Final Exam Solutions

First Part *(Multiple Choice Questions)* 36 points

For each statement, select a single correct answer.

(Question 1) (3 Pts): Given that $x(t)$ has a Nyquist rate ω_0, which of the following signals has a Nyquist rate of $2\omega_0$?

- $x(t)\cos(\omega_0 t)$
- $x(2t)$
- $x(t-2) + x(t)$
- $x(t) * x(t-1)$

Solution:

- (F) $x(t)\cos(\omega_0 t)$
- (T) $x(2t)$
- (F) $x(t-2) + x(t)$
- (F) $x(t) * x(t-1)$

$x(t)\cos(\omega_0 t)$ has a Fourier Transform $X(\omega - \omega_0)$ and has a Nyquist rate $3\omega_0$.
$x(2t)$ has a Fourier Transform $\frac{\omega}{2}X(\frac{\omega}{2})$ and has a Nyquist rate $2\omega_0$.
$x(t-2) + x(t)$ has a Fourier Transform $X(\omega)e^{-2j\omega} + X(\omega)$ and has a Nyquist rate ω_0.
$x(t) * x(t-1)$ has a Fourier Transform $X(\omega)X(\omega)e^{-j\omega}$ and has a Nyquist rate ω_0.

(Question 2) (3 Pts): A continuous-time signal $x(t) = \mathrm{sinc}(750t)$ is sampled using impulse train sampling and later reconstructed with a low-pass filter with gain T and cut-off frequency ω_c. According to the Sampling Theorem, which of the following systems would perfectly reconstruct this signal?

- A system that samples with an interval $T = 10^{-3}$ and reconstructs with a cut-off frequencey $\omega_c = 800\pi$
- A system that samples with an interval $T = 10^{-2}$ and reconstructs with a cut-off frequencey $\omega_c = 1000\pi$
- A system that samples with a frequency $\omega_s = 500\pi$ and reconstructs with a cut-off frequencey $\omega_c = 2000\pi$
- A system that samples with a frequency $\omega_s = 2000\pi$ and reconstructs with a cut-off frequencey $\omega_c = 500\pi$

Solution:

- (T) A system that samples with an interval $T = 10^{-3}$ and reconstructs with a cut-off frequencey $\omega_c = 800\pi$
- (F) A system that samples with an interval $T = 10^{-2}$ and reconstructs with a cut-off frequencey $\omega_c = 1000\pi$
- (F) A system that samples with a frequency $\omega_s = 500\pi$ and reconstructs with a cut-off frequencey $\omega_c = 2000\pi$

- (F) A system that samples with a frequency $\omega_s = 2000\pi$ and reconstructs with a cut-off frequencey $\omega_c = 500\pi$

(Question 3) (3 Pts): Let $z(t) = 500\,\text{sinc}(500t)$. Which of the following LTI systems is a bandpass filter?

- A system with the impulse response $h(t) = z(t)\cos(500\pi t)$
- A system with the impulse response $h(t) = z(t) * \cos(1000\pi t)$
- A system with the impulse response $h(t) = \delta(t) - z(t)$
- A system with the impulse response $h(t) = z(t)\cos(1000\pi t)$

Solution:

- (F) A system with the impulse response $h(t) = z(t)\cos(500\pi t)$
- (F) A system with the impulse response $h(t) = z(t) * \cos(1000\pi t)$
- (F) A system with the impulse response $h(t) = \delta(t) - z(t)$
- (T) A system with the impulse response $h(t) = z(t)\cos(1000\pi t)$

Note that $sinc(500t)$ has a Rectangular Fourier Transform between -500π and 500π.
$500sinc(500t) * cos(1000\pi t)$'s transform is a product of rectangle with 2 delta functions at $\pm 1000\pi$ resulting in 0.
$\delta(t) - 500sinc(500t)$ will be a high pass filter.
The remaining 2 options are convolutions of rectangle with the 2 delta functions (from cosine). $500sinc(500t).cos(500\pi t)$' tranform will have 2 rectangles of width 1000π centered around $\pm 500\pi$ which actually results in a single rectangle between -1000π and 1000π and is a low pass filter.
$500sinc(500t).cos(1000\pi t)$'s transform will have 2 rectangles of width 1000π centered around $\pm 1000\pi$ and results in a band pass filter.

(Question 4) (3 Pts): Let

$$x_1[n] = \delta[n] + \delta[n-1]$$
$$y_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

and

$$x_2[n] = \delta[n] - \delta[n-1]$$
$$y_2[n] = \delta[n] - \delta[n-2]$$

be input-output pairs for an LTI system $\mathcal{H}$. That is, $y_1[n] = \mathcal{H}\{x_1[n]\}$ and $y_2[n] = \mathcal{H}\{x_2[n]\}$. Which of the following statements is true?

