Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 7

Exercise 1 (to be submitted by Thursday, 7.11.2024, 16:00):

(a) Let N and $M_1, \dots, M_k$ be smooth manifolds, where $k \geq 2$, and let $F_i \colon N \to M_i$ be smooth maps, where $1 \leq i \leq k$. Show that the map

$$G \colon N \to \prod_{i=1}^{k} M_i, \;\; x \mapsto \big(F_1(x), \dots, F_k(x)\big)$$

is smooth and that its differential at any point $p \in N$ is of the form

$$\big(dG_p\big)(v) = \big(d(F_1)_p(v), \dots, d(F_k)_p(v)\big), \;\; v \in T_pN.$$

(b) Show that the quotient map $\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is a smooth submersion, and that the kernel of its differential $d\pi_p \colon T_p\big(\mathbb{R}^{n+1} \setminus \{0\}\big) \to T_{[p]}\mathbb{RP}^n$ is the subspace generated by p.

(c) Let M be a non-empty, compact, smooth manifold. Show that there exists no smooth submersion $F \colon M \to \mathbb{R}^k$ for any $k \in \mathbb{Z}_{\geq 1}$.

Exercise 2:

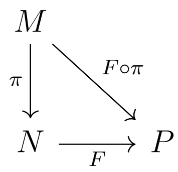
(a) Let M be a smooth manifold. Show that there exists a smooth map $f \colon M \to [0, +\infty)$ that is proper.

[Hint: Use a function of the form $f = \sum_{i=1}^{+\infty} c_i \psi_i$, where $(\psi_i)_{i=1}^{+\infty}$ is a partition of unity and the c_i's are real numbers.]

(b) Let $F \colon M \to N$ be an injective smooth immersion between smooth manifolds. Show that there exists a smooth embedding $G \colon M \to N \times \mathbb{R}$.

[Hint: Use part (a) and *Exercise* 1(a).]

Exercise 3 (Characteristic property of surjective smooth submersions)**:**
Let $\pi\colon M \to N$ be a surjective smooth submersion. Prove the following assertion: For any smooth manifold P, a map $F\colon N \to P$ is smooth if and only if the composite map $F \circ \pi\colon M \to P$ is smooth.



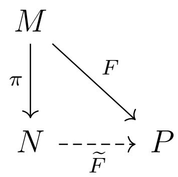
Exercise 4:
Let M and N be smooth manifolds, and let $\pi\colon M \to N$ be a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of *Exercise* 3; in other words, assuming that $\widetilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P, a map $F\colon \widetilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\widetilde{N}$.

Exercise 5 (The converse of *Exercise* 3 is false)**:**
Consider the map

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto xy.$$

Show that π is surjective and smooth, and that for each smooth manifold P, a map $F\colon \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Exercise 6 (*Pushing smoothly to the quotient*)**:**
Let $\pi\colon M \to N$ be a surjective smooth submersion. Prove the following assertion: If P is a smooth manifold and $F\colon M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\widetilde{F}\colon N \to P$ such that $\widetilde{F} \circ \pi = F$.



Exercise 7 (*Uniqueness of smooth quotients*)**:**
Let $\pi_1\colon M \to N_1$ and $\pi_2\colon M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Show that there exists a unique diffeomorphism $F\colon N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$: