leq m < k$, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for every $1 \leq i \leq m$.

(c) If $A \subseteq M$ is a closed subset and if $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ which are smooth in the sense described in *Lemma 6.10*, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for every $1 \leq i \leq k$.

[Hint: Use *Lemma 6.10*.]

Solution:

(a) Let V_0 be an open neighborhood of p in M over which there exists a smooth local trivialization $\Phi\colon \pi^{-1}(V_0) \to V_0 \times \mathbb{R}^k$ of E. As $\Phi\big(\sigma_1(p)\big), \dots, \Phi\big(\sigma_m(p)\big) \in \{p\} \times \mathbb{R}^k$ are linearily independent, there are vectors $v_{m+1}, \dots, v_k \in \mathbb{R}^k$ such that the set

$$\big\{\Phi\big(\sigma_1(p)\big), \dots, \Phi\big(\sigma_m(p)\big), (p, v_{m+1}), \dots, (p, v_k)\big\}$$

is a basis of $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$. For each $m < i \leq k$, define $\sigma_i\colon V_0 \to E$ by $\sigma_i(q) = \Phi^{-1}(q, v_i)$ and note that σ_i is smooth, as both $q \mapsto (q, v_i)$ and Φ^{-1} are so. Now, consider the function

$$d\colon V_0 \to \mathbb{R},\ q \mapsto \det\left(\mathrm{pr}_2\left(\Phi\big(\sigma_1(q)\big)\right), \dots, \mathrm{pr}_2\left(\Phi\big(\sigma_k(q)\big)\right)\right).$$

We have $d(p) \neq 0$, since by construction the set

$$\left\{\mathrm{pr}_2\left(\Phi\big(\sigma_1(p)\big)\right), \dots, \mathrm{pr}_2\left(\Phi\big(\sigma_k(p)\big)\right)\right\}$$

is a basis of $\mathbb{R}^k$. As d is continuous, there exists a neighborhood V of p such that $d|_V$ is nowhere zero. Hence, $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(b) We may complete $(v_1, \dots, v_m)$ to a basis $(v_1, \dots, v_k)$ of $E_p \cong \mathbb{R}^k$. Let U be an open neighborhood of $p \in M$ such that there exists a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over U. As in part (a), we define $\sigma_i\colon U \to E$ by $\sigma_i(q) := \Phi^{-1}(q, v_i)$, and again by continuity of the determinant, this gives a smooth local frame on some open neighborhood $V \subseteq U$ of p.

(c) By hypothesis and by *Lemma 6.10* (applied for $U = M$), for each $i \in \{1, \dots, k\}$ there exists a smooth global section τ_i of E such that $\tau_i|_A = \sigma_i$. Therefore, for every $p \in A$ the set $\big\{\tau_1(p), \dots, \tau_k(p)\big\}$ is a basis of E_p, and by continuity of the determinant there exists an open neighborhood U_p of p in M such that $\big\{\tau_1(q), \dots, \tau_k(q)\big\}$ is a basis of E_q for each $q \in U_p$. Thus, $U := \bigcup_{p \in A} U_p$ is an open subset of M containing A and additionally for every $x \in U$ the set $\big\{\tau_1(x), \dots, \tau_k(x)\big\}$ is a basis of the fiber E_x; in other words, $(\tau_1, \dots, \tau_k)$ is a smooth local frame for E over the open neighborhood U of A.

Exercise 5 (*Correspondence between smooth local frames and smooth local trivializations*)**:** Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth n-manifold M.

(a) Given a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over U, construct a smooth local frame (σ_i) for E over U. (We say that the smooth local frame (σ_i) is associated with the smooth local trivialization Φ.)

(b) Show that every smooth local frame (σ_i) for E is associated with a smooth local trivialization Φ of E.

[Hint: Define the inverse of Φ using (σ_i) and show that it is a bijective local diffeomorphism to conclude.]

(c) Deduce that E is smoothly trivial if and only if it admits a smooth global frame. Interpret this result in case that E is a smooth *line bundle*, i.e., when $k = 1$.

(d) Let (U, φ) be a smooth coordinate chart for M with coordinate functions (x^i) and assume that there exists a smooth local frame (σ_i) for E over U. Consider the map

$$\widetilde{\varphi}\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^k,\ v^i\sigma_i(p) \mapsto \big(x^1(p), \dots, x^n(p), v^1, \dots, v^k\big).$$

Show that $\big(\pi^{-1}(U), \widetilde{\varphi}\big)$ is a smooth coordinate chart for E.

Solution:

(a) Let $e_1, \dots, e_k$ be the standard basis of $\mathbb{R}^k$. As in *Exercise* 4(b), the smooth local sections $\sigma_i\colon U \to E$ defined by $\sigma_i(q) = \Phi^{-1}(q, e_i)$ determine a smooth local frame for E over U.

