Signals & Systems
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Ecole Polytechnique Fédérale, Lausanne
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Practice Final Solutions

- **Rules:** This exam is closed-book and closed-notes; calculators, computing and communication devices are not permitted. Two handwritten and not photocopied double-sided A4 sheets of notes are allowed. Moreover, copies of the tables in Sections 4.A, 4.B, 4.C, 4.D, 6.A, 6.B, 7.A, and 7.B in the lecture notes will be attached to the exam sheet.
- You have 180 minutes to complete this exam.
- Unless explicitly stated otherwise, detailed derivations of the results are required for full credit on all **open problems**.
- For the **multiple choice with unique answer** questions, we give
  - +3 points if your answer is correct,
  - 0 points if your answer is incorrect.
- For the **multiple choice with multiple answers** questions, we give
  - +4 points for all correct answers,
  - +2 points for one incorrect answer and three correct answers,
  - 0 points for other possibilities of answers.

1.*(3 Pts)* The integral $\int_{-\infty}^{\infty}\delta(t-6)\int_{-\infty}^{\infty}f(\tau)\delta(t-\tau)d\tau dt$ evaluates to **(check the unique answer!)**

1. $f(6t)$,
2. 0,
3. $f(6)$,
4. $f(0)$.

Solution: The correct answer is (3).

This question checks your understanding of the basic properties of the Dirac-delta function, as used in class, namely:

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t), \tag{1}$$

and thus,

$$\int_{-\infty}^{\infty}\delta(t-6)\int_{-\infty}^{\infty}f(\tau)\delta(t-\tau)d\tau dt = \int_{-\infty}^{\infty}\delta(t-6)f(t)dt = f(6). \tag{2}$$

2. *(4 Pts)* Which of the following claims about properties of signals and systems are true? **(check all that apply!)**

1. The discrete-time signal $x[n]=\frac{1}{\sqrt{n}}u[n]$ is not an energy signal.
2. The discrete-time signal $z[n]=\cos(2n)$ has fundamental period π.
3. The continuous-time system $\mathcal{H}\{x(t)\}=\frac{d}{dt}x(t)$ is linear and time-invariant.
4. The continuous-time system $\mathcal{H}\{x(t)\}=(x(t)-\mu)^3$ is memoryless but not causal.

Solution:

(1) and (3) are true.

1. This is true. The energy of this signal is infinite, and so it is not an energy signal.
2. This is false. $z[n]$ is not even a periodic signal so it does not have a fundamental period. See Problem Set 1, Problem 3.
3. This is true. This system was analyzed in Problem Set 2, Problem 4.
4. This is false. A memoryless system is always casual.

3. *(4 Pts)* Which of the following systems are linear? **(check all that apply!)**

1. $y(t)=x(t)$.
2. $y(t)=(x(t))^n$ for any $n\in\mathbb{Z}^+$.
3. $y(t)=x(t^2)$.
4. $y[n]=\sum_{k=-\infty}^{n}(-1)^k x[k]$.

Solution:

Systems (1), (3), and (4) are linear.

1. This system is linear. This is the identity system. It is straightforward to check that it is linear.
2. This system is not linear if $n > 1$. Let $n = 2$. Then

   $\mathcal{H}\left\{a_1x_1(t) + a_2x_2(t)\right\} = a_1^2x_1^2(t) + 2a_1a_2x_1(t)x_2(t) + a_2^2x_2^2(t) \neq a_1x_1^2(t) + a_2x_2^2(t) = a_1\mathcal{H}\left\{x_1(t)\right\} + a_2\mathcal{H}\left\{x_2(t)\right\}$

3. This system is linear. This is very similar to Problem 4(c) on Problem Set 1.
4. This system is linear:

$$\begin{aligned}\mathcal{H}\left\{a_1x_1[n] + a_2x_2[n]\right\} &= \sum_{k=-\infty}^{n}(-1)^k(a_1x_1[k] + a_2x_2[k]) \\ &= a_1\sum_{k=-\infty}^{n}(-1)^kx_1[k] + a_2\sum_{k=-\infty}^{n}(-1)^kx_2[k] \\ &= a_1\mathcal{H}\left\{x_1[n]\right\} + a_2\mathcal{H}\left\{x_2[n]\right\}\end{aligned}$$

