Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 7 – Solutions

Exercise 1 (to be submitted by Thursday, 7.11.2024, 16:00):

(a) Let N and $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$, and let $F_i \colon N \to M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G \colon N \to \prod_{i=1}^{k} M_i, \; x \mapsto \big(F_1(x), \ldots, F_k(x)\big)$$

is smooth and that its differential at any point $p \in N$ is of the form

$$\big(dG_p\big)(v) = \big(d(F_1)_p(v), \ldots, d(F_k)_p(v)\big), \; v \in T_pN.$$

(b) Show that the quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion, and that the kernel of its differential $d\pi_p \colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is the subspace generated by p.

(c) Let M be a non-empty, compact, smooth manifold. Show that there exists no smooth submersion $F \colon M \to \mathbb{R}^k$ for any $k \in \mathbb{Z}_{\geq 1}$.

Solution:

(a) The fact that G is smooth follows immediately from part (b) of [*Exercise Sheet* 3, *Exercise* 4], and the fact that the differential of G at $p \in N$ has the above form follows readily from part (b) of [*Exercise Sheet* 4, *Exercise* 1] and [*Exercise Sheet* 4, *Exercise* 3].

(b) From the solution to [*Exercise Sheet* 3, *Exercise* 5], we know that for every $0 \leq i \leq n$ there exists a smooth map $\Phi_i \colon U_i \to \mathbb{R}^{n+1} \setminus \{0\}$ such that $\pi \circ \Phi_i = \iota_{U_i}$, where ι_{U_i} is the inclusion of U_i into $\mathbb{RP}^n$. Write $p = (p^1, \ldots, p^{n+1})$ and for each $0 \leq i \leq n$ set $\widetilde{\Phi}_i = p^i \cdot \Phi_i$. Then we still have $\pi \circ \widetilde{\Phi}_i = \iota_{U_i}$, and moreover $\widetilde{\Phi}_i\big([p]\big) = p$. Hence,

$$d\pi_p \circ d(\widetilde{\Phi}_i)_{[p]} = d(\iota_{U_i})_{[p]},$$

and as the right hand side is an isomorphism, we infer that $d\pi_p$ is surjective. Therefore, π is a submersion.

Let us show that $p \in \ker\big(d\pi_p\big)$. Let $f \in C^\infty(\mathbb{RP}^n)$ be arbitrary, and let $D_p\big|_p$ be the directional derivative at p with direction p defined in [*Exercise Sheet* 4*, Exercise* 2]. Then

$$d\pi_p\left(D_p\big|_p\right)(f) = D_p\big|_p(f \circ \pi) = \frac{d}{dt}\bigg|_{t=0}\big(f \circ \pi\big)(p + tp) = 0$$

as $t \mapsto \pi(p + tp)$ is constant. By [*Exercise Sheet* 4, *Exercise* 2], $D_p\big|_p$ corresponds to p under the natural identification $T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \cong \mathbb{R}^{n+1}$. Thus, the kernel of $d\pi_p$ is generated by p.

(c) Assume by contradiction that there exist a positive integer k and a smooth submersion $F\colon M \to \mathbb{R}^k$. Since M is compact and F is continuous, the image $F(M) \subseteq \mathbb{R}^k$ is also compact, and since $\mathbb{R}^k$ is Hausdorff, we infer that $F(M)$ is a closed subset of $\mathbb{R}^k$. On the other hand, since F is a smooth submersion, it is an open map by *Proposition 4.16*, so the image $F(M)$ is an open subset of $\mathbb{R}^k$. But since $\mathbb{R}^k$ is connected and $M \neq \emptyset$, it follows that $F(M) = \mathbb{R}^k$, which implies that $\mathbb{R}^k$ is compact, a contradiction.

Exercise 2:

(a) Let M be a smooth manifold. Show that there exists a smooth map $f\colon M \to [0, +\infty)$ that is proper.