(b) Let (σ_i) be a smooth local frame for E over an open subset U of M. Consider the map

$$\Psi\colon U \times \mathbb{R}^k \to \pi^{-1}(U),\ \Psi(q, v) := v_1 \cdot_q \sigma_1(q) + \ldots + v_k \cdot_q \sigma_k(q) \in E_q.$$

It is straightforward to check that $\pi \circ \Psi = \mathrm{pr}_1$.

Let us first show that Ψ is bijective. To prove its injectivity, suppose that $\Psi(q, v) = \Psi(q', v')$. By applying π we see that $q = q'$, and thus

$$v_1 \cdot_q \sigma_1(q) + \ldots + v_k \cdot_q \sigma_k(q) = v_1' \cdot_q \sigma_1(q) + \ldots + v_k' \cdot_q \sigma_k(q)$$

inside E_q. As $\sigma_1(q), \dots, \sigma_k(q)$ is a basis of E_q, we infer that $v = v'$, and thus we establish the injectivity of Ψ. Now, to prove the surjectivity of Ψ, let $e \in \pi^{-1}(U)$ be arbitrary. Set $q = \pi(e)$ and let $v = (v_1, \dots, v_k)$ be such that

$$e = v_1 \cdot_q \sigma_1(q) + \ldots + v_k \cdot_q \sigma_k(q)$$

inside E_q. Then $e = \Psi(q, v)$, so we are done.

It remains to check that Ψ is a local diffeomorphism. Let $p \in U$ and let $\Phi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$ be a smooth local trivialization of E, where V is an open neighborhood of p contained in U. Since Φ is a diffeomorphism, if we could show that $\Phi \circ \Psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism from $V \times \mathbb{R}^k$ to itself, then we would infer that $\Psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism from $V \times \mathbb{R}^k$ to its image $\pi^{-1}(V)$.

Since $\Phi \circ \sigma_i|_V\colon V \to V \times \mathbb{R}^k$ is smooth and since post-composition with pr_1 equals Id_V, we see that it is of the form

$$\Phi \circ \sigma_i|_V(q) = \Big(q, \big(\sigma_i^1(q), \dots, \sigma_i^k(q)\big)\Big)$$

for some smooth functions $\sigma_i^1, \dots, \sigma_i^k \colon V \to \mathbb{R}$. If we denote by $A \colon V \to \mathrm{Mat}(k \times k, \mathbb{R})$ the function sending q to the matrix $\big(\sigma_i^j(q)\big)_{1 \le j,i \le k}$ (where j is the index for the lines and i is the index for the columns of the matrix), then A is smooth, as every component is smooth. Furthermore, the image of A lies in $\mathrm{GL}(k, \mathbb{R})$ because $\sigma_i(q), \dots, \sigma_i(q)$ is a basis of E_q by assumption. Now, by construction of Ψ, it is straightforward to check that for any $(q, v) \in V \times \mathbb{R}^k$ we have

$$\big(\Phi \circ \Psi\big)(q, v) = \big(q, A(q) \cdot v\big) \in V \times \mathbb{R}^k.$$

This is clearly smooth, as A is smooth. We then also see that $(\Phi \circ \Psi|_{V \times \mathbb{R}^k})^{-1}$ sends (q, v) to $(q, A(q)^{-1} \cdot v)$, which is smooth as well (we use here that the map $\mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(k, \mathbb{R})$ sending a matrix to its inverse is smooth). Therefore, $\Psi|_{V \times \mathbb{R}^k} \colon V \times \mathbb{R}^k \to \pi^{-1}(V)$ is a diffeomorphism, as desired.

In conclusion, Ψ is a bijective local diffeomorphism, and hence a global diffeomorphism by *Proposition 4.9*(f). It is now straightforward to check that $\Phi = \Psi^{-1} \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a smooth local trivialization of E over U such that the given smooth local frame (σ_i) is associated with Φ.

(c) Recall that E is smoothly trivial if and only if it admits a smooth global trivialization. Thus, E is smoothly trivial if and only if it admits a smooth global frame by (a) and (b).

Assume now that E is a smooth vector bundle of rank $k = 1$ over M. Then E is smoothly trivial if and only if it admits a smooth global frame. Such a frame consists of a single smooth global section $\sigma \colon M \to E$ with the property that for each $p \in M$, the element $\sigma(p) \in E_p$ is a basis of the 1-dimensional $\mathbb{R}$-vector space E_p, and hence $\sigma(p) \neq 0$. Conversely, every smooth global section $\sigma \colon M \to E$ of E such that $\sigma(p) \in E_p \setminus \{0\}$ determines a smooth global frame for E. In conclusion, the smooth *line bundle* $E \to M$ is smoothly trivial if and only if it admits a *nowhere vanishing* smooth global section.

