Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 9 – Solutions

Exercise 1: Let M be a smooth manifold. Show that if S is an embedded submanifold of M, then the subspace topology on S and the smooth structure on S described in *Theorem 5.6* are the only topology and smooth structure with respect to which S is an embedded (or immersed) submanifold.

[Hint: Use [*Exercise Sheet* 8, *Exercise* 5(c)].]

Solution: Consider some other topology and smooth structure on S, denote by $\widetilde{S}$ the resulting smooth manifold, and suppose that $\widetilde{\iota}\colon \widetilde{S} \hookrightarrow M$ is a smooth immersion. (For the exercise as stated, one can suppose that $\widetilde{\iota}$ is a smooth embedding, but the weaker assumption that it is a smooth immersion is actually sufficient). By [*Exercise Sheet* 8, *Exercise* 5(c)] we infer that the corestriction $\widetilde{\iota}|^S\colon \widetilde{S} \to S$ is smooth as well. If we denote by $\iota\colon S \hookrightarrow M$ the inclusion of S into M, then we have $\iota \circ \left(\widetilde{\iota}|^S\right) = \widetilde{\iota}$, so given $p \in \widetilde{S}$, by taking differentials we obtain

$$d\iota_p \circ d\left(\widetilde{\iota}|^S\right)_p = d\widetilde{\iota}_p.$$

Since $d\iota_p$ and $d\widetilde{\iota}_p$ are injective, we deduce that $d\left(\widetilde{\iota}|^S\right)_p$ is injective as well. Hence, $\widetilde{\iota}|^S$ is a smooth immersion, and as it is also bijective, by the *Global Rank Theorem* we conclude that $\widetilde{\iota}|^S$ is a diffeomorphism. Since it is the identity on the underlying set S, we deduce that the topology and smooth structure of $\widetilde{S}$ are identical to the ones of S.

Remark. Thanks to this uniqueness result, we now know that a subset $S \subseteq M$ is an embedded submanifold if and only if it satisfies the local slice condition, and if so, its topology and smooth structure are uniquely determined. Because the local slice condition is a local condition, if every point $p \in S$ has a neighborhood $U \subseteq M$ such that $U \cap S$ is an embedded k-submanifold of U, then S is an embedded k-submanifold of M.

Exercise 2: Let M be a smooth manifold. Show that if S is an immersed submanifold of M, then for the given topology on S, there exists only one smooth structure making S into an immersed submanifold.

[Hint: Use [*Exercise Sheet* 8, *Exercise* 5(b)].]

Solution: Denote by ι the inclusion map $S \hookrightarrow M$ of the immersed submanifold S of M and by $\widetilde{S}$ the topological space S endowed now with another smooth structure such that the inclusion map $\widetilde{\iota}\colon \widetilde{S} \hookrightarrow M$ is a smooth immersion. Note that $\widetilde{S}$ is an immersed submanifold of M. Since S and $\widetilde{S}$ have the same topology by assumption, both maps $\iota\colon S \to \widetilde{S}$ and $\widetilde{\iota}\colon \widetilde{S} \to S$ are continuous, and hence smooth by [*Exercise Sheet* 8*, Exercise* 5(*b*)], so they are inverses of each other. Therefore, S is diffeomorphic to $\widetilde{S}$.

Remark. It is certainly possible for a given subset S of a smooth manifold M to have more than one topology making it into an immersed submanifold of M. However, for *weakly embedded submanifolds* we have the following uniqueness result, which can be proved similarly to *Exercise* 2: *If M is a smooth manifold and if S is a weakly embedded submanifold of M, then S has only one topology and smooth structure with respect to which it is an immersed submanifold of M.*

Exercise 3:

(a) Let M be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_pM$ is in T_pS if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma\colon J \to M$ be a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$.

