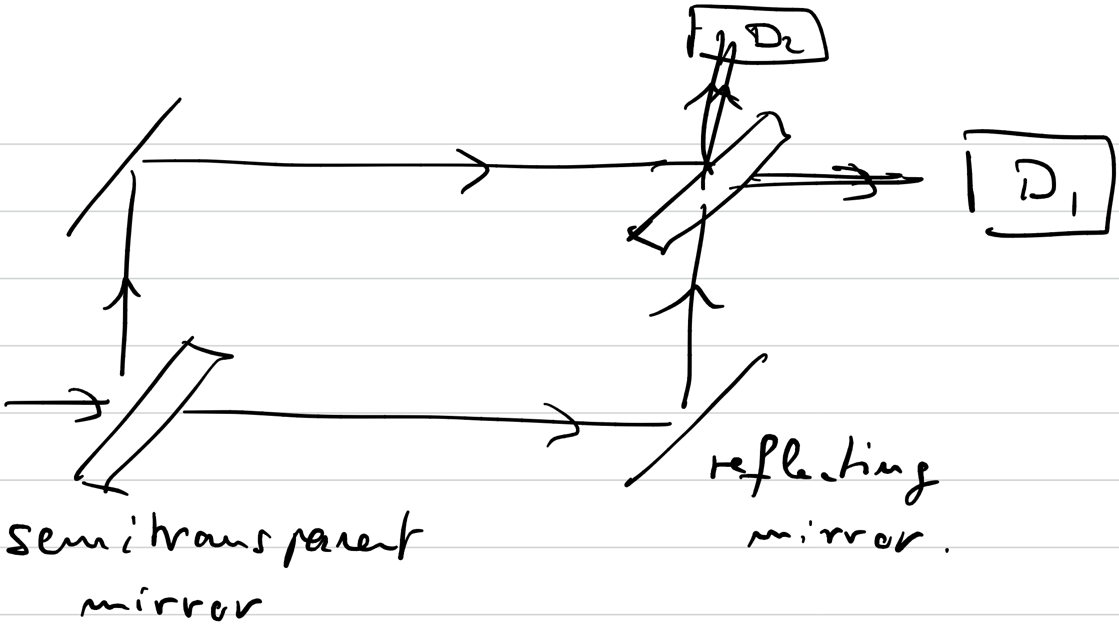
APPLICATION OF THE PRINCIPLES TO INTERFERENCE EXPERIMENTS.

In this chapter we illustrate the application of the principles to a few interference experiments. This will allow to better understand them on one hand, as well as have concrete examples of qubits, before we go on to communication protocols in the next few classes.

I. Examples of qubits.

- Path states in the Mach-Zehnder experiments. Recall the simple MZ interferometer of the first lecture.



A "toy model" for the Hilbert space is here $\mathcal{H} = \mathbb{C}^2 = \{ \alpha |H\rangle + \beta |V\rangle \}$

where $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = horizontal path state of photon.

$|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ = vertical path state of photon.

Recall $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$.

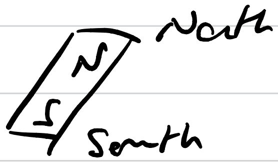
- **Polarization of photons.**

Photons carry with them a polarization vector which is analogous to the polarization of the electromagnetic waves.

We will see that this is described by vectors of $\mathbb{C}^2 = \mathcal{H}$. Thus the polarization of photons is a quantum bit. The description in terms of qubit is "exact" this time (i.e not a toy model approximation).

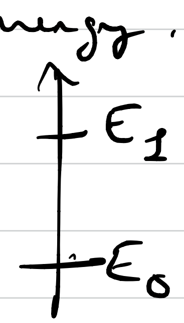
- **Spin 1/2** (of electrons, protons, some nuclei; some ions, atoms, ...)

Many particles carry a small magnetic dipole



whose state vector

is described by a vector in $\mathcal{H} = \mathbb{C}^2$.
This is analogous (but different) to polarization of photons. We will come back to spin later in class when we study its interaction with magnetic fields. Again here we have an "exact" qubit.



- <u>Two level systems</u>

Just like atoms have energy levels corresponding to different orbital states, many systems are described by some set of "energy levels". Often one can isolated two important levels from the rest of the spectrum and <u>model</u> the system to a good approximation by a qubit with Hilbert space $\mathbb{C}^2$.

