Final Exam Solutions

- **Rules:** This exam is closed-book and closed-notes; calculators, computing and communication devices are not permitted. Two handwritten and not photocopied double-sided A4 sheets of notes are allowed. Moreover, copies of the tables in Sections 4.A, 4.B, 4.C, 4.D, 6.A, 6.B, 7.A, and 7.B in the lecture notes will be attached to the exam sheet.
- You have 180 minutes to complete this exam.
- Unless explicitly stated otherwise, detailed derivations of the results are required for full credit on all **open problems**.
- For the **multiple choice with unique answer** questions, we give
  - +3 points if your answer is correct,
  - 0 points if your answer is incorrect.
- For the **multiple choice with multiple answers** questions, we give
  - +4 points for all correct answers,
  - +2 points for one incorrect answer and three correct answers,
  - 0 points for other possibilities of answers.

1.*(3 Pts)* Let the signal $y(t) = \int_{-\infty}^{t}(\delta(\tau+1) - \delta(\tau-1))d\tau$. Then **(check the unique answer!)**

1. $y(t) = 2$.
2. $y(t) = u(t+1) - u(t-1)$.
3. $y(t) = \delta(t+1) - \delta(t-1)$.
4. $y(t) = 2u(t)$.

Solution:

The correct answer is (2).

2. *(4 Pts)* Which of the following claims about properties of signals and systems are true? **(check all that apply!)**

1. The discrete-time signal $x[n] = u[n] - u[n-2]$ is an energy signal.
2. The discrete-time signal $z[n] = \cos(\frac{n}{2})$ has fundamental period 4π .
3. The continuous-time system $\mathcal{H}\{x(t)\} = x(t^2 - 1)$ is linear and time-invariant.
4. The continuous-time system $\mathcal{H}\{x(t)\} = \sqrt{|(x(t) - \mu)|}$ is memoryless and causal.

Solution: (1) is correct, since $x[n] = u[n] - u[n-2] = \delta[n] + \delta[n-1]$ is energy signal.

(2) is wrong because the the fundamental period has to be an integer for discrete-time signals.

(3) is wrong, since the system is linear but not time-invariant. (see HW 1, Problem 4 (c))

(4) is correct because the output only depends on the current input.

3. *(3 Pts)* The Basel problem asks for the sum of reciprocals of the squares of the nature numbers, i.e., $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The sum can be considered as half of the *energy* of a discrete time signal $x[n]$ such that

$$x[n] = \begin{cases} \dfrac{\cos(\pi n)}{jn}, & n \neq 0, \\ 0, & n = 0. \end{cases}$$

Which one of the following values is equal to the sum, $\sum_{n=1}^{\infty} \frac{1}{n^2}$? **(check the unique answer!)**

1. $\frac{\pi^2}{2}$.
2. $\frac{\pi^2}{3}$.
3. $\frac{\pi^2}{4}$.
4. $\frac{\pi^2}{6}$.

Solution: The correct answer is (4)

As has been hinted in the description of the problem, we should compute the energy of the given signal. However, it is not easy to compute it directly. Hence, we may use the Parseval's relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 d\omega.$$

Now we need to find the DTFT of $x[n]$, which is not easy to compute if we are not familiar with integral. But if we notice that the differentiation in frequency property of DTFT, we can use an auxiliary signal $y[n]$ such that

$$\begin{aligned} y[n] = nx[n] = &\begin{cases} \dfrac{\cos(\pi n)}{j}, & n \neq 0, \\ 0, & n = 0. \end{cases} \\ =&\frac{\cos(\pi n)}{j} - \frac{\delta[n]}{j} \\ =&\frac{1}{j}(e^{-j\pi n} - \delta[n]) \end{aligned}$$

Now, we can use the DTFT pair table to get the DTFT for both $e^{-j\pi n}$ and $\delta[n]$.

$$Y(e^{j\omega}) = \frac{1}{j}(2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi - 2\pi l) - 1)$$

