Final Exam Solutions

Problem 1 *(Impulse Response)* 12 points

Determine the impulse response for each system described below. Justify your answer.

(a) (3 pts) A causal continuous-time LTI system with transfer function $H(s) = \frac{1}{s}$

Solution: Since the system is causal, the ROC of $H(s)$ must be right-sided with $Re(s) > 0$. The inverse Laplace transform of this signal is $h(t) = u(t)$ and this is the impulse response of the system.

(b) (3 pts) A stable discrete-time LTI system with transfer function $H(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$

Solution: Since the system is stable, the ROC of $H(z)$ must include the unit circle and is given by $|z| > \frac{1}{4}$. The impulse response is the inverse z-Transform of $H(z)$ and it is given by $h[n] = \left(\frac{1}{4}\right)^n u[n]$.

(c) (3 pts) An LTI system described by the input-output relationship

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau+2) d\tau.$$

By definition of the impulse response

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau+2) d\tau = e^{-(t+2)} \int_{-\infty}^{t} \delta(\tau+2) d\tau = e^{-(t+2)} u(t+2)$$

(d) (3 pts) An LTI system with a step response $s[n] = \alpha^n u[n]$ where recall that the step response is given by $s[n] = \mathcal{H}\{u[n]\}$.

We have that $\delta[n] = u[n] - u[n-1]$ and since the system is LTI

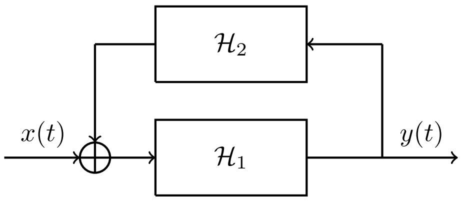
$$h[n] = \mathcal{H}\{\delta[n]\} = \mathcal{H}\{u[n] - u[n-1]\} = s[n] - s[n-1] = \alpha^n u[n] - \alpha^{n-1} u[n-1]$$

This could be further simplified to

$$h[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \alpha^n - \alpha^{n-1}, & n > 0 \end{cases}$$

Problem 2 *(System Composition)* 10 points

Consider the feedback composition of two continuous-time LTI systems with transfer functions $H_1(s)$ and $H_2(s)$:



(a) (5 pts) Suppose $H_1(s) = 1$ and $H_2(s) = \frac{1}{s}$, what is the transfer function $H(s)$ of the overall system $\mathcal{H}$?

The overall system $\mathcal{H}$ is known to be causal, is it also stable? Find its impulse response $h(t)$.

Solution: The transfer function of the overall system is given by

$$H(s) = \frac{H_1(s)}{1 - H_1(s)H_2(s)}.$$

Substituting the given $H_1(s)$ and $H_2(s)$ we obtain

$$H(s) = \frac{1}{1 - 1/s} = \frac{s}{s-1} = 1 + \frac{1}{s-1}.$$

This system has a pole at $s = 1$. Since it is causal, the ROC is right-sided and cannot include the imaginary axis. Thus, the system is NOT stable.

The impulse response is $h(t) = \delta(t) + e^t u(t)$.

(b) (5 pts) Suppose the overall system $\mathcal{H}$ is known to be causal and to have a causal inverse system $\mathcal{G}$. Find the transfer function $G(s)$ and the impulse response $g(t)$ of the inverse system.

Solution:

The inverse system must satisfy

$$G(s)H(s) = 1$$

and thus

$$G(s) = \frac{s-1}{s} = 1 - \frac{1}{s}.$$

Since $\mathcal{G}$ is causal $G(s)$ has a right-sided ROC $Re(s) > 0$. Thus, the impulse response of the inverse system is

$$g(t) = \delta(t) - u(t).$$

Problem 3 *(Difference Equations)* 10 points

An LTI system is described by the following difference equation:

$$y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n].$$

(a) (4 pts) Find the transfer function $H(z)$.

Solution: Taking the z-Transform of both sides, and applying the shift-in-time property we obtain

$$\begin{aligned} Y(z)[z^{-1} - \frac{10}{3} + z] &= X(z) \\ H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} - \frac{10}{3} + z} = \frac{z^{-1}}{1 - \frac{10}{3}z^{-1} + z^{-2}} \\ H(z) &= \frac{-3/8}{1 - \frac{1}{3}z^{-1}} + \frac{3/8}{1 - 3z^{-1}} \end{aligned} \tag{1}$$

(b) (3 pts) If the system is known to be stable, determine the impulse response. Is this system causal, anti-causal or neither?