II. Analysis of MZ interferometers.

$\mathcal{H} = \mathbb{C}^2$, Basis $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = horizontal path state

$|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ = vertical path state.

semi-transparent mirror: a unitary matrix, for example

$$U = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ "Hadamard matrix"

$$H = \frac{1}{\sqrt{2}} \Big(|H\rangle\langle H| + |H\rangle\langle V| + |V\rangle\langle H| - |V\rangle\langle V| \Big)$$

perfectly reflecting mirror:

$$U = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

also called a NOT gate or matrix.

$$X = |H\rangle\langle V| + |V\rangle\langle H|$$

Analysis:

- initial state $|H\rangle$
- after first semi-transparent mirror:

$$H\,|H\rangle = \frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right)$$

- after reflecting mirrors

$$X H\,|H\rangle = \frac{1}{\sqrt{2}}\left(X\,|H\rangle + X\,|V\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|V\rangle + |H\rangle\right)$$

- after second semi-transparent mirror:

$$H X H\,|H\rangle = \frac{1}{\sqrt{2}}\left(H\,|V\rangle + H\,|H\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\left(|H\rangle - |V\rangle\right) + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right)$$

$$= |H\rangle$$

So we see that the photon comes out of the interferometer as $|H\rangle$!.

- Note the total unitary evolution is :

$$U_{total} = H \times H .$$

- Measurement process :

The detectors constitute a measurement apparatus modelled by the basis $\{|H\rangle, |V\rangle\}$

We thus apply the Born rule and get :

$$\text{Prob}(\text{photon obs in } D_1) = \text{Prob}(\text{output state is } |H\rangle)$$

$$= |\langle \underset{\substack{\nearrow \\ \text{output}}}{H} | \underset{\substack{\uparrow \\ \text{input}}}{H} \rangle|^2 = 1$$

$$\text{Prob (photon obs in } D_2) = \text{Prob (output state is } |V\rangle)$$

$$= |\langle V | H \rangle|^2 = 0$$

$$\underset{\text{output}}{\langle V|} \quad \underset{\text{input}}{|H\rangle}$$

Recall this is the result of the experiment described in the first lecture.

Remark : One can introduce the "Clic" observable, an hermitian matrix :

$$C = (+1)\, |H\rangle\langle H| + (-1)\, |V\rangle\langle V|$$

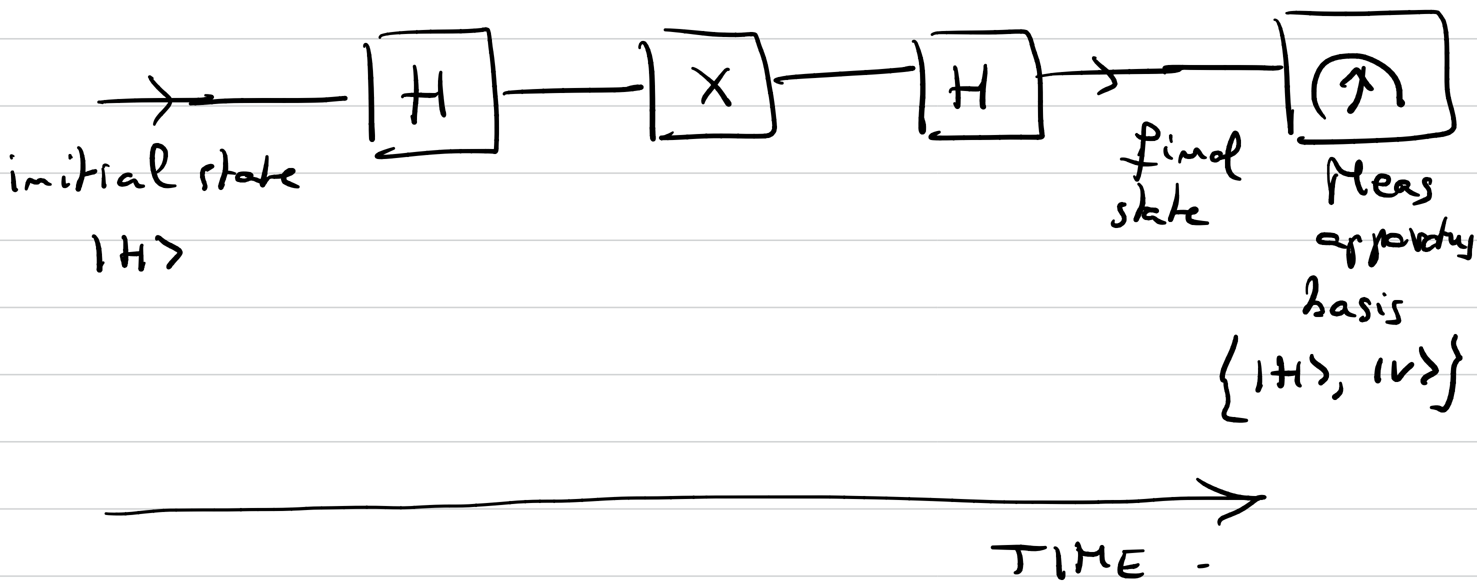
with eigenvalues / eigenvectors ± 1, $|H\rangle$, $|V\rangle$:

$$\begin{cases} C\,|H\rangle = (+1)\,|H\rangle \\ C\,|V\rangle = (-1)\,|V\rangle \end{cases}$$