Solution:

(a) Assume that the given vector $v \in T_pM$ lies also in T_pS, which is identified with $d\iota_p(T_pS)$, so that $v = d\iota_p(w)$ for some $w \in T_pS$. By [*Exercise Sheet* 4*, Exercise* 5(*a*)] there exists a smooth curve $\gamma\colon J \to S$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = w$. Since S is an immersed (or embedded) submanifold of M, the inclusion map $\iota\colon S \hookrightarrow M$ is a smooth immersion, so the composite map $\iota \circ \gamma\colon J \to M$ is a smooth curve in M whose image is clearly contained in S, it satisfies $0 \in J$, $(\iota \circ \gamma)(0) = p$, and additionally by [*Exercise Sheet* 4*, Exercise* 5(*b*)] we have

$$(\iota \circ \gamma)'(0) = d\iota_{\gamma(0)}\big(\gamma'(0)\big) = d\iota_p(w) = v.$$

The converse follows immediately from [*Exercise Sheet* 4*, Exercise* 5(*a*)] taking the identification of T_pS with $d\iota_p(T_pS)$ into account.

(b) By assumption and by [*Exercise Sheet* 8*, Exercise* 5(*c*)] the given map γ is also smooth as a map from J to S, so the statement follows immediately from part (a).

Remark. If $S \subseteq M$ is merely immersed, then the conclusion of *Exercise* 3(*b*) is not true in general. Indeed, here is a counterexample:

Consider the smooth map

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t).$$

According to *Example 4.5*(2), *Proposition 5.13* and *Exercise* 5(a) below, its image $S := \beta\big(-\pi,\pi\big)$, the figure-eight curve, is an immersed but not embedded submanifold of $\mathbb{R}^2$. Observe that $\beta(0) = (0,0)$ and that

$$d\beta_0\left(\left.\frac{d}{dt}\right|_{t=0}\right) = 2\cos(0)\left.\frac{\partial}{\partial x}\right|_{(0,0)} + \cos(0)\left.\frac{\partial}{\partial y}\right|_{(0,0)} = 2\left.\frac{\partial}{\partial x}\right|_{(0,0)} + \left.\frac{\partial}{\partial y}\right|_{(0,0)}.$$

Since $\beta\colon(-\pi,\pi)\to S$ is a diffeomorphism, the tangent vector $2\left.\frac{\partial}{\partial x}\right|_{(0,0)}+\left.\frac{\partial}{\partial y}\right|_{(0,0)}$ constitutes a basis for $T_{(0,0)}S$. Consider now the smooth map

$$\gamma\colon(-\pi,\pi)\to\mathbb{R}^2,\ t\mapsto(\sin 2t,\,-\sin t)$$

and observe that its image lies in S. Moreover, $\gamma(0)=(0,0)$ and we have

$$d\gamma_0\left(\left.\frac{d}{dt}\right|_{t=0}\right) = 2\left.\frac{\partial}{\partial x}\right|_{(0,0)} - \left.\frac{\partial}{\partial y}\right|_{(0,0)},$$

which clearly does not lie in the subspace $T_{(0,0)}S$ of $T_{(0,0)}\mathbb{R}^2$.

Finally, note that the same (counter)example shows that the characterization of T_pS given in *Proposition 5.19* does not work in the merely immersed case.

Exercise 4:

(a) Let M be a smooth manifold and let $S\subseteq M$ be an embedded submanifold. Show that if $\Phi\colon U\to N$ is a local defining map for S, then it holds that

$$T_pS\cong\ker\big(d\Phi_p\colon T_pM\to T_{\Phi(p)}N\big)\ \text{ for every } p\in S\cap U.$$

(b) Let M be a smooth manifold. Suppose that $S\subseteq M$ is a level set of a smooth submersion $\Phi=(\Phi_1,\dots,\Phi_k)\colon M\to\mathbb{R}^k$. Show that a vector $v\in T_pM$ is tangent to S if and only if $v\Phi_1=\ldots=v\Phi_k=0$.

Solution:

(a) Recall that we identify T_pS with its image $d\iota_p(T_pS)\subseteq T_pM$, where $\iota\colon S\hookrightarrow M$ is the inclusion map, which is a smooth embedding by assumption. Note that by hypothesis we have $S\cap U=\Phi^{-1}(q)$ for some $q\in N$. Therefore, we have $\Phi\circ\iota|_{S\cap U}=c_q$, where $c_q\colon S\cap U\to N$ is the constant map on $S\cap U$ with value $q\in N$. Thus, if $p\in S\cap U$ is arbitrary, then

$$0=d(c_q)_p=d\Phi_p\circ d(\iota|_{S\cap U})_p.$$

Hence, the differential $d(\iota|_{S\cap U})_p$ induces an injective map from T_pS to $\ker d\Phi_p$ (because ι is an embedding).

