Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 8 – Solutions

Exercise 1:

(a) *Sufficient conditions for properness*: Let X and Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Prove the following assertions:

  (i) If X is compact and Y is Hausdorff, then F is proper.

  (ii) If F is a topological embedding with closed image, then F is proper.

  (iii) If Y is Hausdorff and F has a continuous *left inverse*, i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$, then F is proper.

(b) Let M be a smooth manifold and let S be an embedded submanifold of M. Show that S is properly embedded if and only if S is a closed subset of M.

(c) *Slice of the product manifold*: If M and N are smooth manifolds, then for each $q \in N$ the subset $M \times \{q\}$, called a *slice of the product manifold*, is an embedded submanifold of $M \times N$ diffeomorphic to M.

(d) *Global graphs are properly embedded*: Let $f\colon M \to N$ be a smooth map between smooth manifolds. Show that the graph $\Gamma(f)$ of f is a properly embedded submanifold of $M \times N$.

Solution:

(a) We deal with the three cases below separately:

  (i) Let K be a compact subset of Y. Since Y is Hausdorff, K is a closed subset of Y. Since F is continuous, $F^{-1}(K)$ is a closed subset of X, and now since X is compact, $F^{-1}(K)$ is also compact, as desired. Therefore, F is a proper map.

  (ii) Let K be a compact subset of Y. By assumption, $F(X)$ is a closed subset of X, so $F(X) \cap K$ is a closed subset of K, and thus compact. Since $F^{-1}\colon F(X) \to X$ is continuous and bijective by assumption and since $F(X) \cap K \subseteq F(X)$, the image $F^{-1}\big(F(X) \cap K\big) = F^{-1}(K)$ is a compact subset of X, as desired.

(iii) Let K be a compact subset of Y. On the one hand, since G is continuous, $G(K)$ is a compact subset of X. On the other hand, since Y is Hausdorff, K is a closed subset of Y, and since F is continuous, $F^{-1}(K)$ is a closed subset of X. Now, we claim that $F^{-1}(K) \subseteq G(K)$, which implies that $F^{-1}(K)$ is compact, as desired. Indeed, given $s \in F^{-1}(K)$, we have $F(s) = t \in K$, so

$$s = \mathrm{Id}_X(s) = (G \circ F)(s) = G(t) \in G(K),$$

which proves the claim, and completes thus the proof of (iii).

(b) Assume first that S is a properly embedded submanifold of M. Then the inclusion map $\iota\colon S \hookrightarrow M$ is proper by definition, and hence closed by *Claim 3* in the proof of *Proposition 4.6*. Since ι is clearly a topological embedding, we deduce that S is a closed subset of M.

Assume now that S is a closed subset of M. Since then the inclusion map $\iota\colon S \hookrightarrow M$ is a topological embedding with closed image $\iota(S) = S$, it follows from (a)(ii) that ι is proper, and thus S is a properly embedded submanifold of M.

(c) The assertion follows immediately from *Example 5.4* by considering the constant function

$$f\colon M \to N,\ x \mapsto q \in N,$$

which is smooth by part (b) of [*Exercise Sheet* 3*, Exercise* 3].

(d) By *Example 5.4* we know that the map

$$\gamma_f\colon M \to M \times N,\ x \mapsto \big(x, f(x)\big)$$

is a smooth embedding with image $\Gamma(f)$ and that the projection

$$\pi_M\colon M \times N \to M,\ (x, y) \mapsto x$$

is a smooth left inverse for γ_f, i.e., $\pi_M \circ \gamma_f = \mathrm{Id}_M$. It follows from (a)(iii) that γ_f is proper, hence closed by *Claim 3* in the proof of *Proposition 4.6*. Therefore, $\Gamma(f)$ is a closed subset of $M \times N$, so *Example 5.4* together with (b) imply that $\Gamma(f)$ is a properly embedded submanifold of $M \times N$.

Exercise 2: Fix $n \geq 0$. Using

(i) the local slice criterion, and

(ii) the regular level set theorem,

show that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$.