According to the Measurement postulate the outcome is ± 1 with probabilities

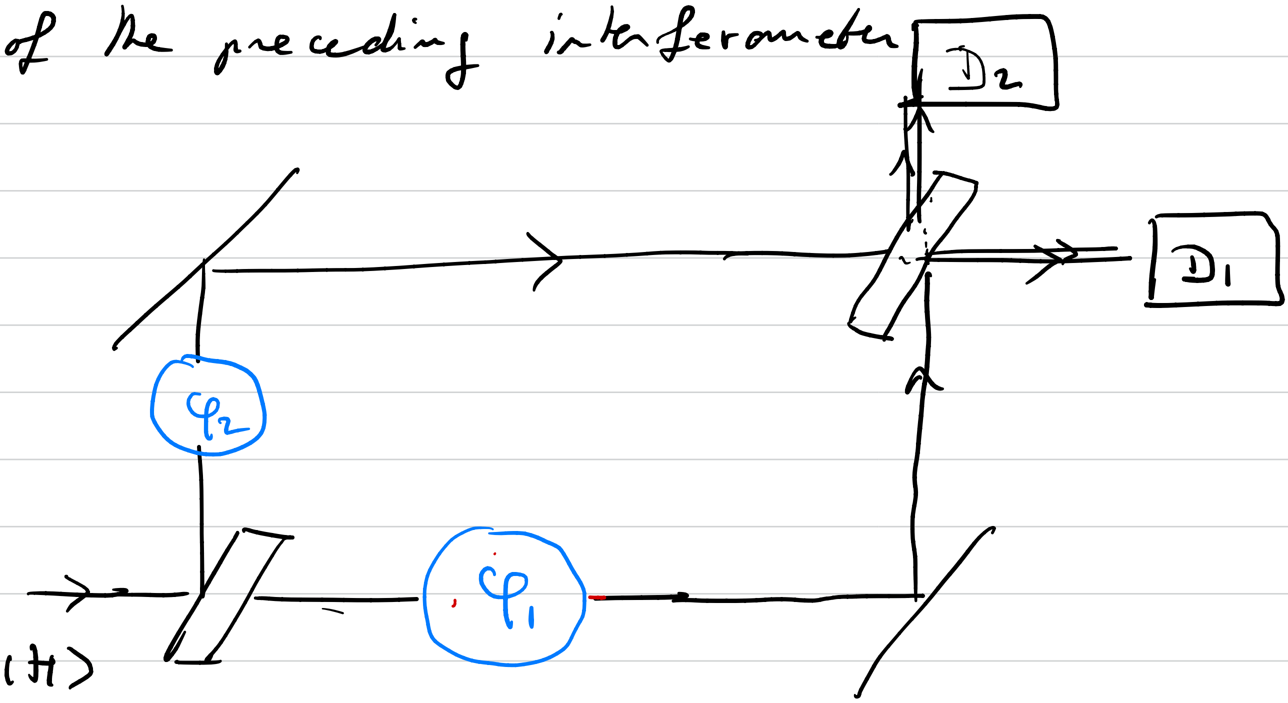
$$\underbrace{\text{Prob}(+1)}_{D_1 \text{ clics}} = |\langle H | H \rangle|^2 = 1 \; ; \; \underbrace{\text{Prob}(-1)}_{D_2 \text{ clics.}} = |\langle V | H \rangle|^2 = 0$$

Remark: The whole experiment has a "circuit interpretation". Quantum circuits constitute a representation of quantum algorithms (see second semester CS-308).



III Another MZ interferometer.

Let us look at the following variation of the preceding interferometer



φ_1 is a "dephaser" which changes the "phase" of a photon by $|H\rangle \to e^{i\varphi_1}|H\rangle$

φ_2 a "dephaser" : $|V\rangle \to e^{i\varphi_2}|V\rangle$

The corresponding unitary operations are

$$\begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & 1 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$$

Thus the dephasers acts on the state in $\mathbb{C}^2$

as

$$U_D = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} & 0 \\ 0 & e^{i\varphi_2} \end{pmatrix}$$

Note these commute so here it does not matter in what order they act (or simultaneously).