In order to conclude, it suffices to show that both spaces have the same dimension. Denote by m, n, s the dimension of M, N, S, respectively. By *Corollary 5.10* the codimension of S in M is n (i.e., $m - s = n$). On the other hand, by the *rank-nullity theorem* from linear algebra and by the surjectivity of $d\Phi_p$ we have

$$n = \dim \operatorname{im} d\Phi_p = \underbrace{\dim T_pM}_{=m} - \dim \ker d\Phi_p$$
$$\implies \dim \ker d\Phi_p = m - n = s.$$

Hence, T_pS and $\ker d\Phi_p$ have the same dimension s, and are thus identified via $d\iota_p$.

(b) Fix $p \in S$. By part (a) we know that $v \in T_pM$ is tangent to S if and only if $d\Phi_p(v) = 0$. Denote by $\mathrm{pr}_1, \dots, \mathrm{pr}_k \colon \mathbb{R}^k \to \mathbb{R}$ the projection maps to the corresponding coordinates. By the description of $T_p\mathbb{R}^k$, note that a vector $w \in T_p\mathbb{R}^k$ is 0 if and only if $w(\mathrm{pr}_i) = 0$ for all $1 \leq i \leq k$. Hence,

$$d\Phi_p(v) = 0 \iff d\Phi_p(v)(\mathrm{pr}_i) = 0,\ \forall 1 \leq i \leq k \iff v\big(\mathrm{pr}_i \circ \Phi\big) = v\Phi_i = 0,\ \forall 1 \leq i \leq k.$$

Exercise 5:

(a) Consider the smooth curve

$$\beta \colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t, \sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

(b) Consider the smooth function

$$\Phi \colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Show that the level set $\Phi^{-1}(0)$ is an immersed submanifold of $\mathbb{R}^2$.

(c) Consider the smooth function

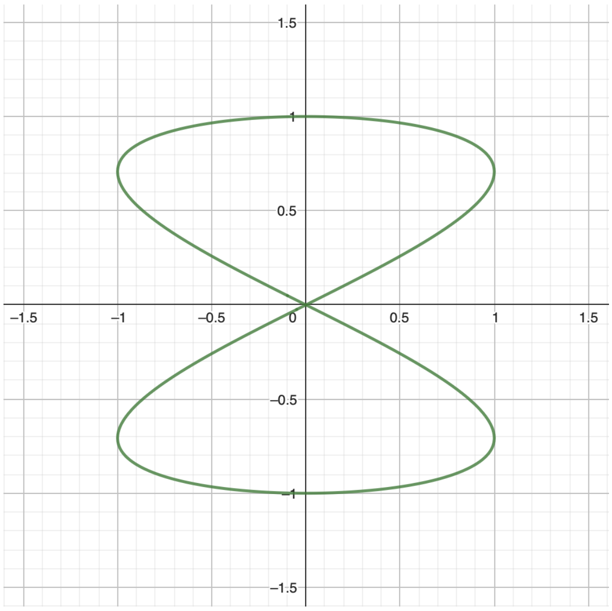
$$\Psi \colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^3.$$

Show that the level set $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

[Hint: Argue by contradiction and use *Exercise* 3(a).]

Solution:

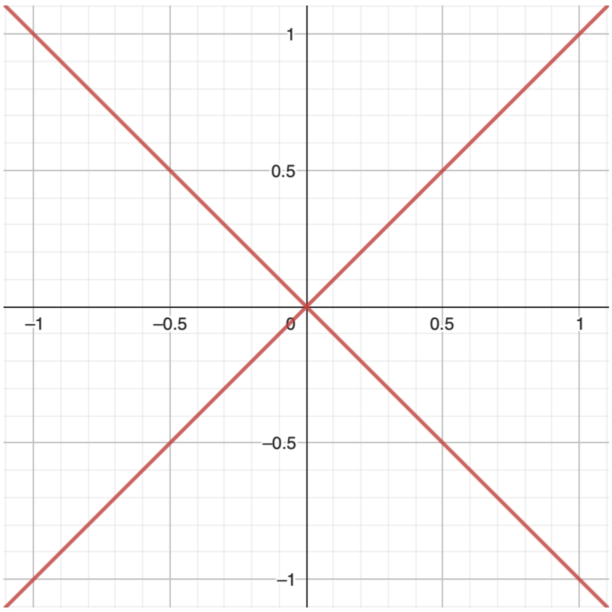
(a) Endowed with the subspace topology inherited from $\mathbb{R}^2$, the image of β (which has been plotted below) is not a topological manifold. Indeed, essentially the same argument as the one presented in the solution of [*Exercise Sheet* 1*, Exercise* 4] shows that $\beta(-\pi, \pi)$ is not locally Euclidean at the (self-intersection) point $(0, 0) \in \beta(-\pi, \pi)$. Therefore, the image of β cannot be an embedded submanifold of $\mathbb{R}^2$.