Solution: We first show that the unit n-sphere $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$ using the local slice criterion. To this end, recall that $\mathbb{S}^n$ is locally the graph of a smooth function; indeed, by *Example 1.3*(2) we already know that each point of $\mathbb{S}^n$ belongs to one of the sets $U_i^{\pm} \cap \mathbb{S}^n$ and that $U_i^{+} \cap \mathbb{S}^n$ is the graph of

$$x^i = f\big(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\big)$$

and $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f\big(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\big),$$

where f is the smooth function

$$f\colon \mathbb{B}^n \to \mathbb{R},\ u \mapsto \sqrt{1 - \|u\|^2}.$$

It follows now from *Example 5.4* and *Theorem 5.6* that $\mathbb{S}^n$ satisfies the local n-slice condition, and hence it is an embedded submanifold of $\mathbb{R}^{n+1}$ again by *Theorem 5.6*.

We now show that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$ using the regular level set theorem. To this end, consider the smooth function

$$f\colon \mathbb{R}^{n+1} \to \mathbb{R},\ x = (x^1, \ldots, x^{n+1}) \mapsto \|x\|^2 - 1 = \sum_{i=1}^{n+1} (x^i)^2 - 1$$

and note that

$$\mathbb{S}^n = f^{-1}(0).$$

The gradient of f is given at an arbitrary point $x = (x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1}$ by

$$\operatorname{grad}(f)(x^1, \ldots, x^{n+1}) = \big(2x^1, \ldots, 2x^{n+1}\big).$$

Since $\operatorname{grad}(f)$ vanishes only at the point $0 = (0, \ldots, 0) \in \mathbb{R}^{n+1}$, which clearly does not belong to $\mathbb{S}^n$, it follows from *Corollary 5.10* that $\mathbb{S}^n = f^{-1}(0)$ is a properly embedded submanifold of $\mathbb{R}^{n+1}$.

Remark.

(1) It follows from *Exercise* 1(b) and *Exercise* 2 (or [*Exercise Sheet* 6, *Exercise* 3(*a*)]) that $\mathbb{S}^n$ is a properly embedded submanifold of $\mathbb{R}^{n+1}$.

(2) One can check that the coordinates for $\mathbb{S}^n$ determined by the slice charts described in *Exercise* 2(i) are precisely the graph coordinates defined in *Example 1.3*(2).

Exercise 3:

(a) Consider the smooth function

$$f\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^3 + xy + y^3.$$

Show that if $c \in \mathbb{R} \setminus \{0, \frac{1}{27}\}$, then the level set $f^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Given $c \in \mathbb{R}$, examine whether the level set $\Phi^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

Solution:

(a) The gradient of f at an arbitrary point $(x, y) \in \mathbb{R}^2$ is given by

$$\operatorname{grad}(f)(x, y) = \left(\frac{\partial f}{\partial x}(x, y),\ \frac{\partial f}{\partial y}(x, y) \right) = \big(3x^2 + y,\ 3y^2 + x\big).$$

It is now easy to check that

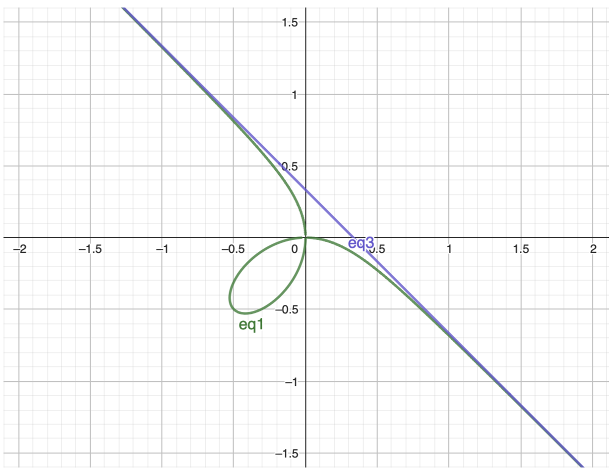
$$\operatorname{grad}(f)(x, y) = (0, 0) \ \text{ if and only if } \ (x, y) \in \left\{ (0, 0),\ \left(-\frac{1}{3}, -\frac{1}{3}\right) \right\}.$$