#

<u>Semi-transparent mirror</u> : we choose the model

$$U_{ST} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

<u>Perfectly reflecting mirrors</u> : we choose the model

$$U_R = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

<u>Analysis:</u> the total unitary operator is

$$U_{total} = U_{SM} \, U_R \, U_D \, U_{SM}$$

$$= \frac{1}{2}\begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} & -ie^{i\varphi_1} + ie^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} & -e^{i\varphi_1} - e^{i\varphi_2} \end{pmatrix}$$

For an initial state $|H\rangle$, the final state is

$$|\psi_{final}\rangle = U_{SM} \, U_R \, U_D \, U_{SM} \, |H\rangle$$

$$= \frac{1}{2}\begin{pmatrix} -e^{i\varphi_1} - e^{i\varphi_2} \\ ie^{i\varphi_1} - ie^{i\varphi_2} \end{pmatrix}$$

$$= -\frac{e^{i\varphi_1} + e^{i\varphi_2}}{2}|H\rangle + i.\frac{(e^{i\varphi_1} - e^{i\varphi_2})}{2}|V\rangle$$

<u>Measurement results</u>

$$\text{Prob}(D_1 \text{ clics}) = |\langle H|\psi_{final}\rangle|^2 = \frac{|e^{i\varphi_1} + e^{i\varphi_2}|^2}{4}$$

$$= \frac{|1+e^{i(\varphi_2-\varphi_1)}|^2}{4}$$

$$= \frac{1}{4}\left[(1+\cos(\varphi_2-\varphi_1))^2 + \sin^2(\varphi_1-\varphi_2)\right]$$

$$= \frac{1}{2}(1+\cos(\varphi_2-\varphi_1))$$

$$= \left[\cos\left(\frac{\varphi_1-\varphi_2}{2}\right)\right]^2$$

Similarly:

$$\text{Prob}(D_2 \text{ clics}) = |\langle v|\psi_{final}\rangle|^2 = \frac{1}{4}|e^{i\varphi_1} - e^{i\varphi_2}|^2$$

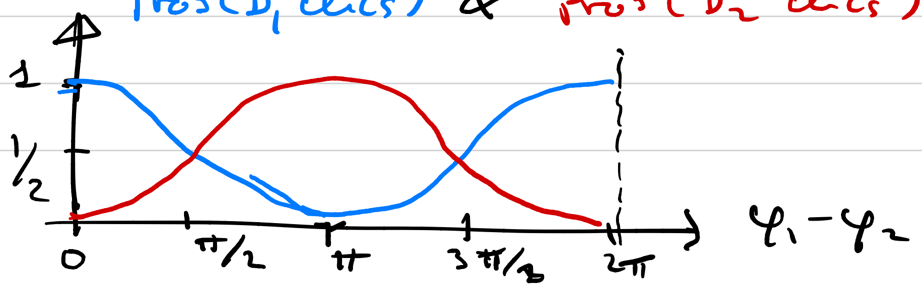
$$= \frac{1}{4}|1 - e^{i(\varphi_2-\varphi_1)}|^2$$

$$= \frac{1}{4}\left[(1-\cos(\varphi_2-\varphi_1))^2 + (\sin(\varphi_2-\varphi_1))^2\right]$$