Since $Y(e^{j\omega})$ is 2π-periodic, for $\omega \in (0, 2\pi]$, we have

$$Y(e^{j\omega}) = \frac{1}{j}(2\pi\delta(\omega - \pi) - 1)$$

According to the table, the $X(e^{j\omega})$ and $Y(e^{j\omega})$ satisfy

$$Y(e^{j\omega}) = j\frac{dX(e^{j\omega})}{d\omega} \tag{1}$$

As $X(e^{j\omega})$ is also 2π-periodic, for $\omega \in (0, 2\pi]$, we have

$$X(e^{j\omega}) = \frac{1}{j}\int Y(e^{j\omega})d\omega = w - 2\pi \int \delta(\omega - \pi)d\omega = \omega - 2\pi u(\omega - \pi).$$

Therefore, the energy of $x[n]$ is

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} |x[n]|^2 =&\frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega &(2) \\ =&\frac{1}{2\pi}\int_0^{\pi} |\omega|^2 d\omega + \int_{\pi}^{2\pi} |\omega - 2\pi|^2 d\omega &(3) \\ =&\frac{\pi^2}{3}. &(4) \end{aligned}$$

Recall the the sum is equal to half of the energy. Hence it is $\frac{\pi^2}{6}$.

4. *(4 Pts)* Consider a discrete-time LTI system with frequency response $H(e^{j\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$. Which of the following claims are true? **(check all that apply!)**

1. The input $x[n]$ and output $y[n]$ satisfy $y[n] + \frac{1}{2}y[n-1] = x[n]$.
2. If the input is $x[n] = e^{-j\frac{\pi}{2}n}$, then the output is $y[n] = \frac{e^{-j\frac{\pi}{2}n}}{1-\frac{j}{2}}$.
3. The system is causal and memoryless.
4. The system is stable.

Solution: (1) is correct.

(2) is wrong. The denominator should be $1 + \frac{j}{2}$.

(3) is wrong, because the system is not memoryless. The output $y[n]$ depends on the previous output $y[n-1]$.

(4) is correct.

5. *(3 Pts)* Let $X(\omega)$ denote the Fourier transform of $x(t)$. Then, the Fourier transform of $x_1(t-t_0) + x_2(-t)$ is **(check the unique answer!)**

1. $e^{-jt_0\omega} X_1(\omega) - X_2(-\omega)$.
2. $e^{jt_0\omega} X_1(\omega) + X_2(\omega)$.
3. $e^{jt_0\omega} X_1(\omega) - X_2(\omega)$.
4. $e^{-jt_0\omega} X_1(\omega) + X_2(-\omega)$.

Solution: (4) is correct.

6. *(3 Pts)* Let $x_n(t) = \frac{d^n}{dt^n} x(t)$ and $y_k(t) = \frac{d^k}{dt^k} y(t)$. Let $z(t) = (x * y)(t)$. Then, we have **(check the unique answer!)**

1. $(x_n * y_k)(t) = \frac{d^{nk}}{dt^{nk}} z(t)$.
2. $(x_n * y_k)(t) = \frac{d^{n+k}}{dt^{n+k}} z(t)$.
3. $(x_n * y_k)(t) = \frac{d^{\min(n,k)}}{dt^{\min(n,k)}} z(t)$.
4. $(x_n * y_k)(t) = \frac{d^{\max(n,k)}}{dt^{\max(n,k)}} z(t)$.

Solution:

(2) is correct

7. *(4 Pts)* A sampling system that samples continuous-time signals with a sampling frequency $\omega_s = 800\pi$ is applied to the signal

$$x(t) = e^{j\pi t}.$$

The result is the following discrete time signal: **(check all that apply!)**

1. $x[n] = 1$.
2. $x[n] = e^{j\frac{800\pi}{400}n}$.
3. $x[n] = e^{j\frac{\pi}{400}n}$.
4. $x[n] = e^{j\frac{801\pi}{400}n}$.

Solution: The correct answer is (3) and (4).

The sampled signal will have the form $x[n] = e^{j\pi T n}$ where T is the sampling interval. The answer follows by recalling that $T = \frac{2\pi}{\omega_s}$.