(b) The level set

$$\begin{aligned}\Phi^{-1}(0) &= \left\{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 0\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \mid (y-x)(y+x) = 0\right\}\end{aligned}$$

has been plotted below.



Even though it is not an embedded submanifold of $\mathbb{R}^2$, as already demonstrated in the solution of [*Exercise Sheet* 8*, Exercise* 3(b)], we will show that $\Phi^{-1}(0)$ is the image of an injective smooth immersion, and hence $\Phi^{-1}(0)$ can be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$; see *Proposition 5.13.*

Recall that the problem lies at the point where the two lines cross, which is the origin in our case. The idea is to view $\Phi^{-1}(0)$ as the disjoint union of two lines (where we remove the origin from one of them). To make this precise, let us start with a general construction for smooth manifolds, namely the *disjoint union*.

For $i \in \{0,1\}$, let $(M_i, \mathcal{T}_i, \mathcal{A}_i)$ be a smooth manifold, and assume that both of them have the same dimension. The set theoretic disjoint union of M_0 and M_1 is the set

$$M_0 \sqcup M_1 \coloneqq \big\{(m,i) \mid i \in \{0,1\},\ m \in M_i\big\}.$$

We can endow the set $M_0 \sqcup M_1$ with a natural topology $\mathcal{T}$, called the *disjoint union topology* as follows. For each $i \in \{0,1\}$, denote by ι_i the natural inclusion

$$\begin{aligned} \iota_i \colon M_i &\to M_0 \sqcup M_1 \\ m &\mapsto (m,i) \end{aligned}$$

and define $\mathcal{T}$ by

$$\mathcal{T} \coloneqq \big\{U \subseteq M_0 \sqcup M_1 \mid \forall i \in \{0,1\} :\ \iota_i^{-1}(U) \in \mathcal{T}_i\big\}.$$

It is straightforward to check that $\mathcal{T}$ is a topology on $M_0 \sqcup M_1$, and in fact it is the finest topology on $M_0 \sqcup M_1$ for which the inclusions ι_i are continuous. Furthermore, one can observe that ι_i is an open map, and as it is injective, ι_i is a homeomorphism onto the open subset $\iota_i(M_i)$ of $M_0 \sqcup M_1$. Therefore, we can identify $(M_i, \mathcal{T}_i)$ with $\iota_i(M_i)$ endowed with the subspace topology (note also that the open subsets of $M_0 \sqcup M_1$ are precisely the subsets of the form $\iota_0(U_0) \cup \iota_1(U_1)$ where $U_0 \subseteq M_0$ resp. $U_1 \subseteq M_1$ are open). In particular, $M_0 \sqcup M_1$ is locally Euclidean and second countable, since we have the open cover $M_0 \sqcup M_1 = \iota_0(M_0) \cup \iota_1(M_1)$, where both open subsets are locally Euclidean and second countable. Finally, $M_0 \sqcup M_1$ is also Hausdorff, because if $(m,i), (n,j)$ are distinct elements of $M_0 \sqcup M_1$, then either $i \neq j$, in which case they can be separated by the disjoint open subsets $\iota_i(M_i)$ and $\iota_j(M_j)$, or we have $i = j$, in which case there exist disjoint open subsets $U, V \subseteq M_i$ with $m \in U$ and $n \in V$, so that (m,i) and (n,i) are separated by $\iota_i(U)$ and $\iota_i(V)$.