$$= \frac{1}{2}(1-\cos(\varphi_2-\varphi_1))$$

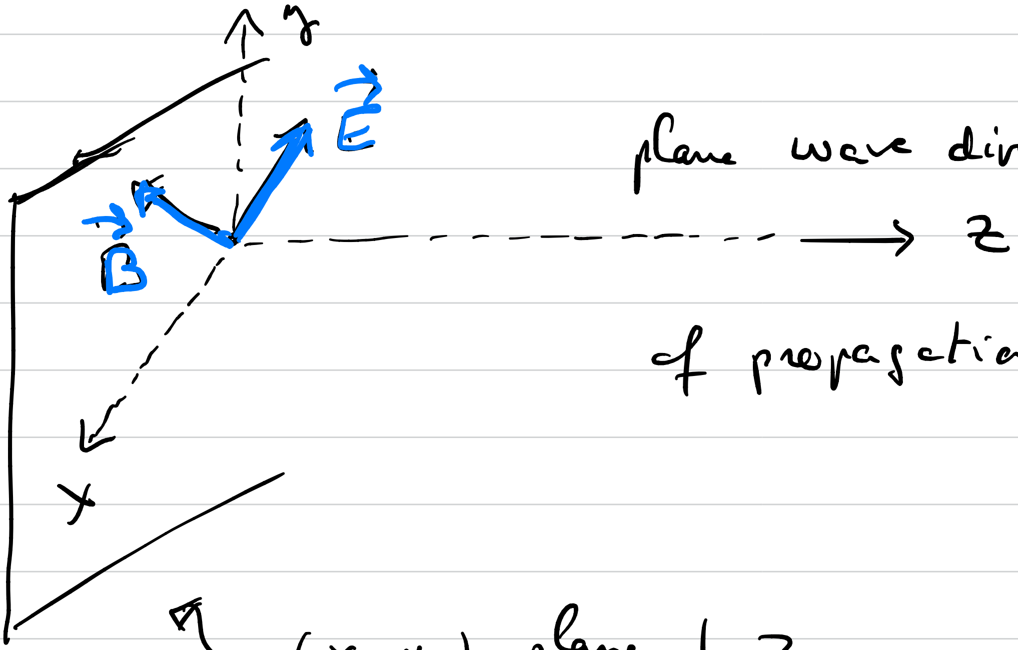
$$= \left(\sin\frac{(\varphi_1-\varphi_2)}{2}\right)^2$$

Prob(D_1 clics) & Prob(D_2 clics)



IV Photon polarization

Electromagnetic waves carry a degree of freedom called "Polarization".



$(\vec{E}, \vec{B})$ fields oscillate in the plane perpendicular to the direction of propagation.

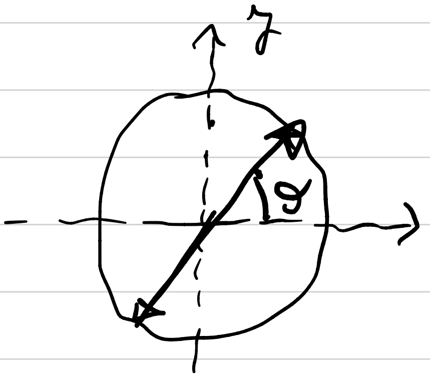
$$\vec{E}(z,t) = \text{Real part}\left\{ \vec{E}_0 \, e^{i(Kz - \omega t)} \right\}$$

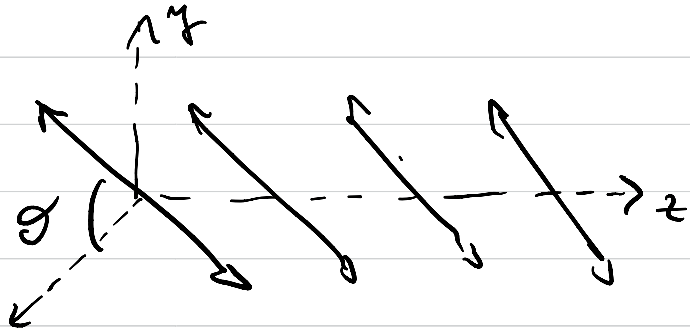
$K = \dfrac{2\pi}{\lambda}$, $\omega = 2\pi\nu$ $\quad$ λ = wave length, ν = frequency $\quad$ $\lambda\nu = c$

with $$\vec{E}_0 = \underset{\in \mathbb{R}}{\epsilon_0} \begin{pmatrix} \cos\vartheta \\ (\sin\vartheta)\, e^{i\varphi} \\ 0 \end{pmatrix}$$

<u>Linear polarization</u>: $\varphi = 0$ so the

<u>$\vec{E}$-field oscillates along ϑ direction in (xy) plane</u>



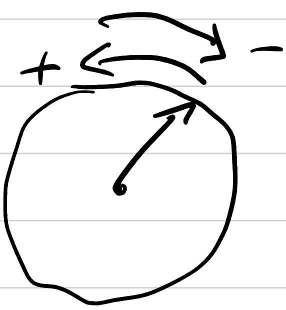


$$\vec{E} \propto \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{pmatrix} e^{ikz-\omega t}$$

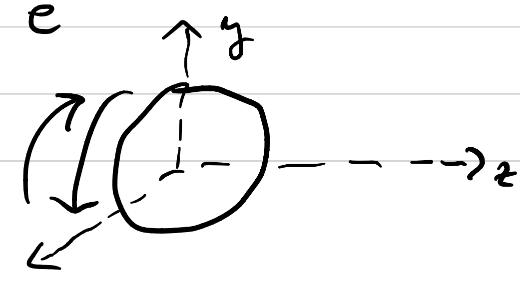
$\longrightarrow$ dir of propagation

<u>Circular polarization</u>: $\vartheta = \frac{\pi}{4}$, $\varphi = \pm\frac{\pi}{2}$