[Hint: Use a function of the form $f = \sum_{i=1}^{+\infty} c_i\psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a partition of unity and the c_i's are real numbers.]

(b) Let $F\colon M \to N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G\colon M \to N \times \mathbb{R}$.

[Hint: Use part (a) and *Exercise* 1(a).]

Solution:

(a) Since M is paracompact, there exists a countable, locally finite family $\mathfrak{U} = (U_i)_{i\in\mathbb{N}}$ of relatively compact (i.e., $\overline{U}_i$ is compact for each $i \in \mathbb{N}$) open subsets of M such that $M = \bigcup_{i\in\mathbb{N}} U_i$. Consider now a smooth partition of unity $(\psi_i)_{i\in\mathbb{N}}$ subordinate to $\mathfrak{U}$ and the sequence $(c_i = i)_{i\in\mathbb{N}}$ of non-negative real numbers such that $\lim_{i\to\infty} c_i = +\infty$, and define the smooth function

$$f\colon M \to \mathbb{R},\ x \mapsto \sum_{i\in\mathbb{N}} c_i\,\psi_i(x) = \sum_{i\in\mathbb{N}} i\,\psi_i(x).$$

We will show that f is proper. Observe that for any $k \in \mathbb{N}$ and any $x \notin U_1 \cup \ldots \cup U_k$, since $\operatorname{supp}\psi_i \subseteq U_i$ and $\sum_{i\in\mathbb{N}} \psi_i(x) = 1$, we have

$$f(x) = \sum_{i>k} i\psi_i(x) \geq (k+1)\sum_{i>k} \psi_i(x) = k + 1,$$

which implies (by contraposition) that $f^{-1}\big([0, k]\big) \subseteq U_1 \cup \ldots \cup U_k$ for any $k \in \mathbb{N}$. Thus, for any compact subset $K \subseteq \mathbb{R}$, the preimage $f^{-1}(K)$ is a closed subset of M and there exists $r_k \in \mathbb{N}$ such that $f^{-1}(K)$ is contained in some finite union $U_1 \cup \ldots \cup U_{r_k}$, and hence in the compact subset $\overline{U}_1 \cup \ldots \cup \overline{U}_{r_k}$ of M. It follows that $f^{-1}(K)$ is compact, as desired.

(b) By part (a) there exists a smooth proper function $f\colon M \to \mathbb{R}$. Consider now the map

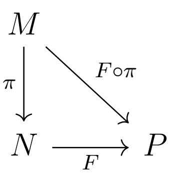
$$G\colon M \to N \times \mathbb{R},\ x \mapsto \big(F(x), f(x)\big),$$

which is smooth and whose differential has the form $dG = \big(dF, df\big)$ by *Exercise* 1(a). Since F is injective, one immediately sees that G is also injective. Moreover, since F is a smooth immersion, and thus its differential dF_p is injective at every point $p \in M$, it follows readily that $dG_p = \big(dF_p, df_p\big)$ is also injective at every point $p \in M$. Consequently, G is an injective smooth immersion.

Next, we claim that G is a proper map. Given a compact subset $K \subseteq N \times \mathbb{R}$, we will show that $G^{-1}(K)$ is a compact subset of M. To this end, since $N \times \mathbb{R}$ is a Hausdorff space, K is in particular a closed subset of $N \times \mathbb{R}$, and since G is continuous, the preimage $G^{-1}(K)$ is a closed subset of M. Now, since the projection to the second factor $\mathrm{pr}_2\colon N \times \mathbb{R} \to \mathbb{R}$ is continuous, the image $\mathrm{pr}_2(K)$ is a compact subset of $\mathbb{R}$, and since f is proper by assumption, the preimage $f^{-1}\big(\mathrm{pr}_2(K)\big)$ is a compact subset of M, which contains the closed set $G^{-1}(K)$. Hence, $G^{-1}(K)$ is a compact subset of M, as claimed.

