Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
Lecturer: Dr. N. Tsakanikas
Assistant: L. E. Rösler

Exercise Sheet 6 – Solutions

Exercise 1:

(a) Prove the following assertions:

(i) A composition of smooth submersions is a smooth submersion.
(ii) A composition of smooth immersions is a smooth immersion.
(iii) A composition of smooth embeddings is a smooth embedding.

(b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

Solution:

(a) First, we show (i). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth submersions and fix $p \in M$. Then the composite map $G \circ F\colon M \to P$ is smooth by part (e) of [*Exercise Sheet* 3, *Exercise* 3], and by part (d) of [*Exercise Sheet* 4, *Exercise* 1] its differential at p is the linear map

$$d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P,$$

which is surjective, since both linear maps

$$dF_p\colon T_pM \to T_{F(p)}N \quad \text{and} \quad dG_{F(p)}\colon T_{F(p)} \to T_{(G\circ F)(p)}$$

are surjective by assumption. Since $p \in M$ was arbitrary, we conclude that $G \circ F$ is a smooth submersion.

Next, to prove (ii), we argue exactly as in (i), except that the word "surjective" is replaced by the word "injective".

Finally, we show (iii). Let $F\colon M \to N$ and $G\colon N \to P$ be smooth embeddings. By (ii) we know that the composite map $G \circ F\colon M \to P$ is a smooth immersion, so it remains to show that $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology. To this end, note that F is a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology, and that G is a homeomorphism onto its image $G(N) \subseteq P$ in the subspace topology, so the restriction $G|_{F(M)}\colon F(M) \to G\big(F(M)\big)$ is also a homeomorphism. Therefore, the composite map $G \circ F$ is a homeomorphism onto its image $(G \circ F)(M) \subseteq P$ in the subspace topology, as required. In conclusion, $G \circ F$ is a smooth embedding.

(b) Consider the maps

$$\gamma\colon (0, 2\pi) \to \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

and

$$\pi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto y.$$

By *Exercise* 2(a), π is a surjective smooth submersion. Moreover, observe that

$$\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$$

and

$$\|\gamma'(t)\| = \|(-\sin t, \cos t)\| = 1 \quad \text{for all } t \in (0, 2\pi),$$

so γ is an injective smooth immersion; see *Example 4.4*(1). Hence, both γ and π are smooth maps of constant rank. However, the composite map

$$\pi \circ \gamma\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto \sin t$$

does not have constant rank, because its derivative

$$(\pi \circ \gamma)'\colon (0, 2\pi) \to \mathbb{R},\ t \mapsto -\cos t$$

vanishes for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

Exercise 2:

(a) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Show that each of the projection maps $\pi_i\colon M_1 \times \ldots \times M_k \to M_i$ is a smooth submersion.

(b) Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \ldots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j\colon M_j \to M_1 \times \ldots \times M_k,\ x \mapsto (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k).$$

Show that each ι_j is a smooth embedding.

(c) Examine whether the following plane curves are smooth immersions:

(i) $\alpha\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3,\ t^2)$.

(ii) $\beta\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (t^3 - 4t,\ t^2 - 4)$.

If so, then examine also whether they are smooth embeddings.

(d) Show that the map

$$G\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (u, v) \mapsto \big((2 + \cos 2\pi u) \cos 2\pi v,\ (2 + \cos 2\pi u) \sin 2\pi v,\ \sin 2\pi u\big)$$

is a smooth immersion.

Solution:

(a) Fix $i \in \{1, \dots, k\}$ and $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$. By [*Exercise Sheet* 3*, Exercise* 4] we know that $\pi_i \colon M_1 \times \dots \times M_k \to M_i$ is a smooth map, while by [*Exercise Sheet* 4*, Exercise* 3] we know that

$$\begin{aligned} T_p\big(M_1 \times \dots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \dots, d(\pi_i)_p(v), \dots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism. Using the above identification, we infer that the differential of π_i at p,

$$d(\pi_i)_p \colon T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k \to T_{p_i}M_i,$$

is surjective. Since $p \in M_1 \times \dots \times M_k$ was arbitrary, we conclude that π_i is a smooth submersion.

