



Differential Geometry II - Smooth Manifolds

Winter Term 2024/2025

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Exercise Sheet 6

Exercise 1:

- (a) Prove the following assertions:
- (i) A composition of smooth submersions is a smooth submersion.
 - (ii) A composition of smooth immersions is a smooth immersion.
 - (iii) A composition of smooth embeddings is a smooth embedding.
- (b) Show by means of a counterexample that a composition of smooth maps of constant rank need not have constant rank.

Exercise 2 (to be submitted by Thursday, 31.10.2024, 16:00):

- (a) Let M_1, \dots, M_k be smooth manifolds, where $k \geq 2$. Show that each of the projection maps $\pi_i: M_1 \times \dots \times M_k \rightarrow M_i$ is a smooth submersion.
- (b) Let M_1, \dots, M_k be smooth manifolds, where $k \geq 2$. Choosing arbitrarily points $p_1 \in M_1, \dots, p_k \in M_k$, for each $1 \leq j \leq k$ consider the map

$$\iota_j: M_j \rightarrow M_1 \times \dots \times M_k, \quad x \mapsto (p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_k).$$

Show that each ι_j is a smooth embedding.

- (c) Examine whether the following plane curves are smooth immersions:

- (i) $\alpha: \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (t^3, t^2)$.
- (ii) $\beta: \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (t^3 - 4t, t^2 - 4)$.

If so, then examine also whether they are smooth embeddings.

- (d) Show that the map

$$G: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (u, v) \mapsto ((2 + \cos 2\pi u) \cos 2\pi v, (2 + \cos 2\pi u) \sin 2\pi v, \sin 2\pi u)$$

is a smooth immersion.

Exercise 3:

- (a) Show that the inclusion map $\iota: \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding.
- (b) Consider the map

$$F: \mathbb{R} \rightarrow \mathbb{R}^2, t \mapsto (2 + \tanh t) \cdot (\cos t, \sin t).$$

- (i) Show that F is an injective smooth immersion.
- (ii) Show that F is a smooth embedding.

[Hint: Show that $F: \mathbb{R} \rightarrow U = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 3\}$ is a proper map.]

Exercise 4 (*Inverse function theorem for manifolds*):

Let $F: M \rightarrow N$ be a smooth map. Show that if $p \in M$ is a point such that the differential dF_p of F at p is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0}: U_0 \rightarrow V_0$ is a diffeomorphism.

[Hint: Reduce to the ordinary inverse function theorem for functions between Euclidean spaces.]

Exercise 5:

Let M and N be smooth manifolds and let $F: M \rightarrow N$ be a map. Prove the following assertions:

- (a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.
- (b) If $\dim M = \dim N$ and if F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

Exercise 6:

Let M , N and P be smooth manifolds, and let $F: M \rightarrow N$ be a local diffeomorphism. Prove the following assertions:

- (a) If $G: P \rightarrow M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.
- (b) If F is surjective and if $H: N \rightarrow P$ is any map, then H is smooth if and only if $H \circ F$ is smooth.