(d) By part (b), there exists a smooth local trivialization $\Phi \colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that (σ_i) is associated with Φ. In particular, we have $\big(\Phi \circ \sigma_i\big)(q) = (q, e_i)$ for all $q \in U$, where $e_1, \dots, e_k$ is the standard basis of $\mathbb{R}^k$. Now let $\psi \colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^k$ be the composition $\psi = (\varphi \times \mathrm{Id}_{\mathbb{R}^k}) \circ \Phi$, where $\varphi \times \mathrm{Id}_{\mathbb{R}^k} \colon U \times \mathbb{R}^k \to \varphi(U) \times \mathbb{R}^k$ is defined by applying φ on U and $\mathrm{Id}_{\mathbb{R}^k}$ on $\mathbb{R}^k$. Note that ψ is a diffeomorphism as both Φ and $\varphi \times \mathrm{Id}_{\mathbb{R}^k}$ are diffeomorphisms. To see how ψ acts on the points of $\pi^{-1}(U)$, let $e \in \pi^{-1}(U)$ be arbitrary, and set $q = \pi(e) \in U$. As $\sigma_1(q), \dots, \sigma_k(q)$ is a basis of E_q, there exist real numbers $v^1, \dots, v^k$ such that $e = v^i \sigma_i(q)$. As $\big(\Phi \circ \sigma_i\big)(q) = (q, e_i)$, we obtain $\Phi(e) = (q, (v^i))$, so $\psi(e) = \big(\varphi(q), (v^i)\big)$. Therefore, $\psi = \widetilde{\varphi}$. As ψ is a diffeomorphism, this proves that $\big(\pi^{-1}(U), \widetilde{\varphi}\big)$ is a smooth coordinate chart for E.

Remark. By arguing as in the solution to *Exercise* 3(a)(b) – essentially by replacing the words "smooth" with "continuous" and "diffeomorphism" with "homeomorphism" – one can also show that, more generally, there is a correspondence between (continuous) local frames and (continuous) local trivializations for any (topological) vector bundle. This allows one to prove the topological case of *Proposition 6.14* with an essentially identical argument to the smooth case (which was treated in the lecture).

Exercise 6 (*Uniqueness of the smooth structure on* TM)**:** Let M be a smooth n-manifold. Show that the topology and smooth structure on the tangent bundle TM constructed in *Proposition 3.12* are the unique ones with respect to which $\pi\colon TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.

[Hint: Use *Exercise* 5(d).]

Solution: Denote by $(TM)'$ a smooth manifold with underlying set $TM = \bigsqcup_{p\in M} T_pM$, but with possibly different topology and smooth structure than the usual tangent bundle TM, such that the map $\pi'\colon (TM)' \to M$, sending $v \in T_pM$ to $p \in M$, gives $(TM)'$ the structure of a smooth vector bundle over M such that all the coordinate vector fields are smooth local sections. (Note that, as set-theoretic maps, π and π' are the same, but we denote them differently to emphasize that their source may have different topology and smooth structure.) By assumption, if $\big(U, (x^1, \ldots, x^n)\big)$ is a smooth chart for M, then the maps

$$\begin{aligned} \frac{\partial}{\partial x^i}\colon U &\to (TM)' \\ p &\mapsto \left.\frac{\partial}{\partial x^i}\right|_p \end{aligned}$$

are smooth local sections of π'. Since for every $p \in U$ the vectors $\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$ form a basis of T_pM, it follows that $\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)_{1\le i\le n}$ is a smooth local frame for $(TM)'$, and according to *Exercise* 5(d), the map

$$\begin{aligned} \widetilde{\varphi}\colon (\pi')^{-1}(U) &\to \varphi(U) \times \mathbb{R}^n \\ \left(p, v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) &\mapsto \big(\varphi(p), v^1, \ldots, v^n\big) \end{aligned}$$

is a smooth chart for $(TM)'$. But the same holds for TM (as we saw in the proof of *Proposition 3.12*). It follows that the identity map $TM \to (TM)'$ is a diffeomorphism. In particular, it is a homeomorphism, and thus also the topology agrees.

Remark. We somewhat used in *Exercise* 6 that the smooth structure actually determines the topology. That is, we have the following:

Let M be a set and let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on M, both endowing it with the structure of a topological manifold. Supposed that $\mathcal{A}$ is an atlas for both topologies, such that both $(M, \mathcal{T}, \mathcal{A})$ and $(M, \mathcal{T}', \mathcal{A})$ are smooth manifolds. Then $\mathcal{T} = \mathcal{T}'$. Indeed, the identity $\mathrm{Id}_M\colon (M, \mathcal{T}, \mathcal{A}) \to (M, \mathcal{T}', \mathcal{A})$ is smooth, as we have the same atlas on both sides; in particular, it is continuous, so $\mathcal{T}' \subseteq \mathcal{T}$. A symmetric argument also shows that the reverse inclusion holds. Therefore, $\mathcal{T} = \mathcal{T}'$, as claimed.