In conclusion, G is a smooth embedding by the above and by *Proposition 4.6*(b), as asserted.

Exercise 3 (Characteristic property of surjective smooth submersions)**:** Let $\pi\colon M \to N$ be a surjective smooth submersion. Prove the following assertion: For any smooth manifold P, a map $F\colon N \to P$ is smooth if and only if the composite map $F \circ \pi\colon M \to P$ is smooth.



Solution: If F is smooth, then $F \circ \pi$ is also smooth by [*Exercise Sheet* 3*, Exercise* 3]. Conversely, assume that $F \circ \pi$ is smooth and let $q \in N$. Since π is surjective, there is $p \in M$ such that $\pi(p) = q$, and then *Theorem 4.15* guarantees the existence of a neighborhood U of q in N and a smooth local section $\sigma : U \to M$ of π such that $\sigma(q) = p$. Then $\pi \circ \sigma = \mathrm{Id}_U$ implies

$$F\big|_U = F\big|_U \circ \mathrm{Id}_U = F\big|_U \circ (\pi \circ \sigma) = \big(F\big|_U \circ \pi\big) \circ \sigma,$$

which is a composition of smooth maps. It follows from [*Exercise Sheet* 3*, Exercise* 2] and [*Exercise Sheet* 3*, Exercise* 3] that F is smooth.

Exercise 4: Let M and N be smooth manifolds, and let $\pi\colon M \to N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of *Exercise* 3; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P, a map $F\colon \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$.

Solution: Denote by Id_N, respectively $\mathrm{Id}_{\widetilde{N}}$, the identity map of N, respectively $\widetilde{N}$, with the smooth structure of N, respectively $\widetilde{N}$, on both the source and the target. Denote also by $\mathrm{Id}_{N,\widetilde{N}}$, respectively $\mathrm{Id}_{\widetilde{N},N}$, the identity map, where on the source, respectively on the target, we put the smooth structure of N, and where on the target, respectively on the source, we put the smooth structure of $\widetilde{N}$. In addition, denote by π_N, respectively $\pi_{\widetilde{N}}$, the surjective smooth submersion with the smooth structure of N, respectively of $\widetilde{N}$, on the target. Now, note that

$$\mathrm{Id}_{N,\widetilde{N}} \circ \pi_N = \pi_{\widetilde{N}},$$

which is smooth, so by the assumption on N applied to $P = \widetilde{N}$ and $F = \mathrm{Id}_{N,\widetilde{N}}$ we conclude that $\mathrm{Id}_{N,\widetilde{N}}$ is smooth. On the other hand, we also have

$$\mathrm{Id}_{\widetilde{N},N} \circ \pi_{\widetilde{N}} = \pi_N,$$

which is smooth, so by the assumption on $\widetilde{N}$ applied to $P = N$ and $F = \mathrm{Id}_{\widetilde{N},N}$ we conclude that $\mathrm{Id}_{\widetilde{N},N}$ is smooth. Hence, $\mathrm{Id}_{N,\widetilde{N}}$ is a diffeomorphism with inverse $\mathrm{Id}_{\widetilde{N},N}$.

Exercise 5 (The converse of *Exercise* 3 is false)**:** Consider the map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P, a map $F\colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Solution: Both the smoothness and the surjectivity of π is clear. Therefore, if a map $F\colon \mathbb{R} \to P$ is smooth, then $F \circ \pi$ is also smooth by [*Exercise Sheet* 3*, Exercise* 3]. Now, assume that we have a smooth manifold P and a map of sets $F\colon \mathbb{R} \to P$ such that $F \circ \pi$ is smooth. Consider the map

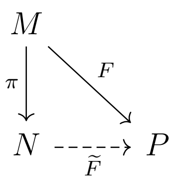
$$\iota\colon \mathbb{R} \to \mathbb{R}^2,\ x \mapsto (x,1),$$

which is clearly smooth and additionally satisfies $\pi \circ \iota = \mathrm{Id}_{\mathbb{R}}$. Hence, the map

$$F = F \circ \mathrm{Id}_{\mathbb{R}} = (F \circ \pi) \circ \iota$$

is smooth. Finally, note that the Jacobian of π is given by $(y\ x)$, which vanishes at $(x,y) = 0$, so π is not a smooth submersion.