In conclusion, $(M_0 \sqcup M_1, \mathcal{T})$ is a topological manifold. Let us now endow it with a smooth structure. As ι_i is an open injection, we can consider the following collection $\iota_{i,*}(\mathcal{A}_i)$ of charts on $M_0 \sqcup M_1$:

$$\iota_{i,*}(\mathcal{A}_i) \coloneqq \big\{\big(\iota_i(U), \varphi \circ \iota_i^{-1}\big) \mid (U,\varphi) \in \mathcal{A}_i\big\}\,,\ i \in \{0,1\}.$$

It is straightforward to check that $\iota_{0,*}(\mathcal{A}_0) \cup \iota_{1,*}(\mathcal{A}_1)$ is a smoothly compatible atlas on $M_0 \sqcup M_1$, and therefore induces a smooth structure $\mathcal{A}$ on $M_0 \sqcup M_1$ by *Proposition 1.8*(a). As a final remark on this abstract construction, note that ι_i is a diffeomorphism onto the open subset $\iota_i(M_i)$ of $M_0 \sqcup M_1$. This essentially follows from the fact that for any smooth chart (U,φ) on M_i we have the smooth chart $\big(\iota_i(U), \varphi \circ \iota_i^{-1}\big)$ on $M_0 \sqcup M_1$.

With this construction at hand, it is straightforward to solve the exercise. Consider $M_0 = \mathbb{R}$ and $M_1 = \mathbb{R} \setminus \{0\}$, both endowed with the standard smooth structure. Let $M = M_0 \sqcup M_1$ be the smooth manifold which is their disjoint union. Consider the map

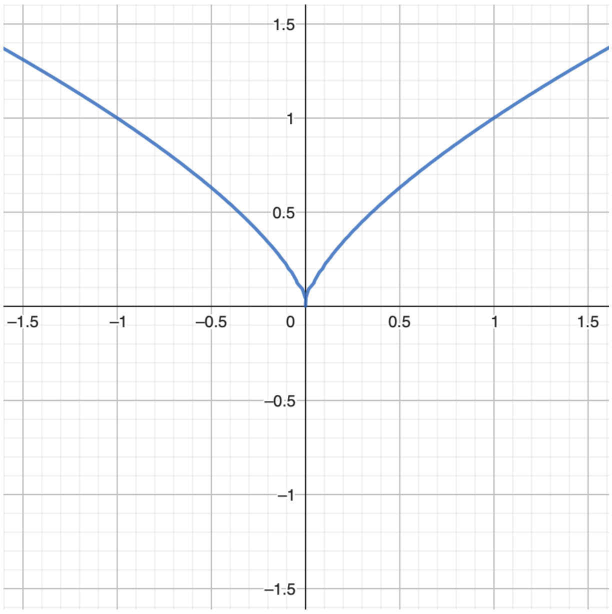
$$\begin{aligned} F \colon M &\to \mathbb{R}^2 \\ (m,i) &\mapsto \big(m, (-1)^i m\big). \end{aligned}$$

Notice that $F \circ \iota_0 \colon \mathbb{R} \to \mathbb{R}^2$ is the map $x \mapsto (x, x)$ and $F \circ \iota_1 \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}^2$ is the map $x \mapsto (x, -x)$, and that both of these maps are smooth embeddings. As being continuous resp. smooth resp. a smooth immersion can be checked on an open cover, and ι_i are diffeomorphisms, we conclude at once that F is continuous, smooth, and even a smooth immersion. Observe also that F is injective. Therefore, by *Proposition 5.13* the image $\Phi^{-1}(0) = F(M)$ is an immersed submanifold of $\mathbb{R}^2$.

(c) The level set

$$\Psi^{-1}(0) = \big\{(x, y) \in \mathbb{R}^2 \mid x^2 - y^3 = 0\big\}$$

has been plotted below.