(b) Fix $j \in \{1, \dots, k\}$ and points $p_1 \in M_1, \dots, p_{j-1} \in M_{j-1}, p_{j+1} \in M_{j+1}, \dots, p_k \in M_k$. We have already seen in the solution of [*Exercise Sheet* 4*, Exercise* 3] that the map

$$\iota_j \colon M_j \to M_1 \times \dots \times M_k, \; x \mapsto (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k)$$

is smooth, and it is also clear that ι_j is a homeomorphism onto its image

$$\iota_j(M_j) = \{p_1\} \times \dots \times \{p_{j-1}\} \times M_j \times \{p_{j+1}\} \times \dots \times \{p_k\}.$$

Moreover, given a point $p_j \in M_j$, using the identification

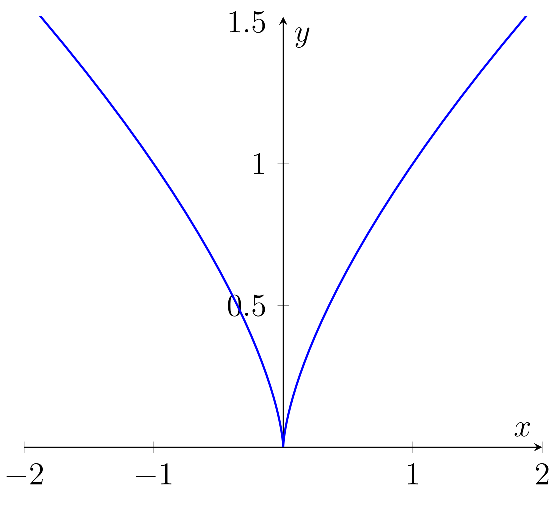
$$T_p\big(M_1 \times \dots \times M_k\big) \cong T_{p_1}M_1 \oplus \dots \oplus T_{p_i}M_i \oplus \dots \oplus T_{p_k}M_k,$$

where $p \coloneqq (p_1, \dots, p_{j-1}, p_j, p_{j+1}, \dots, p_k) \in M_1 \times \dots M_k$, we infer that the differential of ι_j at p,

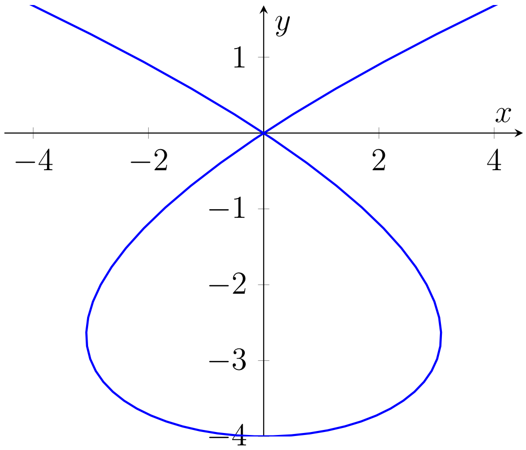
$$d(\iota_j)_{p_j} \colon T_{p_j}M_j \to T_{p_1}M_1 \oplus \dots \oplus T_{p_j}M_j \oplus \dots \oplus T_{p_k}M_k,$$

is injective. In conclusion, ι_j is a smooth embedding.

(c) We first deal with (i). The map $\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, is clearly smooth, but it is not an immersion, since $\alpha'(t) = (3t^2, 2t)$ vanishes at the point $t = 0$. Thus, α cannot be an embedding either.



We now deal with (ii). The map $\beta(t) = (t^3 - 4t, t^2 - 4)$, $t \in \mathbb{R}$, is clearly smooth and its velocity vector $\beta'(t) = (3t^2 - 4, 2t)$, $t \in \mathbb{R}$, is nowhere vanishing, so β is an immersion, see *Example 4.4*(1). However, the image curve $\beta(\mathbb{R})$ has a self-intersection for $t = -2$, $t = 2$, and hence β cannot be an embedding.



(d) The map G with component functions (G^1, G^2, G^3) is clearly smooth with Jacobian matrix

$$J_G(u,v) = \begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix}$$

$$= \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}.$$

The 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \end{pmatrix}$$

of J_G has determinant

$$D_{12}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\sin(2\pi u),$$

the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^1}{\partial u}(u,v) & \frac{\partial G^1}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi\big(2+\cos(2\pi u)\big)\sin(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{13}(u,v) := 4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\sin(2\pi v),$$

and the 2×2 submatrix

$$\begin{pmatrix} \frac{\partial G^2}{\partial u}(u,v) & \frac{\partial G^2}{\partial v}(u,v) \\ \frac{\partial G^3}{\partial u}(u,v) & \frac{\partial G^3}{\partial v}(u,v) \end{pmatrix} = \begin{pmatrix} -2\pi \sin(2\pi u)\sin(2\pi v) & 2\pi\big(2+\cos(2\pi u)\big)\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