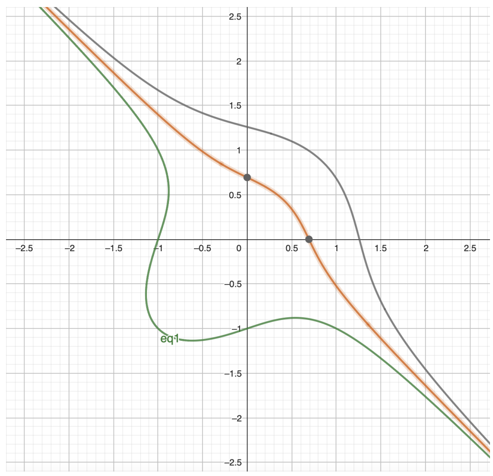
Since

$$f(0, 0) = 0 \quad \text{ and } \quad f\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}$$

and since the fibers of f are disjoint, we conclude that $(0, 0)$ belongs exclusively to the level set $f^{-1}(0)$ and that $\big(-\frac{1}{3}, -\frac{1}{3}\big)$ belongs exclusively to the level set $f^{-1}\big(\frac{1}{27}\big)$. Hence, if $c \in \mathbb{R} \setminus \{0, \frac{1}{27}\}$, then the fiber $f^{-1}(c)$ is a regular level set, and thus a properly embedded submanifold of $\mathbb{R}^2$ by *Corollary 5.10*.

We have plotted in the left figure below the level sets $f^{-1}(0)$ (in green) and $f^{-1}\big(\frac{1}{27}\big)$ (in purple), while in the right one the level sets $f^{-1}(-1)$ (in green), $f^{-1}\big(\frac{1}{3}\big)$ (in orange) and $f^{-1}(2)$ (in grey).





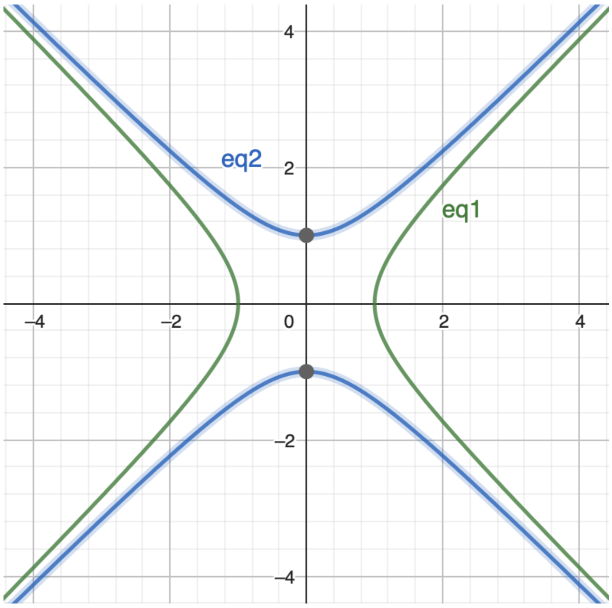
(b) The gradient of Φ at an arbitrary point $(x, y) \in \mathbb{R}^2$ is given by

$$\operatorname{grad}(\Phi)(x, y) = \left(\frac{\partial \Phi}{\partial x}(x, y),\ \frac{\partial \Phi}{\partial y}(x, y) \right) = \big(2x, -2y\big)$$