We assume that $\Psi^{-1}(0)$ can be given a topology and smooth structure making it into an immersed submanifold of $\mathbb{R}^2$ and we will derive a contradiction using *Exercise* 3(a). To this end, observe that $\Psi^{-1}(0)$ must be 1-dimensional; indeed, $\Psi^{-1}(0) \setminus \{(0, 0)\}$ is an embedded 1-submanifold of $\mathbb{R}^2$, as its two connected components, corresponding to $(x, y) \in \Phi^{-1}(0)$ with $x < 0$ (the left branch) and $(x, y) \in \Phi^{-1}(0)$ with $x > 0$ (the right branch), are the graphs of the smooth functions $x \in (-\infty, 0) \mapsto x^{\frac{2}{3}}$ and $x \in (0, +\infty) \mapsto x^{\frac{2}{3}}$, respectively. Therefore, $T_{(0,0)}\Phi^{-1}(0)$ is a 1-dimensional subspace of $T_{(0,0)}\mathbb{R}^2 \cong \mathbb{R}^2$, so by *Exercise* 3(a) there exists a smooth curve $\gamma \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ whose image lies in $\Phi^{-1}(0)$ and which satisfies $\gamma(0) = (0, 0)$ and $\gamma'(0) \neq 0$. Writing $\gamma(t) = \big(x(t), y(t)\big)$, we see that $y(t)$ takes a global minimum at $t = 0$, so $y'(0) = 0$. On the other hand, since $\gamma(t) \in \Phi^{-1}(0)$ for every $t \in (-\varepsilon, \varepsilon)$, we have $x^2(t) = y^3(t)$ for every $t \in (-\varepsilon, \varepsilon)$. Differentiating twice and setting $t = 0$, we obtain $x'(0) = 0$, and since $y'(0) = 0$, we conclude that $\gamma'(0) = 0$, which is a contradiction. Hence, $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

Remark. Here are a few remarks on the above solution to *Exercise* 5(b).

- One could perform the construction of $M_0 \sqcup M_1$ a bit more concretely: if $M_0 \subseteq \mathbb{R}^n$ and $M_1 \subseteq \mathbb{R}^n$ are given as embedded submanifolds of $\mathbb{R}^n$, then it is more or less straightforward to see that $(M_0 \times \{0\}) \cup (M_1 \times \{1\}) \subseteq \mathbb{R}^{n+1}$ is an embedded submanifold of $\mathbb{R}^{n+1}$. The abstract disjoint union $M_0 \sqcup M_1$ is then diffeomorphic to $(M_0 \times \{0\}) \cup (M_1 \times \{1\})$ via the obvious map.

In our example where $M_0 = \mathbb{R}$ and $M_1 = \mathbb{R} \setminus \{0\}$, this shows that $M_0 \sqcup M_1$ is diffeomorphic to the embedded submanifold

$$\{(x,0) \mid x \in \mathbb{R}\} \cup \{(x,1) \mid x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2.$$

However, conceptually it is cleaner to argue just with the abstract disjoint union $M_0 \sqcup M_1$.

- The abstract disjoint union has a universal property, similar to the universal property of a product: for all smooth manifolds N and all smooth maps $f_0\colon M_0 \to N$ and $f_1\colon M_1 \to N$, there exists a unique smooth map $f\colon M_0 \sqcup M_1 \to N$ such that $f \circ \iota_i = f_i$ for all i. In fact, for those who are familiar with the language of category theory, the disjoint union is the coproduct in the category of smooth manifolds.
- By *Proposition 5.13*, there is a topology and smooth structure on $F(M)$ making it an immersed submanifold of $\mathbb{R}^2$. The topology is given by

$$\mathcal{T} = \big\{F(U) \mid U \subseteq M \text{ open}\big\}.$$

This is a strictly finer topology on $F(M)$ than the subspace topology. Indeed, by *Proposition 5.13*, the map $(F(M), \mathcal{T}) \to \mathbb{R}^2$ is continuous, so $\mathcal{T}$ contains the subspace topology. On the other hand, we have $F\big(\iota_0(M_0)\big) \in \mathcal{T}$, but

$$F\big(\iota_0(M_0)\big) = \big\{(x,x) \mid x \in \mathbb{R}\big\},$$

which is not open in the subspace topology on $F(M)$ (otherwise it would have to contain $F(M) \cap B(0,\varepsilon)$ for ε sufficiently small, which is certainly false).

- This exercise also demonstrates that the topology and smooth structure on an immersed manifold might not be unique (cf. *Exercise* 2). Indeed, the map

$$\begin{aligned} G\colon M &\to \mathbb{R}^2 \\ (m,i) &\mapsto \big(m, (-1)^{i+1} m\big) \end{aligned}$$

is also an injective smooth immersion with image $\Phi^{-1}(0)$, and therefore induces a topology and smooth structure on $\Phi^{-1}(0)$ making it an immersed submanifold of $\mathbb{R}^2$; see *Proposition 5.13*. But you can check that it is different from the one induced by F.