of J_G has determinant

$$D_{23}(u,v) := -4\pi^2\big(2+\cos(2\pi u)\big)\cos(2\pi u)\cos(2\pi v).$$

Observe now that for each $(u,v) \in \mathbb{R}^2$, at least one of the determinants $D_{12}(u,v)$, $D_{13}(u,v)$ and $D_{23}(u,v)$ is non-zero, since $\cos(2\pi\theta)$ and $\sin(2\pi\theta)$ do not vanish simultaneously. This implies that $\operatorname{rk}\big(J_G(u,v)\big) = 2$ for all $(u,v) \in \mathbb{R}^2$; see the solution to part (c) of [*Exercise Sheet* 2, *Exercise* 3]. In conclusion, G is a smooth immersion, as claimed.

Exercise 3:

(a) Show that the inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding.

(b) Consider the map

$$F\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (2+\tanh t)\cdot(\cos t, \sin t).$$

(i) Show that F is an injective smooth immersion.

(ii) Show that F is a smooth embedding.
[Hint: Show that $F\colon \mathbb{R} \to U = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 3\}$ is a proper map.]

Solution:

(a) Consider the graph coordinates $\big(U_i^\pm \cap \mathbb{S}^n, \varphi_i^\pm\big)$ for $\mathbb{S}^n$; see *Example 1.10*(2). We have shown in *Example 2.12* that the inclusion map $\iota\colon \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth, because its coordinate representation with respect to any of the graph coordinates is

$$\widehat{\iota}(u^1,\dots,u^n) = \left(u^1,\dots,u^{i-1},\pm\sqrt{1-\|u\|^2},u^i,\dots,u^n\right),$$

which is smooth on its domain, the unit ball $\mathbb{B}^n = \{u = (u^1,\dots,u^n) \in \mathbb{R}^n \mid \|u\| < 1\}$. The Jacobian matrix of the coordinate representation $\widehat{\iota} = \iota \circ (\varphi_i^\pm)^{-1}$ of ι with respect to the graph coordinates has the form

$$\begin{pmatrix}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\
\frac{\mp u^1}{\sqrt{1-\|u\|^2}} & \frac{\mp u^2}{\sqrt{1-\|u\|^2}} & \dots & \frac{\mp u^{i-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^{i}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^{i+1}}{\sqrt{1-\|u\|^2}} & \dots & \frac{\mp u^{n-1}}{\sqrt{1-\|u\|^2}} & \frac{\mp u^{n}}{\sqrt{1-\|u\|^2}} \\
0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1
\end{pmatrix}.$$

In particular, we observe that each of these $(n+1) \times n$ matrices (which represent the differential of ι in coordinate bases) has rank n. Hence, ι is an injective smooth immersion. Since $\mathbb{S}^n$ is compact, by *Proposition 4.6*(c) we conclude that ι is a smooth embedding.

(b) We first deal with (i). Clearly, F is smooth. Recall also that the function

$$t \in \mathbb{R} \mapsto \|F(t)\| = 2 + \tanh t$$

is strictly increasing, which implies that F is injective. Finally, to show that F is a smooth immersion, it suffices to show that $F'(t) \neq 0$ for every $t \in \mathbb{R}$. To this end, recall that

$$\frac{d}{dt} \tanh t = \frac{1}{\cosh^2 t}, \quad t \in \mathbb{R},$$

so we have

$$F'(t) = \left(-(2 + \tanh t) \sin t + \frac{1}{\cosh^2 t} \cos t,\ (2 + \tanh t) \cos t + \frac{1}{\cosh^2 t} \sin t \right), \quad t \in \mathbb{R},$$

and thus

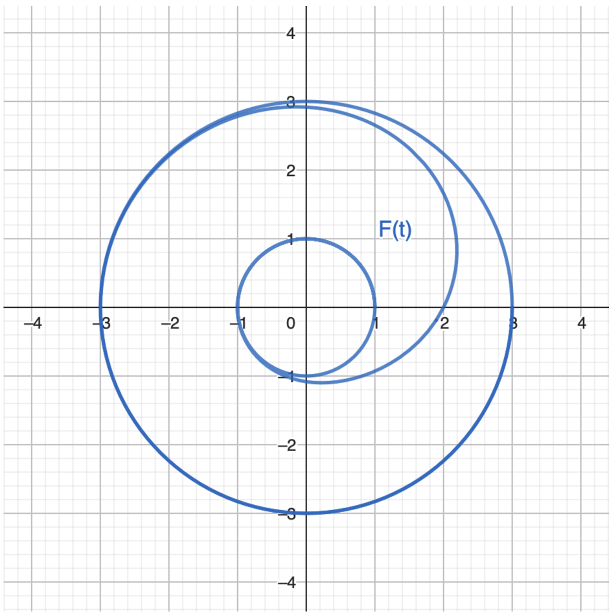
$$\|F'(t)\|^2 = (2 + \tanh t)^2 + \frac{1}{\cosh^4 t} > 0 \quad \text{for all } t \in \mathbb{R},$$

which implies that $F'(t) \neq 0$ for every $t \in \mathbb{R}$, as desired.