- Given the input $x_1[n] + x_2[n]$, the output of $\mathcal{H}$ is $2\delta[n]$
- The impulse response of $\mathcal{H}$ is $\delta[n] + \delta[n-1]$
- $\mathcal{H}$ is not causal
- $\mathcal{H}$ is unstable

Solution:

- (F) The LTI system's response to the input $x_1[n]+x_2[n]$ is $y_1[n]+y_2[n]=2\delta[n]+2\delta[n-1]$.
- (T) The impulse response of the LTI system is its response to $\frac{1}{2}(x_1[n]+x_2[n])=\delta[n]$. Therefore, the impulse response is $\frac{1}{2}(y_1[n]+y_2[n])=\delta[n]+\delta[n-1]$
- (F) The LTI system is causal because $h[n]=\delta[n]+\delta[n-1]$ is zero for $n<0$.
- (F) The LTI system is stable because $h[n]=\delta[n]+\delta[n-1]$ is absolutely summable.

(Question 5) (3 Pts): Suppose that the signal $x[n]$ has a z-transform $X(z)$. Then the z-transform of $nx[n]$ is

- $-z\frac{dX(z)}{dz}$
- $z\frac{dX(z)}{dz}$
- $z^{-1}\frac{dX(z)}{dz}$
- $z\frac{dX(z^{-1})}{dz}$

Solution:

- (T) from Appendix 7.A
- (F)
- (F)
- (F)

(Question 6) (3 Pts): Given the signal $x[n]=(n+1)3^n u[n]$, which of the following statements is true?

- $x[n]$ is absolutely summable
- $x[n]$ is right-sided
- The ROC of $x[n]$ contains the point $z=\frac{1}{3}$
- The ROC of $x[n]$ contains the point $z=1+j$

Solution: The ROC of $x[n]$ is $\{z:|z|>3\}$.

- (F) because the ROC does not include the unit circle.
- (T) because the ROC extends indefinitely outwards.
- (F) $\frac{1}{3}\notin\{z:|z|>3\}$
- (F) $1+j\notin\{z:|z|>3\}$

(Question 7) (3 Pts):
A stable LTI system is characterized by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Which of the following statements is true?

- The impulse response $h(t)$ of the system is unbounded
- The magnitude of the frequency response $|H(\omega)|$ of the system is unbounded
- The system cannot be causal
- The ROC of the transfer function $H(s)$ is $Re(s) > -1$

Solution:

- (F) The impulse response $h(t)$ of the system is unbounded
- (F) The magnitude of the frequency response $|H(\omega)|$ of the system is unbounded
- (F) The system cannot be causal
- (T) The ROC of the transfer function $H(s)$ is $Re(s) > -1$

From the difference equation, we have that:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3}$$

By using inverse Fourier transform, we have that:

$$H(j\omega) = \frac{1}{2} \times \frac{1}{j\omega + 1} + \frac{1}{2} \times \frac{1}{j\omega + 3}$$

Therefore, we have the value of $h(t)$ as:

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Option A and C are wrong because $h(t)$ has its maximum value at t = 0 which is 1 (therefore, it is bounded).

Option B is incorrect because if one tries to write the Laplace transform, then it comes out as

$$H(s) = \frac{1}{2} \times \frac{1}{s+1} + \frac{1}{2} \times \frac{1}{s+3}$$

Therefore, there are two real poles.

For option D, ROC will be the intersection of $\frac{1}{2} \times \frac{1}{s+1}$ and $\frac{1}{2} \times \frac{1}{s+3}$ which is $Re(s) > -1$. Option D is correct.

(Question 8) (3 Pts): Which of the following properties of convolution of signals is false?