Exercise 6: For each $a \in \mathbb{R}$, consider the set

$$M_a \coloneqq \big\{(x,y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x-a)\big\} \subseteq \mathbb{R}^2.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values of a can M_a be given a topology and a smooth structure making it into an immersed submanifold?

[Hint: To answer the second question, for each "singular" value of the parameter $a \in \mathbb{R}$ it is quite useful to plot the corresponding curve $M_a \subseteq \mathbb{R}^2$ in order to get some geometric insights. In particular, for one of those "singular" values of $a \in \mathbb{R}$, it might also be helpful to consider the parametrized curve $\gamma(t) = (t^2, t^3 - t)$ with an appropriate domain of definition $I \subseteq \mathbb{R}$.]

Solution: For each $a \in \mathbb{R}$, consider the function

$$\Phi_a\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto y^2 - x(x-1)(x-a)$$

and observe that $M_a = \Phi_a^{-1}(0)$. The gradient of Φ_a at an arbitrary point $(x,y) \in \mathbb{R}^2$ is given by

$$\operatorname{grad}(\Phi_a)(x,y) = \big(-3x^2 + 2(a+1)x - a,\ 2y\big).$$

Therefore, $\operatorname{grad}(\Phi_a)(x,y) = (0,0)$ for some $(x,y) \in \mathbb{R}^2$ if and only if $y = 0$ and $x \in \mathbb{R}$ satisfies the following system:

$$(\Sigma) : \begin{cases} u^3 - (a+1)u^2 + au = 0 \\ 3u^2 - 2(a+1)u + a = 0. \end{cases}$$

One can now check that the pairs $(u,a) \in \big\{(0,0),(1,1)\big\}$ are the solutions of (Σ).

In conclusion, if $a \in \mathbb{R} \setminus \{0,1\}$, then $0 \in \mathbb{R}$ is a regular value of Φ_a, so $M_a = \Phi_a^{-1}(0)$ is a properly embedded submanifold of $\mathbb{R}^2$ by *Corollary 5.10*, whereas if $a \in \{0,1\}$, then $\operatorname{grad}(\Phi_0)(0,0) = (0,0)$ and $\operatorname{grad}(\Phi_1)(1,0) = (0,0)$, so *Corollary 5.10* cannot be applied; we will address those two cases separately below.

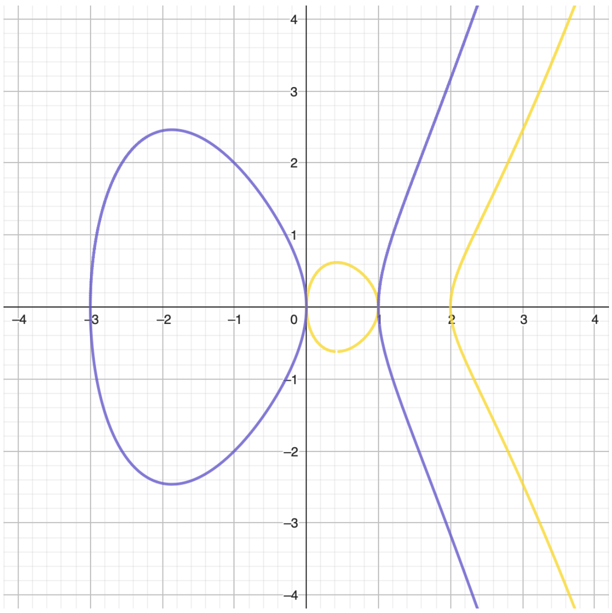
For instance, the curves

$$M_2 = \Phi_2^{-1}(0) = \big\{(x,y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x-2)\big\}$$

and

$$M_{-3} = \Phi_{-3}^{-1}(0) = \big\{(x,y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x+3)\big\}$$