We now deal with (ii). Consider the open annulus

$$U := \left\{ x \in \mathbb{R}^2 \mid 1 < \|x\| < 3 \right\} \subseteq \mathbb{R}^2$$

and note that $F(t) \in U$ for every $t \in \mathbb{R}$. (Incidentally, the image of $F|_{[-4\pi, 4\pi]}$ has been plotted below.)



Thus, F may be viewed as an injective smooth immersion $F : \mathbb{R} \to U$. Since the inclusion map $\iota : U \hookrightarrow \mathbb{R}^2$ is a smooth embedding by *Example 4.4*(3), in view of *Exercise* 1(a)(iii) and *Proposition 4.6*(b), to prove (ii), it suffices to show that $F : \mathbb{R} \to U$ is a *proper* map;

in other words, given a compact subset K of U, we have to show that $F^{-1}(K)$ is a compact subset of $\mathbb{R}$, or equivalently that it is closed and bounded. Since $K \subseteq U$ is compact and $U \subseteq \mathbb{R}^2$ is Hausdorff, K is a closed subset of U, and since F is continuous, $F^{-1}(K)$ is a closed subset of $\mathbb{R}$. Now, denote by m (resp. M) the minimum (resp. the maximum) norm of the points of K, and observe that $[m, M] \subseteq (1, 3)$. Denote also by ℓ (resp. L) the preimage of m (resp. M) under the strictly increasing function

$$g \colon \mathbb{R} \to (1,3),\ t \mapsto \|F(t)\| = 2 + \tanh t$$

and note that $F^{-1}(K) \subseteq [\ell, L]$, which shows that $F^{-1}(K)$ is a bounded subset of $\mathbb{R}$. This finishes the proof of (ii).

Exercise 4 (Inverse function theorem for smooth manifolds)**:** Let $F \colon M \to N$ be a smooth map. Show that if $p \in M$ is a point such that the differential dF_p of F at p is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorphism.

Solution: The idea is to pass to a coordinate representation of F and to use the *inverse function theorem* for open subsets of Euclidean spaces, which is recalled below.

Let $W \subseteq \mathbb{R}^n$ be open and consider a smooth function $G \colon W \to \mathbb{R}^n$. Suppose that there is a point $a \in W$ such that the Jacobian matrix of G at a is invertible. Then there exist connected open sets U and V such that $a \in U \subseteq W$ and $G(U) \subseteq V \subseteq \mathbb{R}^n$, and moreover the restriction $G|_U \colon U \to V$ admits a smooth inverse; that is, $G|_U$ is a diffeomorphism from U to V.

Let (U, φ) and (V, ψ) be charts for M and N around p and $F(p)$, respectively, such that $F(U) \subseteq V$, and assume WLOG that $\varphi(p) = 0$ and $\psi\big(F(p)\big) = 0$. Set $\widehat{U} := \varphi(U)$ and $\widehat{V} := \psi(V)$, and let

$$\widehat{F} = \psi \circ F \circ \varphi^{-1} \colon \widehat{U} \to \widehat{V}$$

be the coordinate representation of F, which is smooth with $\widehat{F}(0) = 0$. Since dF_p is invertible, the tangent space to M at p and to N at $F(p)$ must have the same dimension, and thus $\widehat{U}, \widehat{V} \subseteq \mathbb{R}^n$, where $n = \dim M = \dim N$. Observe now that the differential

$$d\widehat{F}_0 = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_0$$

is invertible, because dF_p is invertible by assumption, and both $d(\varphi^{-1})_0$ and $d\psi_{F(p)}$ are invertible as well, as φ and ψ are diffeomorphisms. Note that the matrix representation of $d\widehat{F}_0$ with respect to the standard coordinates of $\mathbb{R}^n$ is the Jacobian of $\widehat{F}$ at 0. Therefore, by the *inverse function theorem* there are connected open neighborhoods $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ of 0 such that $\widehat{F}|_{\widehat{U}_0} \colon \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Hence, for $U_0 := \varphi^{-1}(\widehat{U}_0) \ni p$ and $V_0 := \psi^{-1}(\widehat{V}_0) \ni F(p)$, the restriction $F|_{U_0} \colon U_0 \to V_0$ is a diffeomorphism, since we can write it as a composition of diffeomorphisms.