$\vec{E}$-field is rotating clockwise or anticlockwise



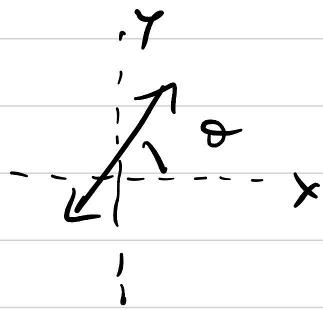
$$\vec{E} \propto \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} e^{ikz-\omega t}$$

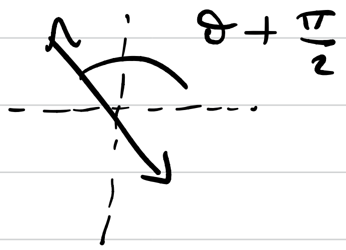


General polarization

Can be described from superpositions of basis vectors of linear pol

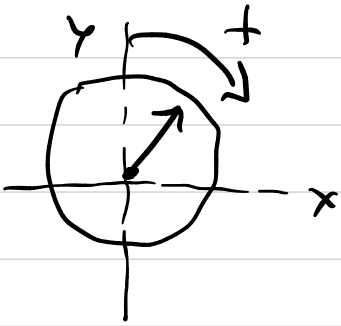
$$\begin{pmatrix} \cos\vartheta \\ \sin\vartheta \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\sin\vartheta \\ +\cos\vartheta \\ 0 \end{pmatrix}$$

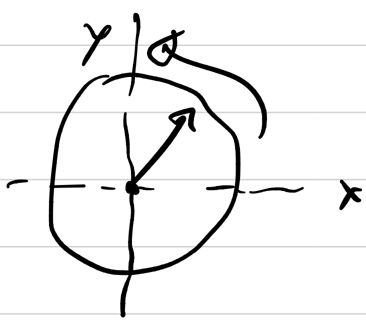




or also by basis vectors of circular pol

$$\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$





Quantum polarization degree of freedom of photon.

It turns out photon (the "quanta of e-m field") carry an analogous degree of freedom,

This is described by vectors of the Hilbert space $\underline{\mathcal{H} = \mathbb{C}^2}$ (i.e qubits)

$$|\vartheta, \varphi\rangle = \begin{pmatrix} \cos\vartheta \\ (\sin\vartheta)\, e^{i\varphi} \end{pmatrix}$$

<u>Linear pol orthonormal basis;</u>

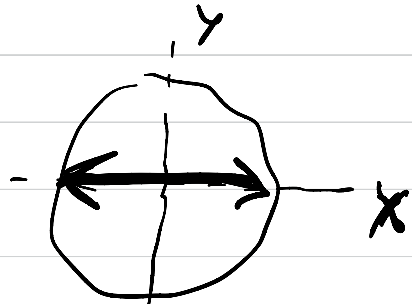
$$\begin{cases} |\vartheta\rangle = \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} = \cos\vartheta\,|x\rangle + \sin\vartheta\,|y\rangle \\ |\vartheta_\perp\rangle = \begin{pmatrix} -\sin\vartheta \\ \cos\vartheta \end{pmatrix} = -\sin\vartheta\,|x\rangle + \cos\vartheta\,|y\rangle \end{cases}$$

horizontal pol   vertical pol.

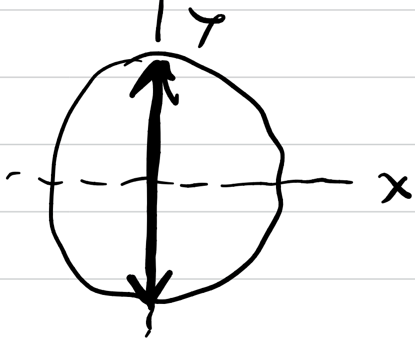
Circular pol orthonormal basis:

$$\begin{cases} |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|x\rangle + i\,|y\rangle\right) \\ |\circlearrowright\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\underbrace{|x\rangle}_{\text{horizontal pol}} - i\,\underbrace{|y\rangle}_{\text{vert pol.}}\right) \end{cases}$$