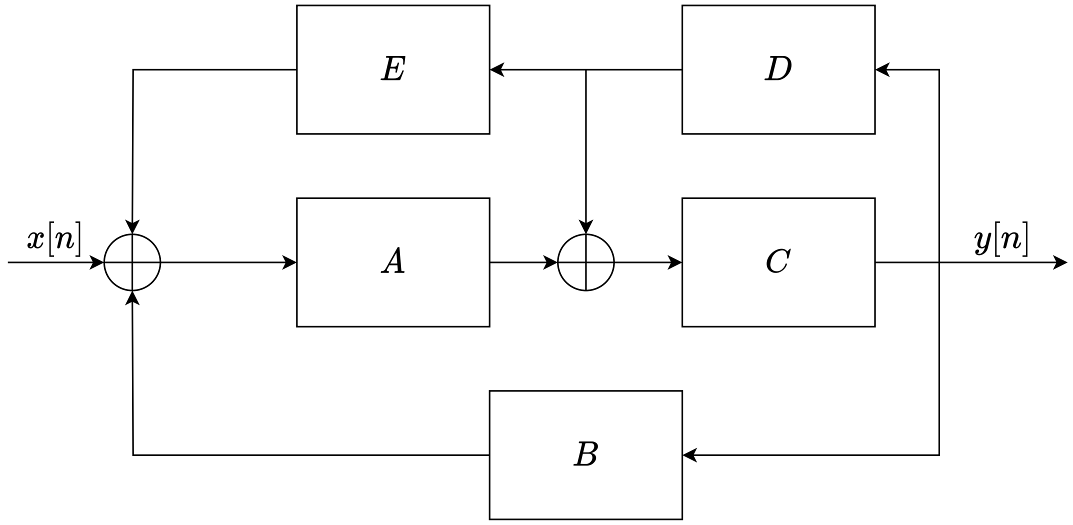


Figure 1: Discrete-time system

- $(x(t) * h(t)) * g(t) = x(t) * (h(t) * g(t))$
- $(x(t) + h(t)) * g(t) = x(t) * g(t) + h(t) * g(t)$
- $(x(t) \times h(t)) * g(t) = (x(t) * g(t)) \times (h(t) * g(t))$
- $x[n] * (k\delta[n]) = kx[n]$, where k is a constant

Solution:

- (T) $(x(t) * h(t)) * g(t) = x(t) * (h(t) * g(t))$
- (T) $(x(t) + h(t)) * g(t) = x(t) * g(t) + h(t) * g(t)$
- (F) $(x(t) \times h(t)) * g(t) = (x(t) * g(t)) \times (h(t) * g(t))$
- (T) $x[n] * (k\delta[n]) = kx[n]$, where k is a constant

(Question 9) (3 Pts): Consider the discrete-time system interconnect shown on Figure 1, where the component systems A, B, C, D, and E are LTI systems with transfer functions $A(z)$, $B(z)$, $C(z)$, $D(z)$ and $E(z)$, respectively.

We are told that the overall system $\mathcal{H}$ has the transfer function $H(z) = \frac{A(z)C(z)}{1-A(z)B(z)C(z)}$. What can we say about the functions $D(z)$ and $E(z)$?

- $D(z) = 0$, but we cannot say anything about $E(z)$
- $E(z) = 0$, but we cannot say anything about $D(z)$
- $D(z) = 0$ and $E(z) = 0$
- Either $D(z) = 0$ or $E(z) = 0$, but not both

Solution:

- (T) $D(z) = 0$, but we cannot say anything about $E(z)$
- (F) $E(z) = 0$, but we cannot say anything about $D(z)$
- (F) $D(z) = 0$ and $E(z) = 0$
- (F) Either $D(z) = 0$ or $E(z) = 0$, but not both

If $D(z) = 0$, then the signal out of D is always zero. Thus, regardless of the value of $E(z)$, the signal coming out of E is always zero. Thus, we are left with two systems A and C composed in series, and the composed with system B via a feedback composition.

(Question 10) (3 Pts):
Two right-sided discrete-time signals $x[n]$ and $y[n]$ are related in the following manner:

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

The z-transform of $y[n]$ is $\frac{z^3 - z^2 + 1}{z(z-1)^2}$. Then

- $X(z) = 1 + \frac{1}{z^2(z-1)}$
- $X(z) = \frac{z^3 - z^2 + 1}{z(z-1)}$
- $X(z) = \frac{z^3 - z^2 + 1}{z^2(z-1)}$
- $X(z) = 1 + \frac{1}{z(z-1)^2}$

Solution:

- (T) $X(z) = 1 + \frac{1}{z^2(z-1)}$
- (F) $X(z) = \frac{z^3 - z^2 + 1}{z(z-1)}$
- (F) $X(z) = \frac{z^3 - z^2 + 1}{z^2(z-1)}$
- (F) $X(z) = 1 + \frac{1}{z(z-1)^2}$

We have,

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Then,

$$Y(z) = \frac{1}{1 - z^{-1}} X(z) = \frac{z}{z-1} X(z)$$

Also, we are given,

$$Y(z) = \frac{z^3 - z^2 + 2}{z(z-1)^2} = \frac{2}{z(z-1)^2} + \frac{z}{z-1}$$

On comparing the above two equations, we have

$$X(z) = 1 + \frac{1}{z^2(z-1)}$$

(Question 11) (3 Pts):
Let $x[n]$ be an absolutely summable discrete-time signal. The z-transform of $x[n]$ is rational function (ratio of two polynomials) with two zeros and two poles. It is known that the poles are located at $z = \pm 3j$. Then, which of the following statements is true about $x[n]$?

- $x[n]$ is a left-sided signal.
- $x[n]$ is a periodic signal.
- $x[n]$ is a finite duration signal.
- $x[n]$ is a right-sided signal.

Solution:

- (T) $x[n]$ is a left-sided signal.
- (F) $x[n]$ is a periodic signal.
- (F) $x[n]$ is a finite duration signal.
- (F) $x[n]$ is a right-sided signal.

We are given that the poles are located at $z = \pm 3j$ i.e., outside the unit circle.
However, since $x[n]$ is absolutely summable its DTFT exists and ROC must contain the unit circle. Thus, the ROC is within the circle of radius $r = 3$, and therefore, the signal is left-sided.