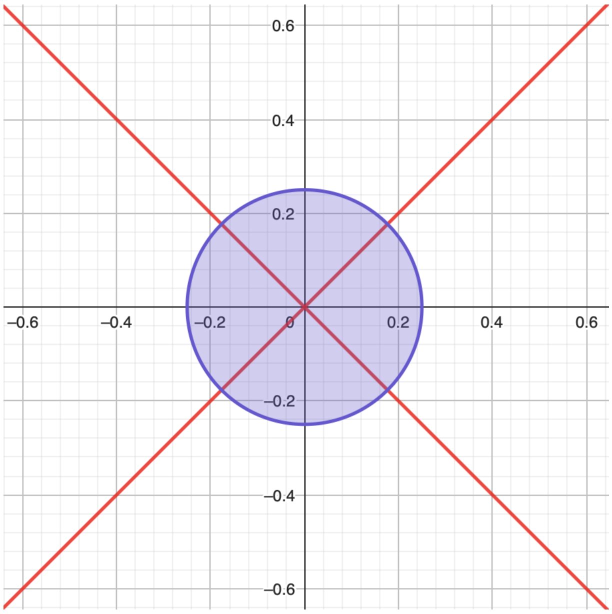
and it is obvious that

$$\operatorname{grad}(\Phi)(x, y) = (0, 0) \ \text{ if and only if } \ (x, y) = (0, 0).$$

As in (a), we conclude that if $c \neq 0$, then the level set $\Phi^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$ according to *Corollary 5.10*. We have plotted below the level sets $\Phi^{-1}(1)$ (in green) and $\Phi^{-1}(-1)$ (in blue).



We now deal with the remaining case $c = 0$. Since $\text{grad}(\Phi)(0,0) = (0,0)$, $c = 0$ is a critical value of Φ, so *Corollary 5.10* cannot be applied; we stress that it does *not* tell us that the level set $\Phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$ either. To examine whether this is true or not, we proceed as follows.



We observe that the level set $\Phi^{-1}(0)$ (plotted above in red) is the union of the lines $y = x$ and $y = -x$ in the plane $\mathbb{R}^2$. By arguing as in [*Exercise Sheet* 1, *Exercise* 4] (for the point $(0,0) \in \Phi^{-1}(0)$), we infer that $\Phi^{-1}(0)$ is not a topological manifold (with the subspace topology inherited from $\mathbb{R}^2$), and hence it cannot be an embedded submanifold of $\mathbb{R}^2$.

Exercise 4: Let S be a subset of a smooth m-manifold M. Show that S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi\colon U \to \mathbb{R}^{m-k}$.

[Hint: Use the local slice criterion.]

Solution: Assume that S is an embedded k-submanifold of M. Then S satisfies the local k-slice criterion by *Theorem 5.6*. Given $p \in S$, if $(x^1, \dots, x^m)$ are slice coordinates for S in an open neighborhood U of p in M, then there are constants $c^{k+1}, \dots, c^m \in \mathbb{R}$ such that (in coordinates we have)

$$U \cap S = \big\{(x^1, \dots, x^m) \in U \mid x^{k+1} = c^{k+1}, \dots, x^m = c^m\big\}.$$

Moreover, the map $\Phi \colon U \to \mathbb{R}^{m-k}$ given in coordinates by

$$\Phi(x^1, \dots, x^m) = (x^{k+1}, \dots, x^m)$$

is a smooth submersion, since its Jacobian is the $(m-k) \times m$-matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

of rank $m-k$, and clearly we have

$$U \cap S = \Phi^{-1}\big(c^{k+1}, \dots, c^m\big).$$

In conclusion, every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi \colon U \to \mathbb{R}^{m-k}$.

Conversely, assume that every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion $\Phi \colon U \to \mathbb{R}^{m-k}$. By *Corollary 5.10*, $U \cap S$ is a properly embedded k-submanifold of U, so it satisfies the local k-slice criterion by *Theorem 5.6*. Therefore, S itself satisfies the local k-slice criterion, and hence it is an embedded k-submanifold of M by *Theorem 5.6*.

Exercise 5:

(a) *Restricting the domain of a smooth map*: If $F \colon M \to N$ is a smooth map and if $S \subseteq M$ is an immersed or embedded submanifold, then the restriction $F|_S \colon S \to N$ is smooth.

(b) *Restricting the codomain of a smooth map*: Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold, and let $G \colon N \to M$ be a smooth map whose image is contained in S. If G is a continuous map from N to S, then $G \colon N \to S$ is smooth.

(c) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Then every smooth map $G \colon N \to M$ whose image is contained in S is also smooth as a map from N to S.

Solution:

(a) The inclusion map $\iota \colon S \to M$ is smooth for both immersed and embedded submanifolds. Hence, the restriction $F|_S = F \circ \iota$ is smooth as well.