8. *(4 Pts)* A signal $x(t)$ is sampled with frequency $\omega_s = 1000\pi$ using the impulse-train sampling procedure covered in lecture, and then reconstructed with a low-pass filter with cut-off frequency $\omega_c = 500\pi$. The reconstructed signal is

$$x_r(t) = \sin(300\pi t).$$

We do not know anything else about $x(t)$. Which of the following signals could be $x(t)$? **(check all that apply!)**

1. $x(t) = \sin(200\pi t)$.
2. $x(t) = \sin(300\pi t)$.
3. $x(t) = \sin(1300\pi t)$.
4. $x(t) = \sin(700\pi t)$.

Solution:

The correct answer is (2) and (3).

(1) and (4) are wrong.

A similar question is covered in the problem set 8.

9.*(4 Pts)* Consider the continuous signals $x(t) = e^{-2t}u(t) - e^{-t}u(-t)$ and $y(t) = e^{t}u(t)$. Which of the following claims are true? **(check all that apply!)**

1. The signal $y(t)$ has a finite energy.
2. The signal $x(t)$ is not absolutely integrable.
3. Laplace transform of $x(t) + y(t)$ exists.
4. Laplace transform of $x(t)y(t)$ exists.

Solution: (1) is wrong because

$$\int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} e^{2t} dt = \infty. \tag{5}$$

(2) is correct because the region of convergence for the Laplace transform is $-2 < Re(s) < -1$, which do not include the imaginary axis. (Or you could just integrate the signal directly and see that because of the $e^{-t}u(-t)$ term it blows up)
(3) is wrong because the region of convergence of $x(t)$, $-2 < Re(s) < -1$ and the region of convergence of $y(t)$, $Re(s) > 1$ do not intersect.
(4) is correct because $x(t)y(t) = e^{-t}u(t)$, and the Laplace transform of this signal exists.

10. *(3 Pts)* Let the Laplace transform of the signal $x(t)$ be

$$X(s) = \frac{1}{s^2 + s}, \tag{6}$$

with the region of convergence $Re(s) > 0$. Then, the corresponding signal $x(t)$ would be, **(check the unique answer!)**

1. $x(t) = (t + e^{t})u(t)$.

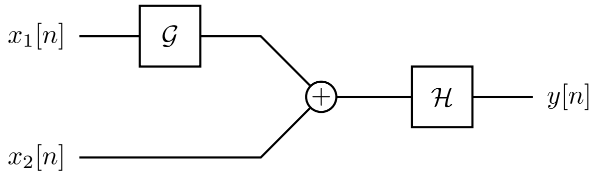
2. $x(t) = (t - 1 + e^{-t})u(t)$.

3. $x(t) = (1 - e^{-t})u(t)$.

4. $x(t) = (t + e^{-t})u(t)$.

Solution: The correct answer should be (3).

11. *(3 Pts)* Consider the following system composite with two input signal $x_1[n]$, $x_2[n]$ and output signal $y[n]$.



Let the transfer functions of LTI systems $\mathcal{G}$ and $\mathcal{H}$ be $G(z) = \frac{4}{1+2z^{-1}}$ and $H(z) = \frac{1}{1+z^{-1}}$, respectively. Find the output $y[n]$ when the inputs $x_1[n] = 2^n, \forall n$ and $x_2[n] = 3^n, \forall n$ are applied to the system. **(check the unique answer!)**

1. $y[n] = \frac{1}{5}6^n, \forall n$.

2. $y[n] = \frac{3}{4}2^n + \frac{4}{3}3^n, \forall n$.

3. $y[n] = \frac{4}{3}2^n + \frac{3}{4}3^n, \forall n$.

4. $y[n] = \frac{5}{6}6^n, \forall n$.

Solution: Observe that the inputs behave as eigenfunctions, thus the output can be expressed as $y[n] = H(3)3^n + G(2)H(2)2^n = \frac{4}{3}2^n + \frac{3}{4}3^n$, $\forall n$. (3) is the correct answer.

12. *(4 Pts)* Suppose that a causal and stable LTI system has a rational transfer function $H(s)$ and that $H(s)$ has exactly two poles and four zeros. Which of the following statements are true? **(check all that apply!)**

1. If $G(s)$ is the transfer function of an inverse of $\mathcal{H}$, it has exactly 4 poles.

2. $\lim_{s\to\infty} H(s) = 0$.