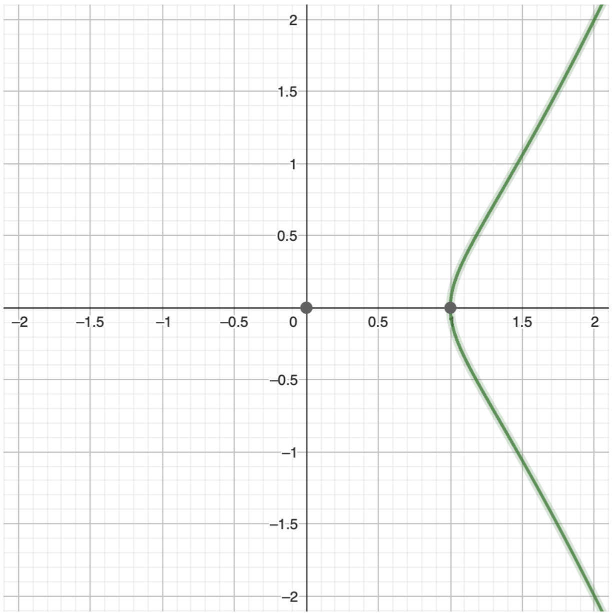
have been plotted below in yellow and purple, respectively.



Next, we deal with the set

$$M_0 = \Phi_0^{-1}(0) = \big\{(x,y) \in \mathbb{R}^2 \mid y^2 = x^2(x-1)\big\},$$

which has been plotted below.

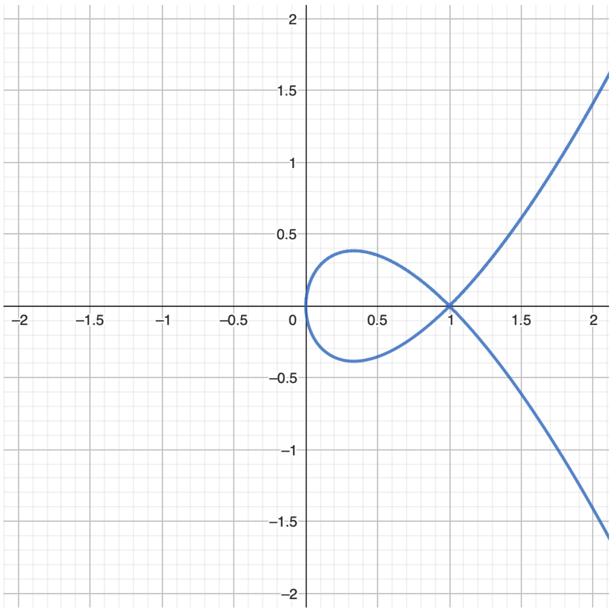


We observe that $(0,0) \in \mathbb{R}^2$ is an isolated point of M_0. Therefore, M_0 cannot be an immersed or embedded submanifold of $\mathbb{R}^2$ (simply because we do not consider spaces of mixed dimension as manifolds).

Finally, we deal with the set

$$M_1 = \Phi_1^{-1}(0) = \big\{(x,y) \in \mathbb{R}^2 \mid y^2 = x(x-1)^2\big\},$$

which has been plotted below.



Observe that the "curve" M_1 has a self-intersection, so it cannot be an embedded submanifold of $\mathbb{R}^2$; see the solution of [*Exercise Sheet* 8, *Exercise* 3(*b*)] for a similar argument. However, we can make M_1 an immersed submanifold of $\mathbb{R}^2$ by giving it an appropriate topology in which it is disconnected; see the solution of *Exercise* 5(b) for a similar idea. To this end, consider the smooth curve

$$\gamma\colon \mathbb{R} \setminus \{-1\} \to \mathbb{R}^2,\ t \mapsto (t^2, t^3 - t)$$

and observe that its image is the set $M_1 \subseteq \mathbb{R}^2$. Note that γ is a smooth immersion by *Example 4.4* (1), since

$$\gamma'(t) = (2t, 3t^2 - 1) \neq 0 \quad \text{for every } t \in \mathbb{R} \setminus \{1\}.$$

It is also injective, since

$$\gamma(t_1) = \gamma(t_2) \implies t_1 = \pm t_2$$

and if $t_1 = -t_2$, then either $t_1 = 0 = -t_2$ or $t_1 = \pm 1 = -t_2$ (which is why we have excluded, for instance, $t = -1$ from the domain of definition of γ, which would also get mapped to the point $(1, 0) \in M_1 \subseteq \mathbb{R}^2$). It follows from *Proposition 5.13* that $M_1 = \gamma\big(\mathbb{R} \setminus \{-1\}\big)$ is an immersed submanifold of $\mathbb{R}^2$.