4. *(4 Pts)* Consider a discrete-time LTI system with frequency response $H(e^{j\omega}) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$. Which of the following claims are true? **(check all that apply!)**

1. The input $x[n]$ and output $y[n]$ satisfy $y[n] - \frac{1}{3}y[n-2] = x[n]$
2. If the input is $x[n] = e^{-j\frac{\pi}{2}n}$, then the output is $y[n] = \frac{e^{-j\frac{\pi}{2}n}}{1-\frac{j}{3}}$.
3. The system is causal and memoryless.
4. The input $x[n]$ and output $y[n]$ satisfy $y[n] - \frac{1}{3}y[n-1] = x[n]$.

Solution: From the given equation we get $Y(e^{j\omega})(1 - \frac{1}{3}e^{-j\omega}) = X(e^{j\omega})$, thus $y[n] - \frac{1}{3}y[n-1] = x[n]$ so (1) is wrong and (4) is correct.

You may immediately see that (3) is wrong because the system is not memoryless. Recall that an impulse response of a memoryless system must be a multiple of $\delta[n]$ and from Appendix 4.D. we see that the impulse response for this system will not be a multiple of $\delta[n]$.

Also for input $x[n] = e^{j\omega_0 n}$, the output is $y[n] = e^{j\omega_0 n}H(e^{j\omega_0 n})$ and for $\omega_0 = -\frac{\pi}{2}$ we deduce that (2) is correct.

5. *(3 Pts)* In the time domain, every LTI system can be characterized by its impulse response, and the input-output relationship can be written as $y(t) = (h * x)(t)$. Let $x(t) = \frac{2}{\pi}\left(\mathrm{sinc}(\frac{2}{\pi}t)\right)^{100}$, and $h(t) = \frac{4}{\pi}\mathrm{sinc}(\frac{4}{\pi}(t-1))$. Find $Y(\omega = 6)$, where $Y(\omega)$ is the Fourier Transform of $y(t)$. **(check the unique answer!)**

1. 22.125,
2. 0,
3. $\frac{\pi}{6}$,
4. None of the other options

Solution: Recall that in the frequency-domain convolution becomes multiplication and $Y(\omega) = X(\omega)H(\omega)$. From Appendix 4.B. we see that $H(\omega = 6) = 0$. Therefore, without computing $X(\omega)$, we compute $Y(\omega) = 0$.

6. *(3 Pts)* Let $x_\tau(t) = x(t - \tau)$ and $y_\mu(t) = y(t - \mu)$. Let $z(t) = (x * y)(t)$. Then, we have **(check the unique answer!)**

1. $(x_\tau * y_\mu)(t) = z(t - \tau - \mu)$,
2. $(x_\tau * y_\mu)(t) = z(t - \tau + \mu)$,
3. $(x_\tau * y_\mu)(t) = z(t + \tau - \mu)$
4. $(x_\tau * y_\mu)(t) = z(t + \tau + \mu)$

Solution: This is perhaps most elegantly tackled using the Fourier transform. Namely, $X_\tau(\omega) = e^{-j\tau\omega}X(\omega)$ and $Y_\mu(\omega) = e^{-j\mu\omega}Y(\omega)$. Now, letting $v(t) = (x_\tau * y_\mu)(t)$, we find $V(\omega) = e^{-j(\tau+\mu)\omega}X(\omega)Y(\omega) = e^{-j(\tau+\mu)\omega}Z(\omega)$. Using the time-shift property (in reverse), this implies that $v(t) = y(t - \tau - \mu)$. The same can be obtained by changing integration variables, but the risk of small (sign) errors is significantly higher (in my opinion).

7. *(4 Pts)* A sampling system that samples continuous-time signals with a sampling frequency $\omega_s = 1000\pi$ is applied to the signal

$$x(t) = \sin(200\pi t).$$

The result is the following discrete time signal: **(check all that apply!)**

1. $x[n] = \sin(200\pi n)$
2. $x[n] = \sin(\frac{2}{5}\pi n)$
3. $x[n] = \sin(5\pi n)$
4. $x[n] = \sin(5n)$

Solution: The correct answer is (2).