(b) Let $p \in M$ and set $q = G(p) \in M$. To prove the smoothness of the corestriction $G|^S\colon N \to S$, we need to find charts of N and S containing p and q, respectively, such that the corresponding coordinate representation of $G|^S$ is smooth. As immersed submanifolds are locally embedded by *Proposition 5.17*, there exists a neighborhood V of q in S such that $\iota_V\colon V \hookrightarrow M$ is a smooth embedding. Thus, there exists a smooth chart (W, ψ) of M containing q which is a slice chart for V (note that it could very well be that $W \cap V \subsetneq W \cap S$, i.e., (W, ψ) might not be a slice chart for S). The fact that (W, ψ) is a slice chart means that $\big(V_0, \widetilde{\psi}\big)$ is a smooth chart for V, where $V_0 = V \cap W$ and $\widetilde{\psi} = \pi \circ \psi$, with $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first $k = \dim S$ coordinates. Since $V_0 = \iota_V^{-1}(W)$ is open in V by continuity of ι_V, it is open in S in its given topology. Hence, $\big(V_0, \widetilde{\psi}\big)$ is also a smooth chart for S. Set $U \coloneqq G^{-1}(V_0)$ and note that U is an open subset of N containing p (this is where we use the hypothesis that G is continuous into S). Choose a smooth chart (U_0, φ) for N such that $p \in U_0 \subseteq U$. Then the coordinate representation of the corestriction $G|^S\colon N \to S$ with respect to the charts (U_0, φ) and $\big(V_0, \widetilde{\psi}\big)$ is

$$\widetilde{\psi} \circ G|^S \circ \varphi^{-1} = \pi \circ \psi \circ G \circ \varphi^{-1},$$

which is smooth, because $G\colon N \to M$ is smooth by assumption. This proves the assertion.

(c) According to (b) we only have to show that the corestriction of any smooth map $G\colon N \to M$ to S remains continuous. This is derived immediately from the following general topological fact: *if $f\colon X \to Y$ is a continuous map between topological spaces X and Y, and if $B \subseteq Y$ and $A \subseteq X$ are arbitrary subsets endowed with the subspace topology, and such that $f(A) \subseteq B$, then $f|_A^B\colon A \to B$ is continuous.*

Let us verify the above result for the sake of completeness. Let $V \subseteq B$ be an open subset of B. By definition of the subspace topology, there exists an open subset $V' \subseteq Y$ such that $V' \cap B = V$. Hence,

$$\big(f|_A^B\big)^{-1}(V) = f^{-1}(V') \cap A,$$

which is open in A, since $f^{-1}(V')$ is open in X by continuity of f and since A is endowed with the subspace topology. Therefore, $f|_A^B$ is continuous.

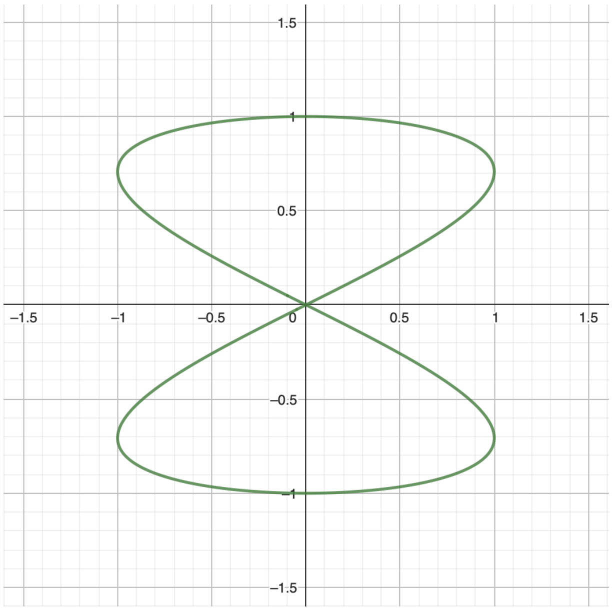
Remark.