Exercise 6 (*Pushing smoothly to the quotient*)**:** Let $\pi\colon M \to N$ be a surjective smooth submersion. Prove the following assertion: If P is a smooth manifold and $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F} \circ \pi = F$.



Solution: We define a set-theoretic function $\widetilde{F}\colon N \to P$ as follows: as π is surjective, there exists a set-theoretic right inverse $s\colon N \to M$, i.e., $\pi \circ s = \mathrm{Id}_N$, and now we set $\widetilde{F} := F \circ s$. Let us verify that we indeed have $\widetilde{F} \circ \pi = F$. Let $x \in M$ be arbitrary. Then x and $s\big(\pi(x)\big)$ both get mapped to $\pi(x)$ by π, and hence both are elements of the fiber $\pi^{-1}\big(\pi(x)\big)$. Since F is constant on the fibers of π by hypothesis, we obtain

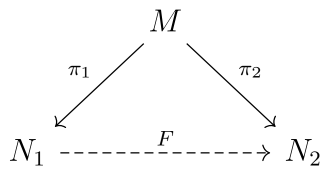
$$\widetilde{F}\big(\pi(x)\big) = F\left(s\big(\pi(x)\big)\right) = F(x).$$

As $x \in M$ was arbitrary, we conclude that $\widetilde{F} \circ \pi = F$, as claimed. Clearly, $\widetilde{F}$ is unique with this property: if $\widetilde{F}'$ is any other such function, then

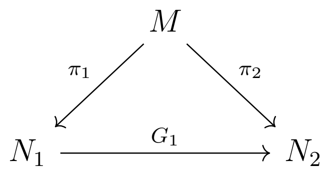
$$\widetilde{F}' = \widetilde{F}' \circ \pi \circ s = F \circ s = F.$$

Finally, as F is smooth, by *Exercise* 3 we conclude that $\widetilde{F}$ is smooth.

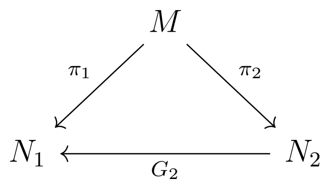
Exercise 7 (*Uniqueness of smooth quotients*)**:** Let $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:



Solution: Since π_1 is a surjective smooth submersion and since π_2 is constant on the fibers of π_1, by *Exercise* 6 there exists a unique smooth map $G_1\colon N_1 \to N_2$ such that $G_1 \circ \pi_1 = \pi_2$:



By reversing now the roles of π_1 and π_2, we see that there exists a unique smooth map $G_2\colon N_2 \to N_1$ such that $G_2 \circ \pi_2 = \pi_1$:



We thus obtain the identities

$$G_2 \circ G_1 \circ \pi_1 = \pi_1 \tag{$*$}$$

and

$$G_1 \circ G_2 \circ \pi_2 = \pi_2. \tag{$**$}$$

Considering the diagram

$$\begin{array}{ccc} & M & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_1} \\ N_1 & & N_1 \end{array}$$

and observing that $\mathrm{Id}_{N_1} \circ \pi_1 = \pi_1$, we deduce by (the uniqueness part of) *Exercise* 6 and $(*)$ that

$$G_2 \circ G_1 = \mathrm{Id}_{N_1} .$$

Considering now the corresponding diagram for π_2 and using $(**)$ instead, we infer similarly that

$$G_1 \circ G_2 = \mathrm{Id}_{N_2} .$$

Hence, $F \coloneqq G_1 \colon N_1 \to N_2$ is a diffeomorphism such that $F \circ \pi_1 = \pi_2$, which is unique (with this property) by construction.