The sampled signal will have the form $x[n] = \sin(200\pi T n)$ where T is the sampling interval. The answer follows by recalling that $T = \frac{2\pi}{\omega_s}$.

You need to be careful here to also rule out (1), (3), and (4) since sometimes two periodic discrete time signals may appear to be different but are actually the same signal. For example $x[n] = \sin(\frac{12}{5}\pi n)$ would have also been correct.

8. *(4 Pts)* A signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using the impulse-train sampling procedure covered in lecture, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \cos(300\pi t).$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$? **(check all that apply!)**

1. $x(t) = \cos(200\pi t)$

2. $x(t) = \cos(300\pi t)$
3. $x(t) = \cos(1300\pi t)$
4. $x(t) = \cos(700\pi t)$

Solution:

(2), (3), and (4) are all correct.

A similar question is covered on Problem Set 8. From the lecture notes we have

$$X_p(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \tag{3}$$

and

$$X_r(\omega) = X_p(\omega)H(\omega). \tag{4}$$

- If $x(t) = \cos(200\pi t)$, then $\omega_s > 2\omega_M$ (ω_M is the largest frequency where frequency response is non-zero), then there is no aliasing and after the low pass filter we get $x_r(t) = \cos(200\pi t) \neq \cos(300\pi t)$ so (1) is wrong.
- If $x(t) = \cos(300\pi t)$, then $\omega_s > 2\omega_M$, then there is no aliasing and after the low pass filter we get $x_r(t) = \cos(300\pi t)$ so (2) is correct.
- If $x(t) = \cos(1300\pi t)$, then $\omega_s < 2\omega_M$. In this case there is aliasing and the sampling theorem does not guarantee exact reconstruction of the sampled signal. After we apply (3) and (4) we get $x_r(t) = \cos(300\pi t)$. Because the Dirac delta at $\omega = 1300\pi$ is shifted by $\omega_s = 1000\pi$ and is now at $\omega = 300\pi$ and after the low pass filer we get $x_r(t) = \cos(300\pi t)$ so (3) is correct.
- If $x(t) = \cos(700\pi t)$, then $\omega_s < 2\omega_M$. tIn this case there is aliasing and the sampling theorem does not guarantee exact reconstruction of the sampled signal. After we apply (3) and (4) we get $x_r(t) = \cos(300\pi t)$. Because the Dirac delta at $\omega = 700\pi$ is shifted by $\omega_s = 1000\pi$ and is now at $\omega = -300\pi$ and after the low pass filer we get $x_r(t) = \cos(300\pi t)$ so (4) is correct.

9.*(3 Pts)* The autocorrelation sequence of a sequence $x[n]$ is defined as

$$r[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k]$$

Let $X(z)$ denote the Z-transform of $x[n]$. Which one of the following expressions is the Z-transform of $r[n]$? **(check the unique answer!)**

1. $R(z) = X(z)X(-z)$.
2. $R(z) = X(z)X(z^{-1})$.
3. $R(z) = X(-z)X(z^{-1})$.
4. $R(z) = X(z^{-1})X(-z^{-1})$.

Solution: We have the autocorrelation

$$r[n] = \sum_{k=-\infty}^{\infty} x[k]x[n+k] = \sum_{k=-\infty}^{\infty} x[-k]x[n-k] = (x_1 * x)[n]$$

where $x_1[n] = x[-n]$. According to the properties of Z-Transform (Appendix 7.A), we have

$$X_1(z) = X(z^{-1})$$

and the Z-transform of autocorrelation should be

$$R(z) = X_1(z)X(z) = X(z^{-1})X(z)$$

10. *(3 Pts)* If two unstable causal LTI systems are placed in a parallel connection, then the overall (end-to-end) system must be **(check the unique answer!)**

1. LTI, causal, and unstable
2. LTI, causal, but may be stable
3. LTI, but may be anti-causal,
4. Causal, but may not be LTI.

Solution: The correct answer is (2). A parallel connection of LTI systems will give an LTI system. A parallel connection of causal LTI systems will give a causal LTI system.