(1) Let $F\colon M \to N$ be a smooth map. *Exercise* 5(a) asserts that if the domain of F is restricted to a smooth submanifold S of M, then the restriction of F to S remains smooth. However, if the codomain of F is restricted, then the resulting map need not be smooth in general, as the following example shows, but *Exercise* 5(b) demonstrates that the failure of continuity is the only thing that can go wrong.

Consider the smooth map

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t),$$

which is an injective smooth immersion according to *Example 4.5*(2). By *Proposition 5.13* its image $S \coloneqq \beta(-\pi, \pi)$ (which has been plotted below) has a unique topology and smooth structure such that S is an immersed submanifold of $\mathbb{R}^2$ and such that β is a diffeomorphism onto its image S.



Consider now the smooth map

$$B\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t)$$

and note that its image lies in S. As a map from $\mathbb{R}$ to S, B is not continuous, because $\beta^{-1} \circ B$ is not continuous at $t = \pi$.

(2) If M is a smooth manifold and if S is an immersed submanifold of M, then S is said to be *weakly embedded in* M if every smooth map $F\colon N \to M$ whose image lies in S is a smooth map as a map from M to S. *Exercise* 5(c) shows that embedded submanifolds are weakly embedded, while the previous example demonstrates that there are immersed submanifolds which are not weakly embedded.

Exercise 6 (*Extension lemma for functions on submanifolds*)**:** Let M be a smooth manifold, let $S \subseteq M$ be a smooth submanifold, and let $f \in C^\infty(S)$. Prove the following assertions:

(a) If S is an embedded submanifold, then there exists a neighborhood U of S in M and a smooth function $\widetilde{f}$ on U such that $\widetilde{f}|_S = f$.

[Hint: Use the local slice criterion and partitions of unity.]

(b) If S is a properly embedded submanifold, then the neighborhood U in (a) can be taken to be all of M.

[Hint: Take the construction in (a) as well as *Exercise* 1(b) into account.]

Solution:

(a) Let $p \in S$ and pick a slice chart (U_p, φ_p) for S in M such that $p \in U_p$. Note that $U_p \cap S$ is a properly embedded submanifold of U_p by *Theorem 5.9* and by the solution of *Exercise* 4; in particular, $U_p \cap S$ a closed subset of U_p by *Exercise* 1(b). By *Exercise* 5(a), the restriction $f|_{U_p \cap S}\colon U_p \cap S \to \mathbb{R}$ of f to $U_p \cap S$ is smooth, and thus by *Lemma 2.22* there exists a smooth function $f_p\colon U_p \to \mathbb{R}$ such that $f_p|_{U_p \cap S} = f|_{U_p \cap S}$ and $\operatorname{supp}(f_p) \subseteq U_p$.

Next, consider the open subset

$$U := \bigcup_{p\in S} U_p$$

of M and observe that U is an open neighborhood of S in M; in particular, U is an open submanifold of M. Let $\{\psi_p\}_{p\in S}$ be a smooth partition of unity subordinate to the open covering $\{U_p\}_{p\in S}$ of U, consider the smooth function

$$\widetilde{f}\colon U \to \mathbb{R},\ \ \widetilde{f}(x) := \sum_{p\in S} \psi_p(x) f_p(x)$$

and note that $\widetilde{f}|_S = f$. Therefore, $\widetilde{f}$ is the desired smooth extension of f.

(b) By *Exercise* 1(b), S is a closed subset of M, so

$$\Big(\bigcup_{p\in S} U_p\Big) \cup \big(M \setminus S\big)$$

is an open covering of M. Therefore, bearing the previous construction in mind, we may now consider a smooth partition of unity $\{\psi_p\}_{p\in S} \cup \{\psi_0\}$ subordinate the open covering $U \cup \big(M \setminus S\big)$ of M, and we may thus construct as above a smooth extension $\widetilde{f}$ of f on the whole of M.

Remark. It can be shown that the results in *Exercise* 6 can be strengthened as follows: Let M be a smooth manifold and let $S \subseteq M$ be a smooth submanifold. The following statements hold:

(a) $S \subseteq M$ is embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to a neighborhood of S in M.

(b) $S \subseteq M$ is properly embedded if and only if every $f \in C^\infty(S)$ has a smooth extension to all of M.