Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 9

Exercise 1:
Let M be a smooth manifold. Show that if S is an embedded submanifold of M, then the subspace topology on S and the smooth structure on S described in *Theorem 5.6* are the only topology and smooth structure with respect to which S is an embedded (or immersed) submanifold.

[Hint: Use [*Exercise Sheet* 8*, Exercise* 5(*c*)].]

Exercise 2:
Let M be a smooth manifold. Show that if S is an immersed submanifold of M, then for the given topology on S, there exists only one smooth structure making S into an immersed submanifold.

[Hint: Use [*Exercise Sheet* 8*, Exercise* 5(*b*)].]

Exercise 3:

(a) Let M be a smooth manifold, let $S \subseteq M$ be an immersed or embedded submanifold, and let $p \in S$. Show that a vector $v \in T_pM$ is in T_pS if and only if there exists a smooth curve $\gamma\colon J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $\gamma\colon J \to M$ be a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$.

Exercise 4:

(a) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Show that if $\Phi\colon U \to N$ is a local defining map for S, then it holds that

$$T_pS \cong \ker\left(d\Phi_p\colon T_pM \to T_{\Phi(p)}N\right) \quad \text{for every } p \in S \cap U.$$

(b) Let M be a smooth manifold. Suppose that $S \subseteq M$ is a level set of a smooth submersion $\varPhi = (\varPhi_1, \ldots, \varPhi_k)\colon M \to \mathbb{R}^k$. Show that a vector $v \in T_pM$ is tangent to S if and only if $v\varPhi_1 = \ldots = v\varPhi_k = 0$.

Exercise 5:

(a) Consider the smooth curve

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

[Be careful: this is not the same as showing that β is not a smooth embedding.]

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Show that the level set $\Phi^{-1}(0)$ is an immersed submanifold of $\mathbb{R}^2$.

(c) Consider the smooth function

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^3.$$

Show that the level set $\Psi^{-1}(0)$ is not an immersed submanifold of $\mathbb{R}^2$.

[Hint: Argue by contradiction and use *Exercise* 3(a).]

Exercise 6 (to be submitted by Thursday, 21.11.2024, 16:00):
For each $a \in \mathbb{R}$, consider the set

$$M_a \coloneqq \big\{(x, y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x-a)\big\} \subseteq \mathbb{R}^2.$$

For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? For which values of a can M_a be given a topology and a smooth structure making it into an immersed submanifold?

[Hint: To answer the second question, for each "singular" value of the parameter $a \in \mathbb{R}$ it is quite useful to plot the corresponding curve $M_a \subseteq \mathbb{R}^2$ in order to get some geometric insights. In particular, for one of those "singular" values of $a \in \mathbb{R}$, it might also be helpful to consider the parametrized curve $\gamma(t) = (t^2, t^3 - t)$ with an appropriate domain of definition $I \subseteq \mathbb{R}$.]