Stability may changes if the poles of the transfer functions for the two systems cancel. Alternatively, the system impulse responses may not be absolutely summable/integrable individually, but their sum could still be absolutely summable/integrable.

11. *(3 Pts)* Which of the following is the impulse response of the causal LTI system described by the differential equation,

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2\frac{d}{dt}x(t) + 3x(t).$$

(check the unique answer!)

1. $h(t) = e^{-4t}u(t) + 2e^{-2t}u(t)$.
2. $h(t) = e^{-t}u(t) + e^{-4t}u(t)$.
3. $h(t) = 2e^{t}u(t) + e^{2t}u(t)$.
4. $h(t) = e^{-t}u(t) + e^{-2t}u(t)$.

Solution: By using Laplace transform we get

$$Y(s)(s^2 + 3s + 2) = X(s)(2s + 3),$$

thus

$$H(s) = \frac{2s+3}{(s+2)(s+1))} \quad (5)$$

$$= \frac{A}{s+2} + \frac{B}{s+1} \quad (6)$$

and even without solving for A and B we get $h(t) = Ae^{-t}u(t) + Be^{-2t}u(t)$, and the only equation of this form the the correct exponential power is (4).

12. *(3 Pts)* Consider a discrete-time LTI system such that if we feed it with an input $x[n] = \left(-\frac{1}{3}\right)^n u[n]$, the corresponding output would be $y[n] = \delta[n] + \left(-\frac{1}{3}\right)^n u[n]$. Suppose now we feed the same system with an input signal $x[n] = \left(\frac{1}{9}\right)^n$, $\forall n$, that is, for $-\infty < n < \infty$. Then, the corresponding output signal $y[n]$ would be, **(check the unique answer!)**

1. $y[n] = 4\left(\frac{1}{9}\right)^n \ , \ \forall n\,.$
2. $y[n] = \left(\frac{1}{3}\right)^{2n} \ , \ \forall n\,.$
3. $y[n] = 5\left(\frac{1}{9}\right)^n \ , \ \forall n\,.$
4. $y[n] = \left(\frac{1}{9}\right)^{n-1} \ , \ \forall n\,.$

Solution: By using Z transform

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} && (7) \\ &= \frac{1 + \frac{1}{1+\frac{1}{3}z^{-1}}}{\frac{1}{1+\frac{1}{3}z^{-1}}} && (8) \\ &= 2 + \frac{1}{3}z^{-1} && (9) \end{aligned}$$

so $H(\frac{1}{9}) = 5$, thus $y[n] = 5\left(\frac{1}{9}\right)^n \ , \ \forall n\,.$ (3) is correct.

Problem 1 *(LTI System)* 12 points

Consider ~~an LTI~~ system with an input and output related through the equation

$$y[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k-n} x[k-1]$$

(a) (4 Pts) What is the impulse response $h[n]$ of the system?

Solution: The impulse response of the system is given by

$$\begin{aligned} h[n] &= \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k-n} \delta[k-1] = \begin{cases} \left(\frac{1}{2}\right)^{-1-n} & \text{if } n \geq 1 \\ 0 & \text{if } n < 1 \end{cases} \\ &= 2^{n+1} u[n-1] \end{aligned}$$

(b) (4 Pts) Is the system stable? Briefly justify your answer.

Solution: The system is **not stable** since $\cancel{\sum_{n=-\infty}^{\infty} |h[n]| = \infty}$ both parts (a) and (c) give an example of a signal with bounded input leading to unbounded output.

Edit: Note that if the system is LTI, then using the condition $\sum_{n=-\infty}^{\infty} |h[n]| = \infty$ is also correct to determine that it is not stable.

(c) (4 Pts)

What is the output $y_1[n]$ of the system when the input

$$x_1[n] = \left(\frac{1}{2}\right)^{n+1} u[n]$$

is applied?

Solution:

$$y_1[n] = \sum_{k=-\infty}^{n} \left(\frac{1}{2}\right)^{-k-n} \left(\frac{1}{2}\right)^{k} u[k-1] = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{-n} = n2^n u[n-1]$$

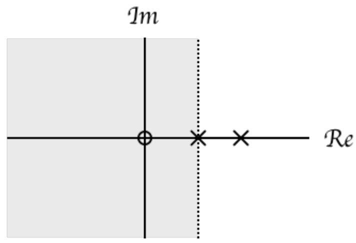
Problem 2 *(System Properties)* 12 points

Determine if each system being described is causal or not. Briefly justify your answer.