(Question 12) (3 Pts):

Two finite-duration continuous-time signals $x(t)$ and $y(t)$ are related in the following manner:

$$y(t) = \int_t^{\infty} x(\tau) d\tau.$$

What is the algebraic expression for the Laplace transform of $y(t)$?

- $Y(s) = X(0) - \frac{1}{s}X(s)$
- $Y(s) = \frac{1}{s}X(-s)$
- $Y(s) = -\frac{1}{s}X(s)$
- $Y(s) = \frac{s-1}{s}X(s)$

Solution: Two answers could be correct for this question:

- (T) $Y(s) = X(0) - \frac{1}{s}X(s)$
- (F) $Y(s) = \frac{1}{s}X(-s)$

- (T) $Y(s) = -\frac{1}{s}X(s)$
- (F) $Y(s) = \frac{s-1}{s}X(s)$

For the first solution:

$$\begin{aligned}
y(t) &= \int_{t}^{\infty} x(\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)e^{-0\tau}d\tau - \int_{-\infty}^{t} x(\tau)d\tau \\
&= X(0) - \int_{-\infty}^{t} x(\tau)d\tau = -\int_{-\infty}^{t} x(\tau)d\tau
\end{aligned}$$

We have that $X(0) = 0$ since we require $y(t)$ and $x(t)$ to be a finite duration signals. Thus, by using the integration in time property $Y(s) = X(0) - \frac{1}{s}X(s) = -\frac{1}{s}X(s)$

Another possible solution is to write

$$\begin{aligned}
y(t) &= \int_{t}^{\infty} x(\tau)d\tau \\
&= \int_{t}^{\infty} x(\tau)u(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = x(t) * u(-t).
\end{aligned}$$

And, by using the convolution in time property together with the step function pair, $Y(s) = -\frac{1}{s}X(s)$.

Question 13 *(System Properties)* 10pts

For each system described below determine if it has the specified property. Justify your answer.

(a) An LTI system is described by the input-output relationship

$$y(t) = \int_{-\infty}^{t-3} x(\tau)d\tau.$$

Is the system causal?

(b) A stable discrete-time LTI system has the frequency response

$$H(e^{j\omega}) = a_0e^{-j\omega} + a_1 + a_2e^{j\omega}$$

where a_0, a_1, a_2 are non-zero reals. Is the system memoryless?

(c) A discrete time system is described by the input-output relationship

$$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k} x[k-1].$$

Is the system time-invariant?

SOLUTION

(a) (3 pts) The system is causal.
First, we can find the impulse response as

$$h(t) = \int_{-\infty}^{t-3} \delta(\tau)d\tau = u(t-3))$$

We have $h(t) \neq 0$ for $t < 0$ so the system is causal.

It is also possible to rewrite the input-output relationship as

$$y(t) = \int_{-\infty}^{t} x(\tau-3)d\tau$$

From this, or the original equation, we see that the system output at time t does not depend on future inputs.

(b) (3 pts) The system is not memoryless.
By taking the inverse FT to get

$$h(t) = a_0\delta[n-1] + a_1\delta[n] + a_2\delta[n+1].$$

Such a system could only be memoryless if the $\delta[n-1]$ and $\delta[n+1]$ components are removed. However, we are told that a_0 and a_2 are non-zero, so the system does have memory.

(c) (4 pts) The system is not time-invariant.

We can check by applying the input $\delta[n]$ to obtain

$$\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k} \delta[k-1] = 2u[n-1].$$

On the other hand, applying the input $\delta[n-n_0]$ with $n_0 \neq 0$ yields

$$\sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k} \delta[k-1-n_0] = 2^{1+n_0} u[n-1-n_0] \neq 2u[n-1-n_0].$$

Question 14 *(Difference Equations)* 14 points

You are working for a chip manufacturing company as an engineer. The analog team has provided you a chip which implements the causal LTI system described by the following difference relation between its input $x[n]$ and output $y[n]$

$$y[n] - \frac{9}{2}y[n-1] + 2y[n-2] = x[n] - \alpha x[n-1]$$

where α is a real value which you can set by changing some chip settings.

(a) Find the transfer function $H(z)$ and the corresponding ROC.

(b) Set $\alpha = \alpha_1 = \frac{1}{2}$. Find the corresponding transfer function $H_1(z)$ with ROC and the corresponding impulse response $h_1[n]$. Is this system stable? Justify your answer.

(c) Find the value $\alpha = \alpha_2$ for which the above system is both, stable and causal. Find the corresponding transfer function be $H_2(z)$ with ROC and the corresponding impulse response $h_2[n]$.

(d) What are the outputs $y_1[n]$ and $y_2[n]$ when a complex exponential signal $x[n] = e^{j\omega_0 n}u[n]$ is given as input to each of the above systems $H_1(z)$ and $H_2(z)$?