3. If $G(s)$ is the transfer function of an inverse of $\mathcal{H}$, $G(s) = \frac{1}{H(s)}$.

4. $H(s)$ must always have a stable inverse system.

Solution: A similar question is covered in the homework.
(1) is true because the poles of $G(s)$ are the zeros of $H(s)$.
(2) is wrong because $H(s)$ has more zeros than poles, and since it is a rational function then $\lim_{s\to\infty} H(s) = \infty$.
(3) is true because of the inverse property.
(4) is wrong because the zeros of $H(s)$ may lie on the imaginary axis and then no stable inverse system exists.

Problem 1 *(LTI System)* 12 points

Consider an LTI system with an input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 5) d\tau$$

(a) (4 Pts) What is the impulse response $h(t)$ of the system?

Solution: The impulse response is given by

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau - 5) d\tau = \int_{-\infty}^{t} e^{-(t-5)} \delta(\tau - 5) d\tau = \begin{cases} e^{5-t}, & t > 5 \\ 0, & t < 5. \end{cases} = e^{5-t} u(t-5)$$

Grading notes:

- one point for demonstrating the knowledge of the definition of impulse response (eg setting up the equation for $h(t)$ by plugging in $\delta(t)$.
- two points for the exact correct answer (one point only if the case $t < 5$ is not considered)

Note that there is another correct way to attempt this problem and this is by making the input-output equation look like a convolution integral and deducing the value of $h(t)$ from this. If the student attempts do this (not very likely) then

- one point for recognizing this approach and trying to make the integral look like a convolution integral (through change of variables, etc)
- two points for the exact correct answer (one point only if the case $t < 5$ is not considered)

(b) (3 Pts) Is the system causal? Justify your answer.

Solution: Yes, the system is causal since $h(t) = 0$ for $t < 0$ and for LTI systems this is equivalent to causality.

It is also correct to use the definition of causality and observe that in the input-output equation the output at time t depends only on the value of the signal in the past.

Grading notes:

- one point for correct answer
- one point for partially correct reasoning, two points for correct reasoning

(c) (5 Pts) What is the output $y(t)$ of the system when the input

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

is applied?

The output is computed by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 5) d\tau$$

Since $x_1(t-5) = 1$ for $5 \le t \le 6$ we get that for $t \ge 6$.

$$y(t) = \int_{5}^{6} e^{-(t-\tau)} d\tau = e^{-t} \int_{5}^{6} e^{\tau} d\tau = e^{-t} [e^{\tau}]_5^2 6 = e^{6-t} - e^{5-t}.$$

For $5 \leq t < 6$ the output is computed by

$$y(t) = \int_5^t e^{-(t-\tau)} d\tau = e^{-t}[e^\tau]_5^t = 1 - e^{5-t}.$$

The output is zero for $t < 5$ and thus

$$y(t) = \begin{cases} e^{6-t} - e^{5-t}, & t \geq 6 \\ 1 - e^{5-t}, & 5 \leq t < 6 \\ 0, & \text{otherwise.} \end{cases}$$

Grading notes:

- two points for setting up the equation correctly (including recognizing that there are three cases) / one point if this is partially correct
- one point for computing each of the three cases correctly

Problem 2 *(System Properties)* 12 points

Determine if each system being described is stable or not. Justify your answer.

(a) (3 Pts) An LTI system with impulse response $h(t) = (1 + e^{-t+1})u(t-1)$

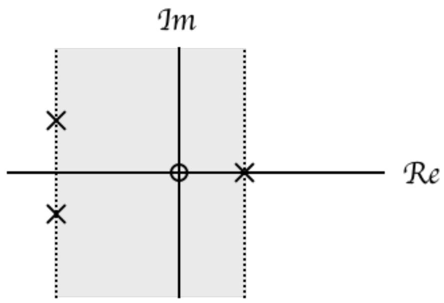
Solution: A CT LTI system is stable if and only if its impulse response is absolutely integrable. This system is **not stable** since

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau = \infty.$$

(b) (3 Pts) A system that given an input $x(t) = \cos(2t)$ produces an output $y(t) = e^{(2j-1)t}$

Solution: Note that $x(t)$ is bounded while $y(t)$ is not since as t goes to $-\infty$, $y(t)$ oscillates between negative and positive infinity. This is an example of a bounded input producing unbounded output and so this system is **not stable**.