(a) (4 Pts) An LTI system with impulse response $h(t) = (1 + e^{-t+1})u(t-1)$

Solution: This system is **causal** since the impulse response is zero for $t < 0$.

(b) (4 Pts) A continuous-time LTI system with system function $H(s)$ that has the following pole-zero plot and ROC



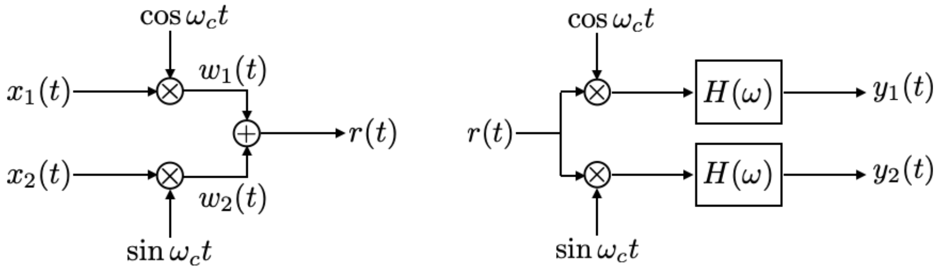
Solution: The system is **not causal**. Since the system function has ROC in the left-hand plane, the impulse response is left sided and non-zero for $t < 0$.

(c) (4 Pts) A discrete-time LTI system with system function $H(z) = 1$ for all z

Solution: The system is **causal**. This is just the identity system that outputs the input without doing anything to it.

Problem 3 *(Communication System)* 10 points

In this problem we will perform a simple analysis of an LTI communication system depicted in the figure below.



(a) (5 Pts) What are the Fourier transforms $W_1(\omega)$ and $W_2(\omega)$ of the signals

$$w_1(t) = x_1(t)\cos\omega_c t \quad \text{and} \quad w_2(t) = x_2(t)\sin\omega_c t \text{ ?}$$

You answers should be in terms of $X_1(\omega)$ and $X_2(\omega)$.

Solution: Using multiplication property we obtain

$$W_1(\omega) = \frac{1}{2\pi}X_1(\omega) * \pi\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right] = \frac{1}{2}\left[X_1(\omega-\omega_c)+X_1(\omega+\omega_c)\right]$$

and

$$W_2(\omega) = \frac{1}{2\pi}X_2(\omega) * \frac{\pi}{j}\left[\delta(\omega-\omega_c)-\delta(\omega+\omega_c)\right] = \frac{1}{2j}\left[X_2(\omega-\omega_c)-X_2(\omega+\omega_c)\right]$$

(b) (5 Pts) Suppose $x_1(t)$ and $x_2(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_1(\omega) = 0, \quad |\omega| \geq \omega_M$$

and

$$X_2(\omega) = 0, \quad |\omega| \geq \omega_M.$$

Moreover, $\omega_M < \omega_c$. Let $r(t) = w_1(t) + w_2(t)$. The same low-pass filter $H(\omega)$ is applied to the signals

$$r(t)\cos\omega_c t \quad \text{and} \quad r(t)\sin\omega_c t$$

in order to obtain y_1 and y_2 (see figure above). Is it possible to design $H(\omega)$ so that $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$? If yes, determine the gain and the cut-off frequency of the desired low-pass filter. If no, briefly explain why.