Solution:

(a) (4 pts)

Applying the shift in time property for z-transform we find that the transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-\alpha z^{-1}}{1-\frac{9}{2}z^{-1}+2z^{-2}} = \frac{1-\alpha z^{-1}}{(1-4z^{-1})(1-\frac{1}{2}z^{-1})}.$$

The ROC is $|z| > 4$ since the transfer function has poles at $z = \frac{1}{2}$ and $z = 4$ and it is a causal system.

(b) (3 pts)

The transfer function is

$$H(z) = \frac{1}{1-4z^{-1}}$$

with ROC $|z| > 4$ and thus has unstable impulse response is $h[n] = (4)^n u[n]$. The system is unstable as the unit circle is not in the ROC.

(c) (3 pts)

With $\alpha_2 = 4$, the transfer function becomes

$$H_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}},$$

with ROC $|z| > \frac{1}{2}$. This system is stable since the unit circle is in the ROC. The impulse response is $h_2[n] = \frac{1}{2}^n u[n]$.

(d) (4 pts)

- Correct $Y_1(z)$ and $Y_2(z)$ in any form - each +1 point

- Correct $y_1[n]$ and $y_2[n]$ - each +1 point

The input $e^{j\omega_0 n}$ has a corresponding z-transform $\frac{1}{1-e^{j\omega_0}z^{-1}}$ and thus

$$Y_1(z) = \frac{1}{1-4z^{-1}}\frac{1}{1-e^{j\omega_0}z^{-1}} = \frac{1}{1-\frac{1}{4}e^{j\omega_0}}\frac{1}{1-4z^{-1}} + \frac{1}{1-4e^{-j\omega_0}}\frac{1}{1-e^{j\omega_0}z^{-1}}$$

with ROC $|z|>4$ and thus the output is $y_1[n] = \frac{1}{1-\frac{1}{4}e^{j\omega_0}}(4)^n u[n] + \frac{1}{1-4e^{-j\omega_0}}e^{j\omega_0 n}u[n]$.

For the second case, the output z-transform is

$$Y_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}}\frac{1}{1-e^{j\omega_0}z^{-1}} = \frac{1}{1-2e^{j\omega_0}}\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{2}e^{-j\omega_0}}\frac{1}{1-e^{j\omega_0}z^{-1}}$$

with ROC $|z|>\frac{1}{2}$. Thus the output is $y_2[n] = \frac{1}{1-2e^{j\omega_0}}(\frac{1}{2})^n u[n] + \frac{1}{1-\frac{1}{2}e^{-j\omega_0}}e^{j\omega_0 n}u[n]$.

Question 15 *(Feedback Composition)* 12 points

We consider the system composition illustrated in Figure 2 with input $x(t)$, output $y(t)$, and the intermediate signal $v(t)$, and their associated Laplace transforms $X(s)$, $Y(s)$, and $V(s)$ respectively. The systems $\mathcal{H}$, $\mathcal{G}$, and $\mathcal{F}$ are LTI, defined by the impulse responses $h(t)$, $g(t)$, and $f(t)$, and their associated transfer functions $H(s)$, $G(s)$, and $F(s)$ respectively.

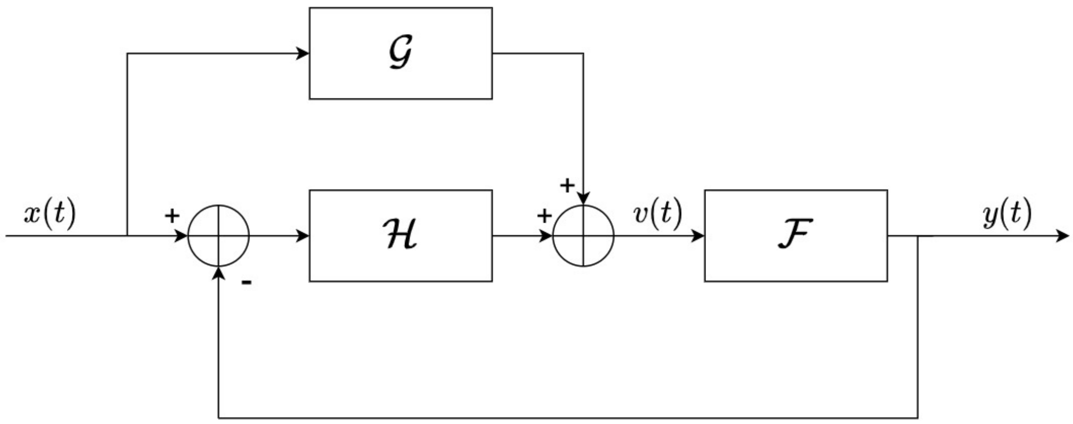


Figure 2: Composed system

(a) Express $V(s)$ in terms of $Y(s)$, $X(s)$, $G(s)$, and $H(s)$.

