Homework 6: Solution
Introduction to Quantum Information Processing

**Exercise 1** *Entanglement by unitary operations*

1) By definition of the tensor product:

$$(H \otimes I) |x\rangle \otimes |y\rangle = H |x\rangle \otimes I |y\rangle = H |x\rangle \otimes |y\rangle .$$

Also, one can use that $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to show that always

$$H |x\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^x |1\rangle).$$

Thus,

$$(H \otimes I) |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y\rangle).$$

Now we apply CNOT. By linearity, we can apply it to each term separately. Thus,

$$\begin{aligned} (\mathrm{CNOT})(H \otimes I) |x\rangle \otimes |y\rangle &= \frac{1}{\sqrt{2}} ((CNOT) |0\rangle \otimes |y\rangle + (-1)^x (\mathrm{CNOT}) |1\rangle \otimes |y\rangle) \\ &= \frac{1}{\sqrt{2}} (|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |y \oplus 1\rangle) \\ &= |B_{xy}\rangle . \end{aligned}$$

2) Let us first start with $H \otimes I$. We use the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{pmatrix},$$

Thus we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

For CNOT, we use the definition:

$$(\mathrm{CNOT}) |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus x\rangle ,$$

which implies that the matrix elements are

$$\langle x'y'| \,\mathrm{CNOT}\, |xy\rangle = \langle x', y'|x, y \otimes x\rangle = \langle x'|x\rangle \langle y'|y \oplus x\rangle = \delta_{xx'} \delta_{y \oplus x, y'} .$$

We obtain the following table with columns $xy$ and rows $x'y'$:

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| 00 | 1 | 0 | 0 | 0 |
| 01 | 0 | 1 | 0 | 0 |
| 10 | 0 | 0 | 0 | 1 |
| 11 | 0 | 0 | 1 | 0 |

For the matrix product (CNOT)$(H \otimes I)$, we find that

$$\begin{aligned}(\text{CNOT})H \otimes I &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ X & -X \end{pmatrix},\end{aligned}$$

where $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus,

$$(CNOT)(H \otimes I) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

One can check that for example $|B_{00}\rangle = (\text{CNOT})(H \otimes I) |0\rangle \otimes |0\rangle$. Finally to check the unitarity, we have to check that $UU^\dagger = U^\dagger U = I$ for $U = H \otimes I$, CNOT and (CNOT)$(H \otimes I)$. We leave this to the reader.

3) Let $U = (\text{CNOT})(H \otimes I)$. We have

$$|B_{xy}\rangle = U |x\rangle \otimes |y\rangle \,, \langle B_{x'y'}| = \langle x'| \otimes \langle y'| U^\dagger.$$

Thus,

$$\begin{aligned}\langle B_{x'y'}|B_{xy}\rangle &= \langle x'| \otimes \langle y'| U^\dagger U |x\rangle \otimes |y\rangle \\ &= \langle x'| \otimes \langle y'| I |x\rangle \otimes |y\rangle \\ &= \langle x'|x\rangle \langle y'|y\rangle = \delta_{xx'}\delta_{yy'}.\end{aligned}$$

**Exercise 2** *Tsirelson bound*

1) Since the eigenvalues of $A$ are $\pm 1$, those of $A^2$ are both $+1$. The eigenvectors are the same. Thus we have $A^2 = |\alpha\rangle\langle\alpha| + |\alpha_\perp\rangle\langle\alpha_\perp| = I_A$ and similarly for the other matrices. The same result can also be obtained by expanding the product $A^2$ in Dirac notation or in matrix component form...

   Now expanding $\mathcal{B}^2$ we get the squared terms of the type $(A \otimes B)^2 = (A \otimes B)(A \otimes B) = A^2 \otimes B^2 = I_A \otimes I_B$. This will yield the term $4 I_A \otimes I_B$.

   Then for the cross terms using $(M_1 \otimes M_2)(N_1 \otimes N_2) = M_1 N_1 \otimes M_2 N_2$ we end up with 4 contributions that dont cancel each other and can be rearranged into the commutator term $[A, A'] \otimes [B, B']$.

2) From the identity we have

$$\begin{aligned}\langle\Psi|\mathcal{B}^2|\Psi\rangle &= 4 - \langle\Psi|[A,A']\otimes[B,B']|\Psi\rangle \\ &\leq 4 + |\langle\Psi|[A,A']\otimes[B,B']|\Psi\rangle|\end{aligned}$$

Now we must prove that the last term on the right-hand-side is less or equal to 4. Using the hint and Cauchy-Schwarz

$$|\langle\Psi|[A,A']\otimes[B,B']|\Psi\rangle| \leq \|[A,A']\otimes I_B|\Psi\rangle\|\|I_A\otimes[B,B']\Psi\rangle\|$$

To estimate each norm on the right-hand-side we first use the triangle inequality

$$\|[A,A']\otimes I_B|\Psi\rangle\| \leq \|AA'\otimes I_B|\Psi\rangle\| + \|A'A\otimes I_B|\Psi\rangle\|$$

and then

$$\begin{aligned}\|AA'\otimes I_B|\Psi\rangle\|^2 &= \langle\Psi|(AA'\otimes I_B)^\dagger(AA'\otimes I_B)|\Psi\rangle \\ &= \langle\Psi|(A'A\otimes I_B)(AA'\otimes I_B)|\Psi\rangle \\ &= \langle\Psi|A'A^2A'\otimes I_B^2|\Psi\rangle \\ &= \langle\Psi|A'^2\otimes I_B|\Psi\rangle \\ &= \langle\Psi|I_A\otimes I_B|\Psi\rangle = 1 \end{aligned} \tag{1}$$

From which we deduce that the right-hand-side of the last inequality above equals $1+1=2$. Thus

$$\|[A,A']\otimes I_B|\Psi\rangle\| \leq 2$$

and

$$|\langle\Psi|[A,A']\otimes[B,B']|\Psi\rangle| \leq 4$$

which completes the proof.

3) This last result can be justified from Cauchy-Schwarz:

$$\langle\Psi|\mathcal{B}|\Psi\rangle^2 \leq \||\Psi\rangle\|^2\|\mathcal{B}|\Psi\rangle\|^2 = \langle\Psi|\mathcal{B}^2|\Psi\rangle$$

Since we proved before that teh r.h.s is less or equal to 8 we get Tsirelson's bound for any state

$$\langle\Psi|\mathcal{B}|\Psi\rangle \leq 2\sqrt{2}$$

Note that the Bell state saturates the bound.

**Exercise 3** *Entanglement swapping*

**(a)** The projector is

$$P = |GHZ\rangle_{135}\langle GHZ|_{135}\otimes I_2\otimes I_4\otimes_6$$

since particles with an even index are not observed (measured).

**(b)** After the measurement the global state is

$$P|\Psi\rangle$$

which gives for the state of particles 246 (after normalization)

$$|GHZ\rangle_{246}$$

Indeed the projector imposes that 135 are in states 000 or 111. But since $|Psi\rangle$ is a product of $|B_{00}\rangle$'s particles 246 must be in the same states as 135.