Solution:

From part (a) we obtain

$$R(\omega) = \frac{1}{2}\left[X_1(\omega-\omega_c)+\frac{1}{j}X_2(\omega-\omega_c)\right] + \frac{1}{2}\left[X_1(\omega+\omega_c)-\frac{1}{j}X_2(\omega+\omega_c)\right].$$

To obtain the Fourier transform of $r(t)\cos\omega_c t$ we applying the multiplication property again

$$\begin{aligned}\frac{1}{2\pi}R(\omega) * \pi\left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right] &= \frac{1}{4}\left[X_1(\omega-\omega_c)+\frac{1}{j}X_2(\omega-\omega_c)\right] * \left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right] \\ &+ \frac{1}{4}\left[X_1(\omega+\omega_c)-\frac{1}{j}X_2(\omega+\omega_c)\right] * \left[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)\right].\end{aligned}$$

Distributing the convolution by the frequency-shifted impulses we obtain

$$\frac{1}{4}\left[X_1(\omega-2\omega_c)+\frac{1}{j}X_2(\omega-2\omega_c)\right]+\frac{1}{2}X_1(\omega)+\frac{1}{4}\left[X_1(\omega+2\omega_c)-\frac{1}{j}X_2(\omega+2\omega_c)\right].$$

And so we see that it is possible to reconstruct $x_1(t)$ by applying a low-pass filter with gain 2 and any cut-off frequency between ω_M and $2\omega_c-\omega_M$.

We can check that the same low-pass filter would work for $x_2(t)$. We again find the Fourier transform of $r(t)\sin\omega_c t$ by applying the multiplication property to obtain

$$\frac{1}{4}\left[\frac{1}{j}X_1(\omega-2\omega_c)-X_2(\omega-2\omega_c)\right]+\frac{1}{2}X_2(\omega)-\frac{1}{4}\left[\frac{1}{j}X_1(\omega+2\omega_c)+X_2(\omega+2\omega_c)\right].$$

It is also possible to trace out the behavior of the system in the time domain by applying Euler's Formula twice to $r(t)\cos\omega_c t$ and $r(t)\sin\omega_c t$.

Problem 4 *(System Composition)* 10 points

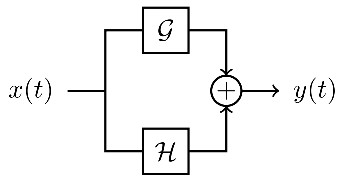
This is a design problem where subparts **do not have a unique answer**.

(a) (5 Pts) Assume that the overall system has the following frequency response

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)}. \tag{10}$$

With respect to the figure below, find stable LTI systems $\mathcal{G}$ and $\mathcal{H}$ such that the overall system in the figure has exactly the frequency response given above.

- Provide the impulse response $g(t)$ and $h(t)$ of the systems $\mathcal{G}$ and $\mathcal{H}$, respectively.
- For full credit, the impulse responses $g(t)$ and $h(t)$ cannot be scaled and/or shifted Dirac delta functions (i.e. $g(t), h(t) \neq \alpha\delta(t-\beta)$ for constants α, β).



Solution: By partial fraction decomposition

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)} = \underbrace{\frac{A}{2+j\omega}}_{G(\omega)} + \underbrace{\frac{B}{3+j\omega}}_{H(\omega)}. \tag{11}$$

Thus, in order for the equality to hold we must satisfy

$$A(3+j\omega) + B(2+j\omega) = 8+3j\omega. \tag{12}$$

For the equality above we obtain the system of linear equations

$$3A + 2B = 8 \tag{13}$$
$$A + B = 3, \tag{14}$$

and it is easy to see that the solution is $A = 2, B = 1$. Therefore, one possible solution is $g(t) = 2e^{-2t}u(t)$ and $h(t) = e^{-3t}u(t)$.

(b) (5 Pts) Assume that the overall system has the following frequency response same as in part (a)

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)}. \tag{15}$$

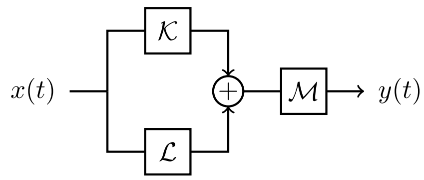
With respect to the figure below, find stable LTI systems $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$ such that the overall system in the figure has exactly the frequency response given above.

- Provide the impulse response $k(t)$, $\ell(t)$ and $m(t)$ of the systems $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{M}$, respectively.
- For full credit, the impulse responses $k(t)$ and $m(t)$ cannot be scaled and/or shifted Dirac delta functions (i.e. $k(t), m(t) \neq \alpha\delta(t-\beta)$ for constants α, β).