(c) (3 Pts) A continuous-time LTI system with transfer function $H(s)$ that has the following pole-zero plot and ROC



Solution: The Transfer function contains the imaginary axes and so the system is **stable**

(d) (3 Pts) Two stable LTI systems (continuous or discrete-time) $\mathcal{G}$ and $\mathcal{H}$ connected in series

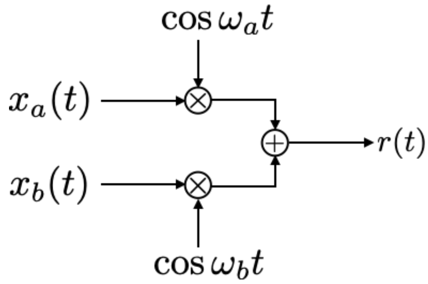
Solution: The overall system is **stable**. There are a number of ways to show this. For example, the ROC for the transfer functions $H(s)$ and $G(s)$ will each contain the imaginary axes, and so their intersection (the ROC of $H(s)G(s)$) must also contain the imaginary axes.

Grading notes (parts (a) - (d)):

- one point for exact correct answer
- one point for partially correct reasoning/ two points for correct reasoning

Problem 3 *(Communication System)* 12 points

In this problem we will perform a simple analysis of an LTI communication system depicted in the figure below.



(a) (4 Pts) What is the Fourier transforms $R(\omega)$ of the carrier signal

$$r(t) = x_a(t)\cos\omega_a t + x_b(t)\cos\omega_b t \ ?$$

You answers should be in terms of $X_a(\omega)$ and $X_b(\omega)$.

Solution: This is an application of the convolution in time property from Appendix 4.A and the cosine Fourier pair from Appendix 4.B:

$$R(\omega) = \frac{1}{2}\left(X_a(\omega-\omega_a) + X_a(\omega+\omega_a) + X_b(\omega-\omega_b) + X_a(\omega+\omega_b)\right)$$

Grading notes:

- one point for demonstrating the knowledge of the convolution in time property
- one point for demonstrating the knowledge for the Fourier transform of cosine
- two points for the exact correct answer

(b) (4 Pts) Suppose that $x_a(t)$ and $x_b(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_a(\omega) = 0, \quad |\omega| \geq \omega_M \quad \text{and} \quad X_b(\omega) = 0, \quad |\omega| \geq \omega_M.$$

A reconstruction system for $x_a(t)$ is proposed below where

$$H_1(\omega) = \begin{cases} 1, & \omega_a - \omega_M < |\omega| < \omega_a + \omega_M \\ 0, & \text{otherwise,} \end{cases}$$

and

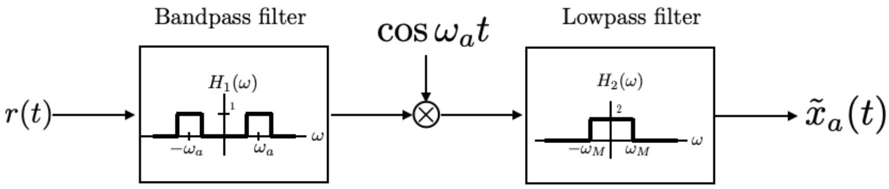
$$H_2(\omega) = \begin{cases} 2, & |\omega| < \omega_M \\ 0, & \text{otherwise.} \end{cases}$$

(A similar system could also be designed to reconstruct $x_b(t)$, by replacing ω_a with ω_b.)

What conditions must ω_a and ω_b satisfy in order to guarantee that $\tilde{x}_a(t) = x_a(t)$?
Assume that $\omega_a < \omega_b$.