Remark. Exercise 4 has the following important corollary: a smooth map $F \colon M \to N$ is a local diffeomorphism if and only if dF_p is invertible for all $p \in M$. This also gives a very useful method to prove that some map is a diffeomorphism, without explicitly constructing a smooth inverse: a smooth bijective map $F \colon M \to N$ whose differential dF_p is invertible for all $p \in M$ is a diffeomorphism.

Exercise 5: Let M and N be smooth manifolds and let $F\colon M \to N$ be a map. Prove the following assertions:

(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.

(b) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Solution: Recall that a local diffeomorphism is a smooth map by part (a) of [*Exercise Sheet* 3*, Exercise* 2].

(a) Assume first that F is a local diffeomorphism. According to part (d) of [*Exercise Sheet* 4*, Exercise* 1], for any $p \in M$, the differential of F at p is an $\mathbb{R}$-linear isomorphism, and thus both injective and surjective. Hence, F is both a smooth immersion and a smooth submersion.

Assume now that F is both a smooth immersion and a smooth submersion. Then for every $p \in M$, its differential dF_p is both injective and surjective, and thus an $\mathbb{R}$-linear isomorphism. It follows from *Exercise* 4 that F is a local diffeomorphism.

(b) Since $\dim M = \dim N$, for any $p \in M$, the differential $dF_p\colon T_pM \to T_{F(p)}N$ is an $\mathbb{R}$-linear map between $\mathbb{R}$-vector spaces of the same dimension. Thus, dF_p is injective or surjective if and only if it an isomorphism. Therefore, F is a smooth immersion if and only if F is a smooth submersion, and hence (b) follows immediately from (a).

Exercise 6: Let M, N and P be smooth manifolds, and let $F\colon M \to N$ be a local diffeomorphism. Prove the following assertions:

(a) If $G\colon P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If F is surjective and if $H\colon N \to P$ is any map, then H is smooth if and only if $H \circ F$ is smooth.

Solution: Recall that a local diffeomorphism is a smooth map by part (a) of [*Exercise Sheet* 3*, Exercise* 2].

(a) If G is smooth, then $F \circ G$ is smooth by part (e) of [*Exercise Sheet* 3*, Exercise* 3]. Conversely, consider the smooth map $H := F \circ G\colon P \to N$ and fix a point $p \in P$. Since F is a local diffeomorphism, there exists an open neighborhood V of $G(p)$ such that $F(V)$ is open in N and $F|_V\colon V \to F(V)$ is a diffeomorphism. Since G is continuous by assumption, $U := G^{-1}(V)$ is an open subset of P, and since $G(p) \in V$, it holds that $p \in U$; in other words, U is an open neighborhood of p in P. Observe now that $G|_U = (F|_V)^{-1} \circ H|_U$ is smooth by part (e) of [*Exercise Sheet* 3*, Exercise* 3], since $(F|_V)^{-1}$ is smooth by assumption and $H|_U$ is smooth by part (b) of [*Exercise Sheet* 3*, Exercise* 2]. It follows from part (a) of [*Exercise Sheet* 3*, Exercise* 2] that G is smooth.

(b) If H is smooth, then $H \circ F$ is smooth by part (e) of [*Exercise Sheet* 3*, Exercise* 3]. Conversely, consider the smooth map $G := H \circ F$ and fix a point $q \in N$. Since F is surjective, there exists a point $p \in M$ such that $F(p) = q$, and since F is a local diffeomorphism, there exists an open neighborhood U of p such that $F(U)$ is open in N

and $F|_U : U \to F(U)$ is a diffeomorphism; in particular, $F(U)$ is an open neighborhood of q in N. Observe now that $H|_{F(U)} = G|_U \circ (F|_U)^{-1}$ is smooth by part (e) of [*Exercise Sheet* 3*, Exercise* 3], since $(F|_U)^{-1}$ is smooth by assumption and $G|_U$ is smooth by part (b) of [*Exercise Sheet* 3*, Exercise* 2]. It follows from part (a) of [*Exercise Sheet* 3*, Exercise* 2] that H is smooth.