The same expression can be rewritten as

$$\frac{8+3j\omega}{(2+j\omega)(3+j\omega)} = \underbrace{\left(3 + \frac{2}{2+j\omega}\right)}_{L(\omega)+K(\omega)} \cdot \underbrace{\frac{1}{3+j\omega}}_{M(\omega)}. \tag{16}$$

Thus, one possible solution is $\ell(t) = 3\delta(t)$, $k(t) = 2e^{-2t}u(t)$ and $m(t) = e^{-3t}u(t)$.



Problem 5 *(Sampling)* 15 points

The signal

$$x(t) = \frac{15}{2\pi}\,\mathrm{sinc}\left(\frac{15}{2\pi}t\right) - \frac{5}{\pi}\,\mathrm{sinc}\left(\frac{5}{\pi}t\right) + \frac{5}{2\pi}\,\mathrm{sinc}\left(\frac{5}{2\pi}t\right)$$

is sampled with a sampling frequency $\omega_s = 10$.

(a) (5 Pts) Find the Fourier Transform $X(\omega)$. Find ω_M such that $X(\omega) = 0$ whenever $|\omega| > \omega_M$.

Solution:

In the frequency domain

$$X(\omega) = \begin{cases} 1 & ,|\omega| \leq 7.5 \\ 0 & ,\text{else} \end{cases} - \begin{cases} 1 & ,|\omega| \leq 5 \\ 0 & ,\text{else} \end{cases} + \begin{cases} 1 & ,|\omega| \leq 2.5 \\ 0 & ,\text{else} \end{cases} \tag{17}$$

so ω_M, which is the largest frequency where frequency response is non-zero is $\omega_M = 7.5$.

(b) (5 Pts) In lecture, we modeled the sampling operation with multiplication by a periodic impulse train. Mathematically this could be written as

$$x_p(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

The reconstruction $x_r(t)$ was obtained from $x_p(t)$ by applying a low-pass filter $h(t)$ with gain T and cutoff frequency $\omega_c = \frac{\omega_s}{2}$, where $\omega_s = \frac{2\pi}{T}$.

In this case, does $x_r(t) = x(t)$ hold? If yes, explain why. If not, find $x_r(t)$.

Solution: The sampling theorem condition is violated as $\omega_s < 2\omega_M$, so sampling theorem cannot be applied here. However, we could still trace out what happens to this signal in the frequency domain.

From the lecture notes we have

$$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{18}$$

and also we have

$$X_r(\omega) = X_p(\omega)H(\omega). \tag{19}$$

With the low-pass filter described in the question, the reconstructed signal in frequency domain is

$$X(\omega) = \begin{cases} 1 & ,|\omega| \leq 5 \\ 0 & ,\text{else} \end{cases} \tag{20}$$

and in time domain $x_r(t) = \frac{5}{\pi}\,\mathrm{sinc}\left(\frac{5}{\pi}t\right)$.

(c) (5 Pts) Let $x(t)$ be as given above and $y(t)$ be some arbitrary signal. The signal

$$z(t) = y(t) * x(t)$$

is sampled with a sampling frequency $\omega_s = 10$.

We again model the sampling operation with multiplication by a periodic impulse train to obtain

$$z_p(t) = z(t)p(t)$$

and apply some filter $g(t)$ to obtain a reconstructed signal $z_r(t)$.

Design the reconstruction filter $g(t)$ such that $z_r(t) = z(t)$ regardless of the value of $y(t)$.

Hint: $g(t)$ will no longer be a simple low-pass filter.

Solution:

(c) In order for the reconstructed signal to be the same as the original signal $z_r(t) = z(t)$, the reconstruction filter must be

$$G(\omega) = \frac{\pi}{5}\begin{cases} 1 & , |\omega| \leq 7.5 \\ 0 & , \text{else} \end{cases} - \frac{\pi}{5}\begin{cases} 1 & , |\omega| \leq 5 \\ 0 & , \text{else} \end{cases} + \frac{\pi}{5}\begin{cases} 1 & , |\omega| \leq 2.5 \\ 0 & , \text{else} \end{cases} \tag{21}$$

and in time domain

$$g(t) = \frac{3}{2}\,\text{sinc}\left(\frac{15}{2\pi}t\right) - \text{sinc}\left(\frac{5}{\pi}t\right) + \frac{1}{2}\,\text{sinc}\left(\frac{5}{2\pi}t\right).$$