Solution:

We can see from part (a) that the modulation procedure creates two copies of $X_a(\omega)$, one centered at ω_a and one centered at $-\omega_a$. Likewise, it creates two copies of $X_b(\omega)$, one centered at ω_b and one centered



at $-\omega_b$. In order for the reconstruction system to recover $x_a(t)$ the Fourier transform $X_a(\omega-\omega_a)$ cannot interfere with the $X_b(\omega-\omega_b)$, the Fourier transform $X_a(\omega+\omega_a)$ cannot interfere with the $X_b(\omega+\omega_b)$, and the Fourier transform $X_a(\omega-\omega_a)$ cannot interfere with $X_a(\omega-\omega_a)$. This leads to two conditions: (1) $\omega_b-\omega_a > 2\omega_M$ and (2) $\omega_a > \omega_M$.

(c) (4 Pts) Finally, suppose that $x_a(t)$ and $x_b(t)$ are both assumed to be band limited continuous-time signals with Fourier transforms that satisfy

$$X_a(\omega)=0, \quad |\omega| \geq \omega_M \qquad \text{and} \qquad X_b(\omega)=0, \quad |\omega| \geq 2\omega_M.$$

That is, $x_b(t)$ has twice as much bandwidth as $x_a(t)$.

What conditions must ω_a and ω_b satisfy now in order to guarantee that $\tilde{x}_a(t)=x_a(t)$?

Solution:

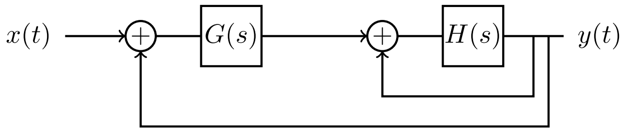
Using the same reasoning as in part (b) we see that the conditions are: (1) $\omega_b-\omega_a > 3\omega_M$ and (2) $\omega_a > \omega_M$.

Grading notes (parts (b) and (c)):

- one or two points for showing partially correct reasoning/ three points for correct reasoning (including no extraneous conditions)
- two points for stating condition (1) in part (b)
- two points for stating condition (1) in part (c)
- one point for stating condition (2) (either in part (b) or part (c))

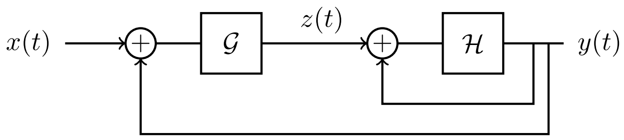
Problem 4 *(Feedback)* 10pts

Consider the interconnection of continuous-time casual LTI systems shown in figure below.



(a) (5 Pts) Express the overall system function for this interconnection in terms of $G(s)$ and $H(s)$.

Solution:
Lets us introduce the auxiliary signal $z(t)$ as shown in the figure.



Erixhen seems to have mixed up G and H here

Then, we can deduce the following identities in the Laplace domain

$$Z(s) = H(s)[X(s) + Y(s)] \tag{7}$$
$$Y(s) = G(s)[Z(s) + Y(s)]. \tag{8}$$

Putting (7) into (8) we get

$$Y(s) = G(s)[H(s)[X(s) + Y(s)] + Y(s)] \tag{9}$$
$$\Rightarrow Y(s)[1 - H(s)G(s) - G(s)] = H(s)G(s)X(s). \tag{10}$$

Thus, the overall transfer function is

$$H_{\text{all}}(s) = \frac{Y(s)}{X(s)} = \frac{H(s)G(s)}{1 - H(s)G(s) - G(s)}. \tag{11}$$

(b) (5 Pts) Let $H(s) = \frac{1}{s+2}$ and $G(s) = 1$. Given the input $x(t) = e^{-2t}u(t)$, find the output $y(t)$.

Solution:
Substituting for $H(s)$ and $G(s)$ we get

$$H_{\text{all}}(s) = \frac{\frac{1}{s+2}}{1 - \frac{2}{s+2}} = \frac{1}{s}. \tag{12}$$

In addition, $X(s) = \frac{1}{s+2}$. So the transfer function of the output would be

$$Y(s) = H_{\text{all}}(s)X(s) \tag{13}$$
$$= \frac{1}{s(s+2)} \tag{14}$$
$$= \frac{A}{s} + \frac{B}{s+2}. \tag{15}$$

By partial fraction decomposition we get $A = \frac{1}{2}, B = -\frac{1}{2}$, thus $y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$.