(b) Prove that the transfer function of the composed system is as follows:

$$\frac{Y(s)}{X(s)} = \frac{(G(s) + H(s))F(s)}{1 + H(s)F(s)}$$

(c) Let $h(t) = u(t)$, $g(t) = 0$, and $f(t) = K\delta(t)$. Find the transfer function of the composed system in this case.

(d) Considering the same impulse responses in *(c)*, and given that the composed system is causal and stable, determine the possible range of K.

Solution:

(a) (3 points) We can see from the diagram that $V(s) = G(s)X(s) - Y(s)H(s) + X(s)H(s)$.

(b) (3 points) We can see from the diagram that

$$\begin{aligned} Y(s) &= F(s)V(s) \\ &= X(s)(H(s) + G(s))F(s) - Y(s)H(s)F(s) \end{aligned}$$

Therefore,

$$\frac{Y(s)}{X(s)} = \frac{(G(s) + H(s))F(s)}{1 + H(s)F(s)}$$

(c) (3 pionts) Letting $h(t) = u(t)$, $g(t) = 0$, and $f(t) = K\delta(t)$ means that $H(s) = \frac{1}{s}$, $G(s) = 0$, and $F(s) = K$. Therefore, the transfer function of the composed system is as follows:

$$\frac{Y(s)}{X(s)} = \frac{K}{K + s}$$

(d) (3 points) From the transfer function in *(c)*, we can see that the composed system has a pole at $-K$. Therefore, the ROC is $\{s : \mathcal{R}e\{s\} > -K\}$ or $\{s : \mathcal{R}e\{s\} < -K\}$. Since the system is causal, then the ROC extends indefinitely to the right. Further, the system is stable then the ROC includes the imaginary axis. Therefore K has to be strictly positive.

Question 16 *(Sampling)* 14pts

An electrocardiogram records the electrical signals in the heart to check for different heart conditions. You can think of it as a sampling problem where $x(t)$ is the true electric potential of the heart, each measurement $x(nT)$ is a sample, and the reconstruction $x_r(t)$ is the waveform observed on the monitor. This electric signals could be modeled by a time signal $x(t)$ which is band limited with some parameter ω_M ; that is, $X(\omega) = 0$ if $|\omega| > \omega_M$.

(a) It is known that the measurement interval $T = 2 \times 10^{-3}$ is sufficiently small and leads to a perfect reconstruction. Use the sampling theorem to provide an upper bound on ω_M . What is the measurement frequency ω_S of the system with $T = 2 \times 10^{-3}$?

(b) It is also known that the measurement interval $T = 8 \times 10^{-3}$ is too large and leads to distortions in the reconstructed signal. Use the sampling theorem to provide a lower bound on ω_M . What is the measurement frequency ω_S of the system with $T = 8 \times 10^{-3}$?

(c) Suppose that the maximum frequency for $x(t)$ is actually $\omega_M = 250\pi$ while the measurement interval used is $T = 2 \times 10^{-3}$. According to the sampling theorem, what is the gain and the cut-off frequency of a lowpass filter needed to correctly reconstruct $x(t)$? For full credit, give the range of all cut-off frequencies that would work.

(d) Actually, to diagnose some heart conditions we do not need to know $x(t)$; we only need to know its high-frequency components. Thus, $x(t)$ is first converted into a so-called *bandpass* signal $y(t)$ where

$$|Y(\omega)| = 0, |\omega| \leq 150\pi \text{ or } |\omega| \geq 250\pi.$$

Suppose $y(t)$ is measured with interval $T = 8 \times 10^{-3}$ using an impulse-train to obtain

$$y_p(t) = \sum_{k=-\infty}^{\infty} y(kT)\delta(t - kT).$$

You are only provided with multiple low-pass filters and a mechanism for adding/subtracting signals. Suggest a way to reconstruct the actual signal $y(t)$ from the measured signal $y_p(t)$ using only the tools provided. Give the gain and the cut-off frequencies of the low-pass filter(s) for full credit.

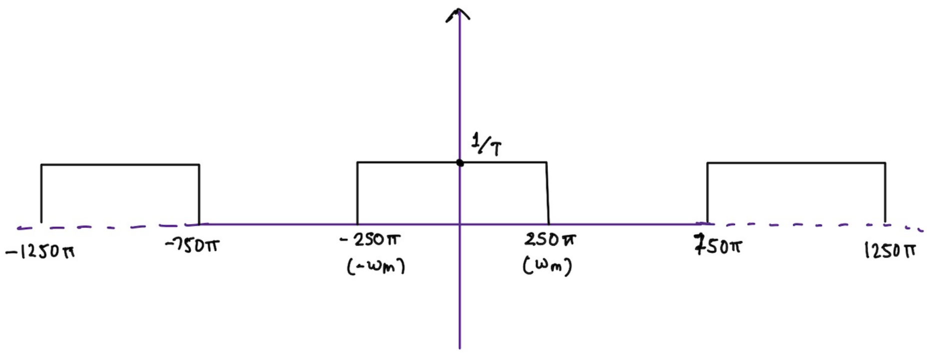
Hint: For part (d), you do not need to state the full range of possible cut-off frequencies that may work. It is sufficient to suggest one set of parameters that would work.

