Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 10

Exercise 1 (to be submitted by Thursday, 28.11.2024, 16:00):

(a) Let $\pi\colon E \to M$ be a smooth vector bundle over a smooth manifold M. Show that π is a surjective smooth submersion.

(b) Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose that $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Show that the transition function $\tau\colon U \cap V \to \mathrm{GL}(k, \mathbb{R})$ between Φ and Ψ is smooth.

(c) Consider the tangent bundle $\pi\colon TM \to M$ of a smooth n-manifold M and let $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^n$ and $\Psi\colon \pi^{-1}(V) \to V \times \mathbb{R}^n$ be the smooth local trivializations of TM associated with two smooth charts (U, φ) and (V, ψ) for M. Determine the transition function $\tau\colon U \cap V \to \mathrm{GL}(n, \mathbb{R})$ between Φ and Ψ.

(d) Consider the tangent bundle $\pi\colon T\mathbb{S}^2 \to \mathbb{S}^2$ of the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Compute the transition function associated with the two local trivializations determined by stereographic coordinates.

(e) Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M, and that for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ of E. For each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, let $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ be the transition function between the smooth local trivializations Φ_α and Φ_β. Show that the following identity is satisfied for all $\alpha, \beta, \gamma \in A$:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p) \quad \text{for all } p \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{$\star$}$$

Exercise 2 (*Smooth vector bundle construction lemma*)**:**
Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M. Suppose that for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that ($\star$) is satisfied for all $\alpha, \beta, \gamma \in A$. Show that there is a smooth vector bundle $E \to M$ of rank

k with smooth local trivializations $\Phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose transitions functions are the given maps $\tau_{\alpha\beta}$.

[Hint: Define an appropriate equivalence relation on $\coprod \left(U_\alpha \times \mathbb{R}^k\right)$ and use the vector bundle chart lemma.]

Exercise 3:

(a) Show that the zero section of every smooth vector bundle is smooth.

[Hint: Consider $\Phi \circ \zeta$, where Φ is a local trivialization.]

(b) Let $\pi\colon E \to M$ be a smooth vector bundle. Show that if $f, g \in C^\infty(M)$ and if $\sigma, \tau \in \Gamma(E)$, then $f\sigma + g\tau \in \Gamma(E)$.

[Hint: Consider $\Phi \circ (f\sigma + g\tau)$, where Φ is a local trivialization of E.]

(c) Let $E := M \times \mathbb{R}^k$ be a product bundle over a topological manifold M. Show that there is a natural one-to-one correspondence between continuous sections of E and continuous functions from M to $\mathbb{R}^k$.

Moreover, if M is a smooth manifold, show that this is a one-to-one correspondence between smooth sections of E and smooth functions from M to $\mathbb{R}^k$. Deduce that there is a natural identification between the space $C^\infty(M)$ and the space of smooth sections of the trivial line bundle $M \times \mathbb{R} \to M$.

(d) Let $\pi\colon E \to M$ be a smooth vector bundle. Show that each element of E is in the image of a smooth global section of E.

[Hint: Use *Lemma 6.10*.]

Exercise 4 (*Completion of smooth local frames for smooth vector bundles*)**:**
Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth manifold M. Prove the following assertions:

(a) If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m-tuple of smooth local sections of E over an open subset $U \subseteq M$, where $1 \leq m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ of E defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(b) If $(v_1, \dots, v_m)$ is a linearly independent m-tuple of elements of the fiber E_p for some $p \in M$, where $1 \leq m < k$, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for every $1 \leq i \leq m$.

(c) If $A \subseteq M$ is a closed subset and if $(\tau_1, \dots, \tau_k)$ is a linearly independent k-tuple of sections of $E|_A$ which are smooth in the sense described in *Lemma 6.10*, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for every $1 \leq i \leq k$.

[Hint: Use *Lemma 6.10*.]

Exercise 5 (*Correspondence between smooth local frames and smooth local trivializations*)**:** Let $\pi\colon E \to M$ be a smooth vector bundle of rank k over a smooth n-manifold M.

(a) Given a smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^k$ of E over U, construct a smooth local frame (σ_i) for E over U. (We say that the smooth local frame (σ_i) is associated with the smooth local trivialization Φ.)

(b) Show that every smooth local frame (σ_i) for E is associated with a smooth local trivialization Φ of E.

[Hint: Define the inverse of Φ using (σ_i) and show that it is a bijective local diffeomorphism to conclude.]

(c) Deduce that E is smoothly trivial if and only if it admits a smooth global frame. Interpret this result in case that E is a smooth *line bundle*, i.e., when $k = 1$.

(d) Let (U, φ) be a smooth coordinate chart for M with coordinate functions (x^i) and assume that there exists a smooth local frame (σ_i) for E over U. Consider the map

$$\widetilde{\varphi}\colon \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^k,\ v^i\sigma_i(p) \mapsto \big(x^1(p), \ldots, x^n(p), v^1, \ldots, v^k\big).$$

Show that $\big(\pi^{-1}(U), \widetilde{\varphi}\big)$ is a smooth coordinate chart for E.

Exercise 6 (*Uniqueness of the smooth structure on* TM)**:**
Let M be a smooth n-manifold. Show that the topology and smooth structure on the tangent bundle TM constructed in *Proposition 3.12* are the unique ones with respect to which $\pi\colon TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.

[Hint: Use *Exercise* 5(d).]