Note: $|x\rangle = \begin{pmatrix}1\\0\end{pmatrix} =$



$|y\rangle = \begin{pmatrix}0\\1\end{pmatrix} =$

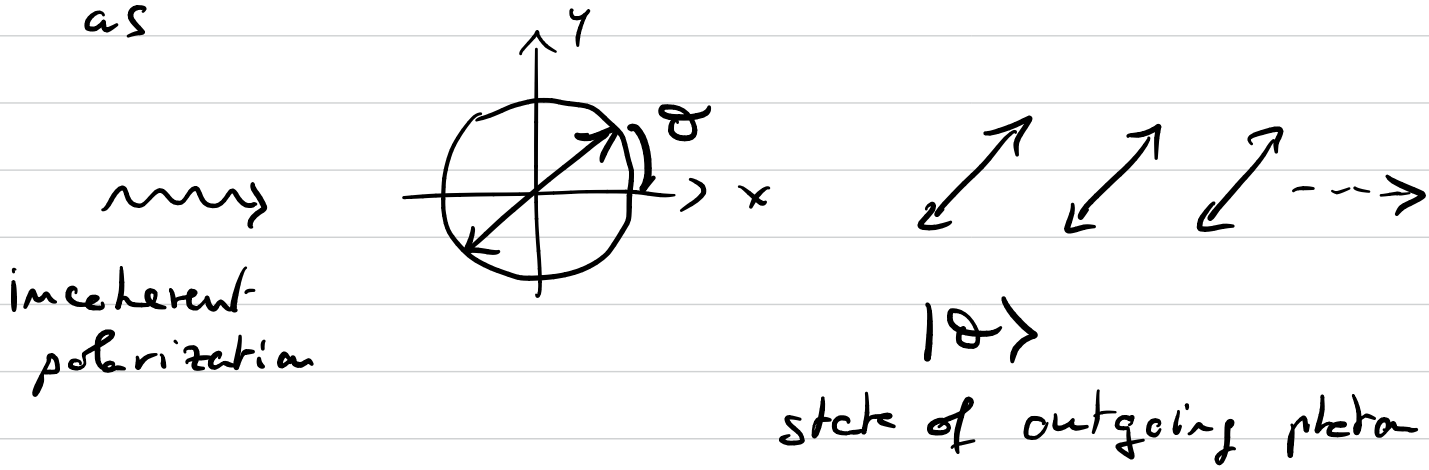


V Experiments with photon polarization.

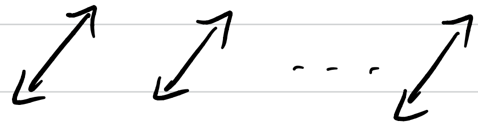
(V.1) Polarizer - Analyzer - Detector setting:

Let us prepare photons in a state of linear polarization $|\vartheta\rangle = \cos\vartheta\,|x\rangle + \sin\vartheta\,|y\rangle$

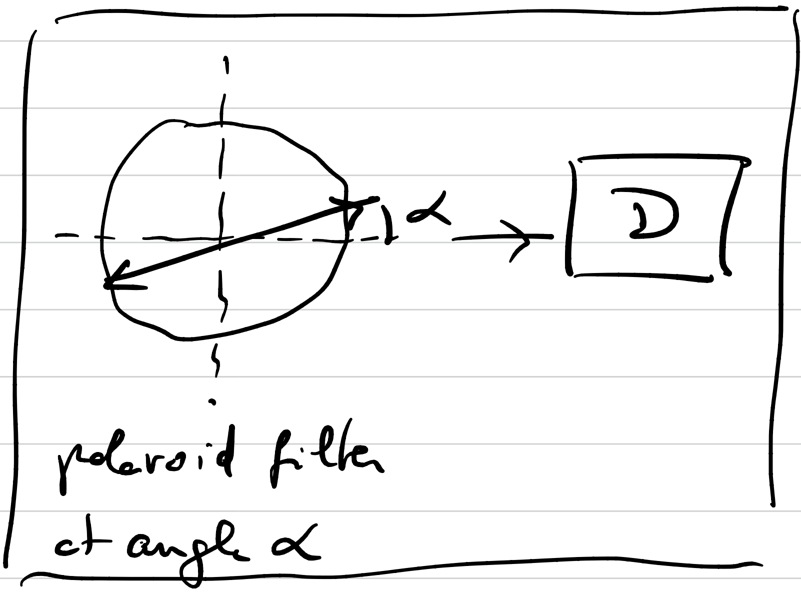
This is done by a "polaroid filter" depicted as



Now we imagine photons in state $|\vartheta\rangle$ enter an "analyzer - detector apparatus"



incoming state $|\theta\rangle$



- If photon goes through analyzer it enters D in state $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$ and D clics ; we register $+1$.