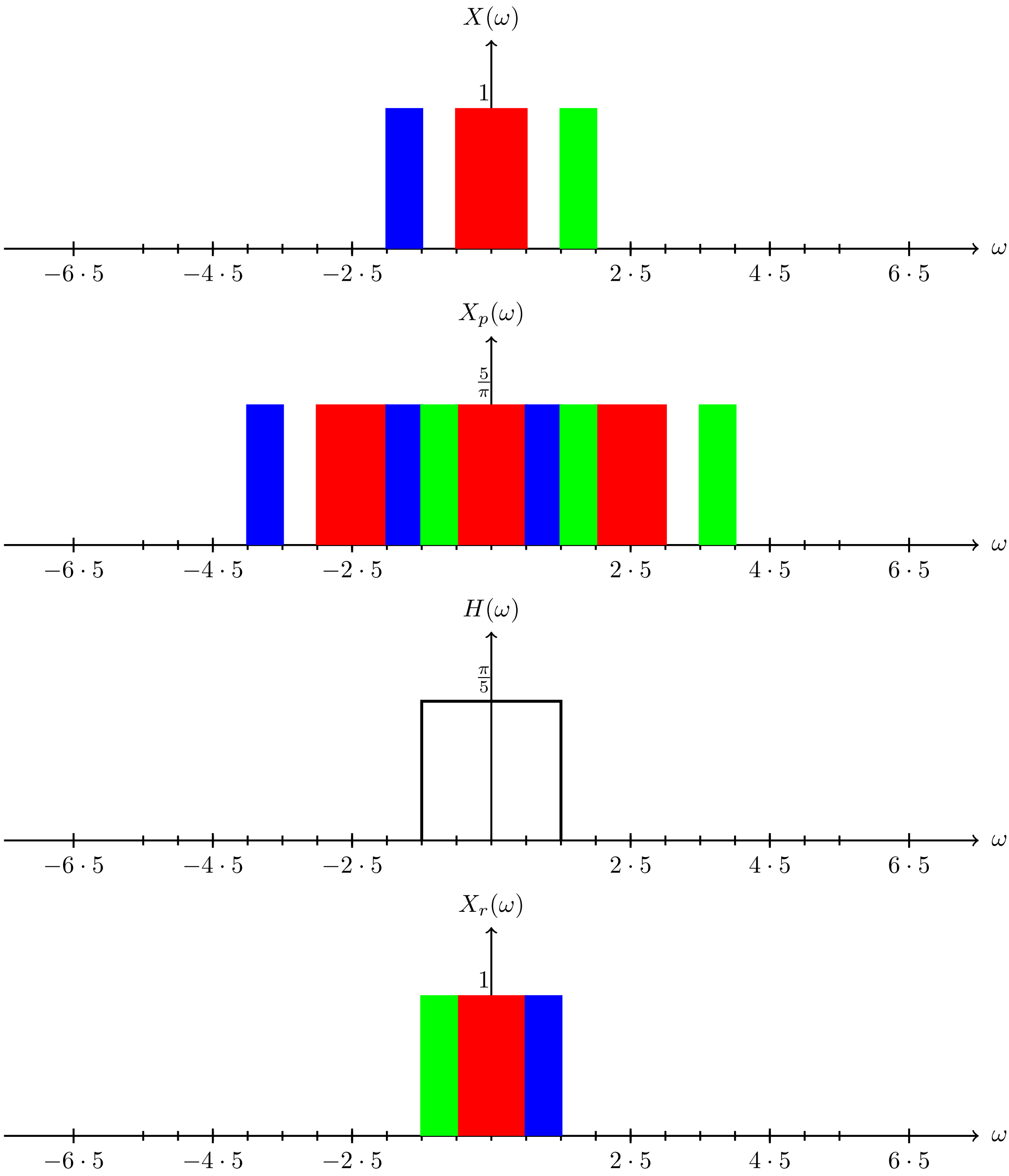


Figure 1: Frequency response.