SOLUTION

(a) (3pts) $\omega_s = \frac{2\pi}{T} = 1000\pi$. From Sampling theorem: $\omega_m < \frac{\omega_s}{2} = 500\pi$.

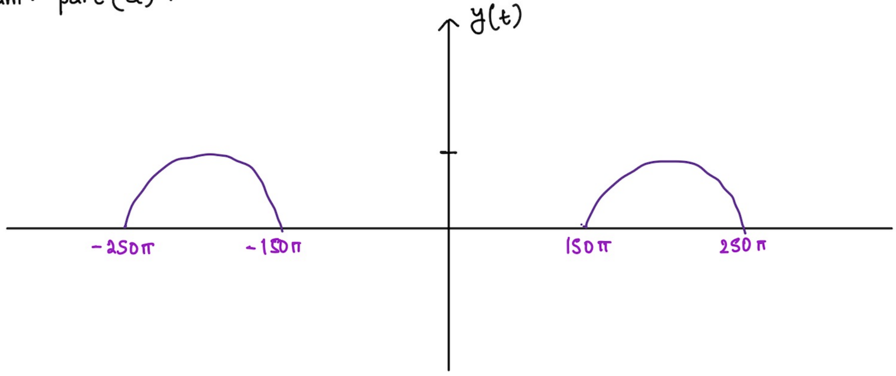
(b) (3 ptst) We have $\omega_s = \frac{2\pi}{T} = 250\pi$. Since the reconstructed signal is distorted, it implies that $\omega_m > \frac{\omega_s}{2}$. i.e., we have $\omega_m > 125\pi$.

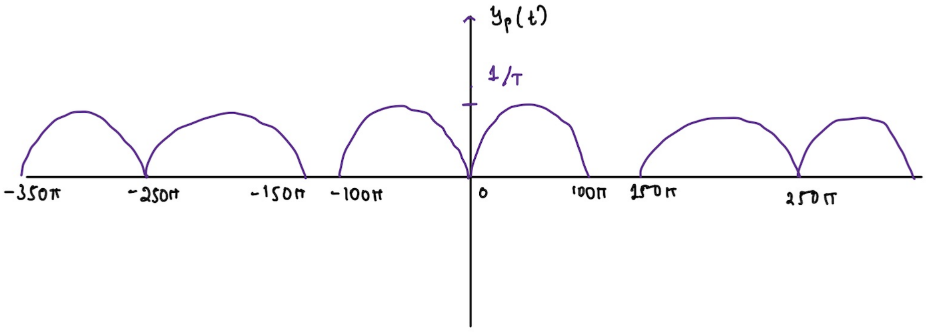
(c) (4 pts) The gain of the low pass filter is $T = 2 \times 10^{-3}$. The range of the cutoff frequency is as follows: $250\pi \leq \omega_c \leq 750\pi$.

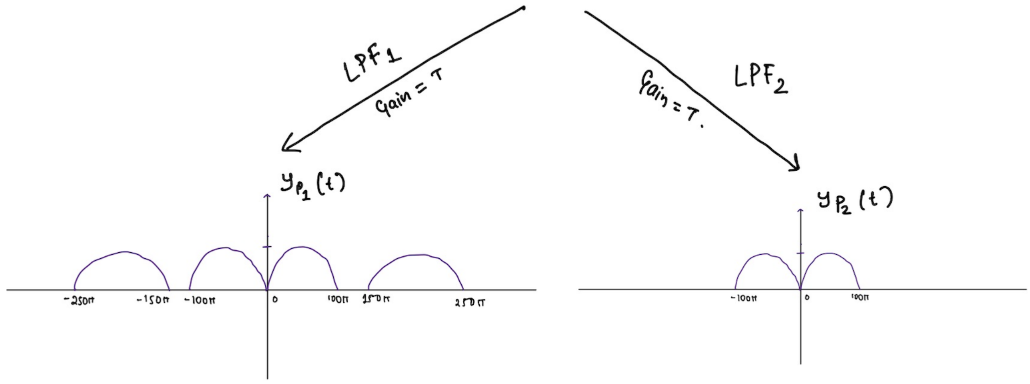


(d) (4 pts) The measured signal can be passed through two low pass filters connected in parallel, say LPF_1 and LPF_2. The reconstructed signal with then be the subtraction of the outputs from the LPF_1 and LPF_2.
Characteristics of LPF_1: $\omega_{c_1} \in [100\pi, 150\pi]$, Gain $= T = 8 \times 10^{-3}$.
Characteristics of LPF_2: $\omega_{c_2} = 250\pi$, Gain $= T = 8 \times 10^{-3}$.