Solution: Thus there are two poles at $z = 3$ and $z = \frac{1}{3}$. For a stable system, the ROC has to include the unit circle. Thus, the ROC is $\frac{1}{3} < z < 3$. The corresponding impulse response is

$$h[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] - \frac{3}{8}\left(3\right)^n u[-n-1] \tag{2}$$

This response is neither causal nor anti-causal.

(c) (3 pts) If the system is known to be causal, determine the impulse response. Is this system stable?

Solution: Since the system is causal, the ROC has to be $|z| > 3$. The impulse response is thus

$$h[n] = -\frac{3}{8}\left(\frac{1}{3}\right)^n u[n] + \frac{3}{8}3^n u[n] \tag{3}$$

This response is clearly unstable.

Problem 4 *(Correlation)* 12pts

In this problem we study an operation called *correlation* which is widely used in signal analysis. The correlation is defined, in discrete-time, as

$$R_{xt}[n] = x[n] \star t[n] = \sum_{m=-\infty}^{\infty} x^*[m]t[n+m]$$

where $x[n]$ is the signal of interest, $t[n]$ is called *the template* signal, and $x^*[n]$ denotes the complex conjugate of $x[n]$.

(a) (4pts) The correlation operation could be transformed into classical convolution. Express $R_{xt}[n]$ as a convolution between functions of $t[n]$ and $x[n]$.

Solution:

We could rewrite it as a convolution by noticing that,

$$x^*[-n] * t[n] = \sum_{m=-\infty}^{\infty} x^*[-m]t[n-m] = \sum_{m=-\infty}^{\infty} x^*[m]t[n+m] = R_{xt}[n] \tag{4}$$

where the last step is obtained from the symmetry of the sum.

It should also be correct to write

$$x^*[n] * t[-n] = \sum_{m=-\infty}^{\infty} t[-m]x^*[n-m] = R_{xt}[n] \tag{5}$$

using the same reasoning as above.

(b) (4pts) Given the signal

$$x[n] = \begin{cases} 1, & if\ 0 \le n \le 2 \\ 2, & if\ 3 \le n \le 5 \\ 0, & otherwise \end{cases}$$

and the template

$$t[n] = max\left(1 - \left|\frac{n}{2}\right|, 0\right)$$

find the correlation product R_{xt}.

Solution : Let

$$z[n] = x^*[-n] = \begin{cases} 1, & if\ \ -2 \le n \le 0 \\ 2, & if\ \ -3 \le n \le -5 \\ 0, & otherwise \end{cases}$$

or

$$t[n] = \frac{1}{2}\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n-1]$$

and observe that

$$t[n] = \begin{cases} 1, & if\ n = 0 \\ \frac{1}{2}, & if\ n = -1, 1 \\ 0, & otherwise \end{cases}$$

Then

$$R_{xt}[n] = z[n] * t[n] = \frac{1}{2}z[n+1] + z[n] + \frac{1}{2}z[n-1]$$

and

$$R_{xt}[n] = \begin{cases} 1, & if\ n = -6 \\ 3, & if\ n = -5 \\ 4, & if\ n = -4 \\ 3.5, & if\ n = -3 \\ 2.5, & if\ n = -2 \\ 2, & if\ n = -1 \\ 1.5, & if\ n = 0 \\ 0.5, & if\ n = 1 \\ 0, & otherwise \end{cases}$$

(c) (4pts) Finally, the cross-correlation operation is defined as the correlation of the signal with itself:

$$R_{xx}[n] = x[n] \star x[n] = \sum_{m=-\infty}^{\infty} x^*[m]x[n+m].$$

Find k such that

$$E = R_{xx}[k]$$

where E denotes the energy of the signal $x[n]$.

Find the energy of the signal in part (b).

Solution:

$$R_{xx}[0] = \sum_{m=-\infty}^{\infty} x^*[m]x[m] = \sum_{m=-\infty}^{\infty} |x[m]|^2 = E \tag{6}$$

the required function is, hence, $E = R_{xx}[0]$ and the energy of $x[n]$ in part (b) is $E = 15$