Problem 5 *(Sampling)* 12 points

Recall that in lecture, we derived the sampling theorem by assuming that $x(t)$ was *band limited.* That is, its Fourier transform satisfies

$$X(\omega) = 0, \quad |\omega| \geq \omega_h.$$

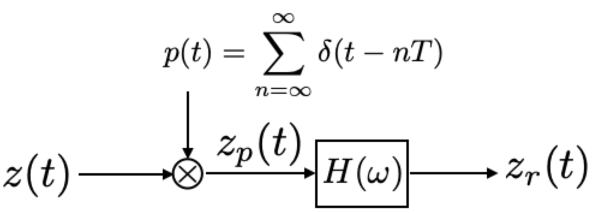
In this problem we study the sampling of a continuous time signal $z(t)$ such that its Fourier transform satisfies

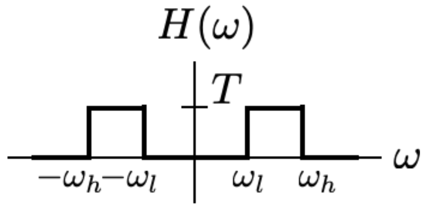
$$Z(\omega) = 0, \quad |\omega| \geq \omega_h \text{ or } |\omega| \leq \omega_l.$$

A signal like this whose energy is concentrated in an energy band is often referred to as a *bandpass signal.*

(a) (4 Pts) Using the sampling theorem, as derived in class, find the Nyquist rate of the signal $z(t)$. In other words, find smallest sampling frequency ω_s needed to reconstruct $z(t)$ exactly from its samples. What is the corresponding sampling interval T?

After graduating from EPFL you get a job at an exciting new startup that promises to design and build the first ever artificially intelligent android (Congratulations!). Your colleague notices that all of the signals sampled by a certain audio circuit are bandpass signals, just like $z(t)$. She proposes a new sampling system (depicted below) which she claims should work better for bandpass signals.





That is, the sampling operation is modeled by an impulse train and is the same as before. However, the reconstruction is now performed by a bandpass filter, rather than a low-pass filter.

(b)(3 Pts) If you use the same sampling frequency as in part (a) with this new reconstruction procedure, does $z_r(t) = z(t)$? Explain why or why not.

Hint: You may find it helpful to recall that in lecture we derived $Z_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Z(\omega - \omega_s)$ *where* ω_s *is the sampling frequency.*

(c) (5 Pts) Assuming that $\omega_h = 2\omega_l$, find the smallest sampling frequency ω_s and the largest sampling interval T such that $z_r(t) = z(t)$ for the new proposed sampling system. How does this compare to part (a)?

Solution:
(a) The Nyquist rate is twice the highest frequency for which the signal is non-zero. For the bandpass signal this is ω_h and so the Nyquist rate is $2\omega_h$. According to the sampling theorem $\omega_s > 2\omega_h$ would be sufficient. The corresponding sampling interval is $T = \frac{2\pi}{\omega_s} < \frac{\pi}{\omega_h}$.

Grading notes:

- two points for correct Nyquist rate
- two points for correct sampling interval

(b) Yes, this new reconstruction procedure will reconstruct $z(t)$ because it is a bandpass signal. That is, the filter zeros out frequencies between $-\omega_l$ and ω_l, but for $z(t)$ these are zero anyway.

Grading notes:

- one point for correctly identifying that $z(t) = z_r(t)$
- two points for correct reasoning

(c) The smallest sampling frequency is $\omega_s = \omega_h$. This is because of the bandlimited assumption, the aliasing that will happen between $-\omega_l$ and ω_l will be removed by the bandpass signal. The sampling interval is $T = \frac{2\pi}{\omega_h}$.

With the extra assumption of band pass signal, the sampling frequency is smaller, and the sampling interval is larger. This system needs to take fewer samples than the one in part (a).

Grading notes:

- two points for correctly identifying $\omega_s = \omega_h$, one point for any other reasonable suggestion
- one point for correct T
- one point for correct reasoning
- one point for recognizing that this system is better for bandpass signals

Question 17 - Correction

The hint for part (b) should say:

Hint: You may find it helpful to recall that in lecture we derived $Z_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Z(\omega - \omega_s)$ *where* ω_s *is the sampling frequency.*