Diagram: part (d):







$$\boxed{y(t) = y_{p_1}(t) - y_{p_2}(t)}$$

Question 17 *(LTI Systems - Echo Cancelation)* 14pts

An important use of inverse systems is to remove distortions of some type from signals. An example of this is the problem of removing echo from acoustic signals. If a room has a perceptible echo, then the initial acoustic impulse will be followed by attenuated versions of the sound at regularly spaced intervals and could be modeled as an LTI system. In this problem, we will design an inverse LTI system that removes the unwanted echo.

(a) Let us begin with a very simple echo system with impulse response

$$h(t) = h_0\delta(t) + h_1\delta(t-T).$$

Suppose that $x(t)$ represents the original acoustic signal and $y(t) = x(t) * h(t)$ is the actual signal that is heard if no processing is done to remove the echos. We would like to design an inverse system $g(t)$ such that

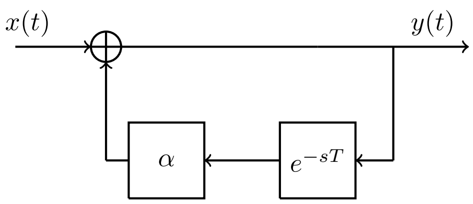
$$h(t) * g(t) = \delta(t).$$

The required impulse response of $g(t)$ is known to be an impulse train:

$$g(t) = \sum_{k=0}^{\infty} g_k\delta(t-kT).$$

Using convolution in the time domain, determine the algebraic equations that the successive g_k must satisfy, and solve these equation for g_0, g_1, and g_2 in terms of h_0 and h_1.

(b) A more complex model for the echo generation could be illustrated with the following causal feedback system:



where α and e^{-sT} in the feedback link denote transfer functions of composed LTI systems, and $\alpha > 0$ is real. Find the algebraic expression and the ROC for the transfer function $H(s)$ for the whole echo generating system.
Hint: $H(s)$ will not be a rational function, but it will still have a pole.

(c) We would like to design an inverse system $G(s)$ such that

$$H(s)G(s) = 1.$$

Find the transfer function $G(s)$ and impulse response $g(t)$ for the inverse of the system in part *(b)*. Construct a realization of this inverse system using adders, coefficient multiplies, and T-delay elements.

(d) For the system in part *(b)* it is reasonable to assume that $\alpha < 1$. Why? What happens to the system if $\alpha > 1$? Would you want to be in a room with such echo characteristics?

SOLUTION

(a) (4 pts) We have that

$$\begin{aligned}
h(t) * g(t) &= h(t) * \left(\sum_{k=0}^{\infty} g_k \delta(t-kT)\right) \\
&= \sum_{k=0}^{\infty} h(t) * (g_k \delta(t-kT)) \\
&= \sum_{k=0}^{\infty} g_k (h_0 \delta(t-kT) + h_1 \delta(t-(k+1)T) \\
&= \sum_{k=0}^{\infty} g_k h_0 \delta(t-kT) + \sum_{k=1}^{\infty} g_{k-1} h_1 \delta(t-kT) \\
&= g_0 h_0 \delta(t) + \sum_{k=1}^{\infty} (g_k h_0 + g_{k-1} h_1) \delta(t-kT) \\
&= \delta(t).
\end{aligned}$$

From this we see that

$$\begin{aligned}
g_0 h_0 = 1 &\implies g_0 = \frac{1}{h_0} \\
g_1 h_0 + g_0 h_1 = 0 &\implies g_1 = -\frac{h_1}{h_0^2} \\
g_2 h_0 + g_1 h_1 = 0 &\implies g_2 = \frac{1}{h_0}\left(\frac{h_1}{h_0}\right)^2
\end{aligned}$$

(b) (4 pts) By using feedback linear system composition we obtain

$$H(s) = \frac{1}{1 - \alpha e^{-sT}}$$

We know that the system is causal and the ROC is right-sided. There is a pole at $s = \frac{\ln \alpha}{T}$. The ROC is thus $Re(s) > \frac{\ln \alpha}{T}$.

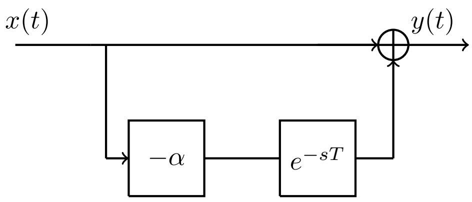
(c) (4 pts)

$$H(s)G(s) = \frac{1}{1 - \alpha e^{-sT}} G(s) = 1 \implies G(s) = 1 - \alpha e^{-sT}$$

with ROC the whole imaginary plane. The impulse response is

$$g(t) = \delta(t) - \alpha \delta(t - T).$$

The realization in the attached picture is the identity system and a scaled time delay by T composed in parallel.



(d) (2 pts)

The the system is causal and the ROC is right-sided. If $\alpha \geq 1$ then the system will be unstable since it will not include the imaginary axis. You would not want to be in a room with such characteristics since any noise you make will lead to an increasingly noisy echos.