- If photon is absorbed by analyzer it collapses in state $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$. D does not clic and we register -1.

Accarding to Meas postulate ;

$$\underset{\substack{\uparrow \\ \text{clic}}}{\mathrm{Prob}(+1)} = |\langle\alpha|\theta\rangle|^2 = (\cos(\theta-\alpha))^2$$

$$\text{Prob}(-1) = |\langle \alpha_\perp | \theta \rangle|^2 = (\sin(\theta-\alpha))^2$$

No click

Observable measured here:

$$P_\alpha = (+1)\,|\alpha\rangle\langle\alpha| + (-1)\,|\alpha_\perp\rangle\langle\alpha_\perp|$$

eigenvectors and eigenvalues; $\begin{cases} P_\alpha|\alpha\rangle = (+1)|\alpha\rangle \\ P_\alpha|\alpha_\perp\rangle = (-1)|\alpha_\perp\rangle \end{cases}$

Remarks.

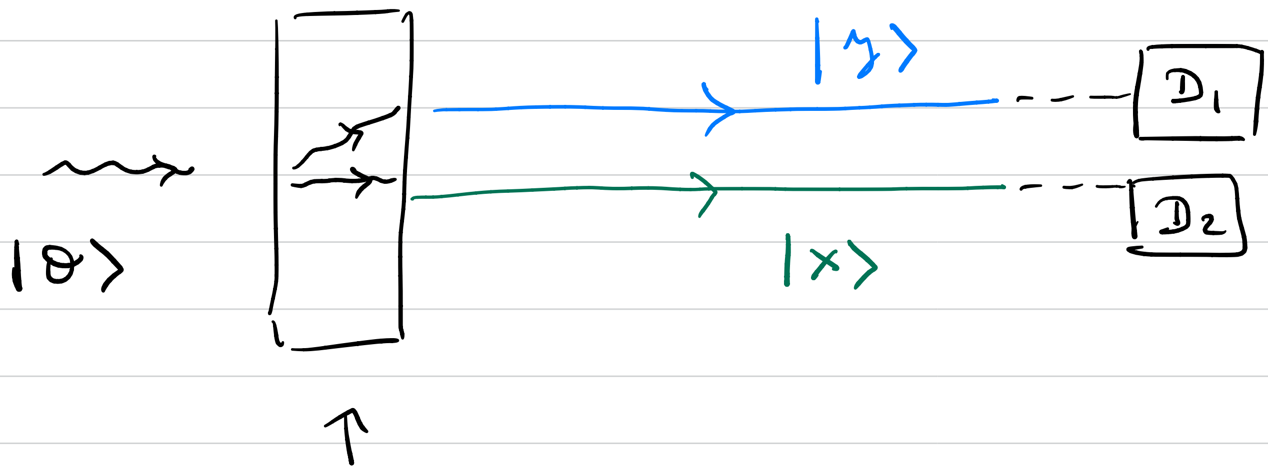
- With classical electromagnetic waves the similar experiments give intensity of electric field detected in D $\propto (\cos(\theta-\alpha))^2$
This is the law of Malus (19th century) and was instrumental in discovering the polarization of waves.

- Another good question is: what is the unitary time evolution of states $|\theta\rangle$ in this experiment? In fact here U is the "simple 1×1 matrix" $e^{i\omega t}$ where $\omega = 2\pi\nu$ the frequency of the photon,

$$U|\theta\rangle = e^{i\omega t}|\theta\rangle.$$

So here this plays a somewhat trivial role and we do not consider it.

(V.2) Path splitting with Birefringent crystals.



birefringent
crystal interacts with radiation.

- incoming photons (or waves) in state $|\theta\rangle$
- outgoing photons (or waves) are split in
  two possible paths :

  in upper path pol state is $|y\rangle$ = vertical

  in lower path pol state is $|x\rangle$ = horizontal.

total quantum state is : $\cos\theta\,|x\rangle + \sin\theta\,|y\rangle$

- Detection process: if one does the experiment one will observe a sequence of clics with probability distr:

$$\mathrm{Prob}\,(D_1 \text{ clics}) = (\sin\vartheta)^2$$

$$\mathrm{Prob}\,(D_2 \text{ clics}) = (\cos\vartheta)^2$$

This is consistent with the quantum prediction from the Measurement Postulate (or Born rule):

$$\mathrm{Prob}\,(\text{state } |y\rangle \text{ is observed})$$

$$= \left| \langle y | \left(\cos\vartheta\, |x\rangle + \sin\vartheta\, |y\rangle \right) \right|^2$$

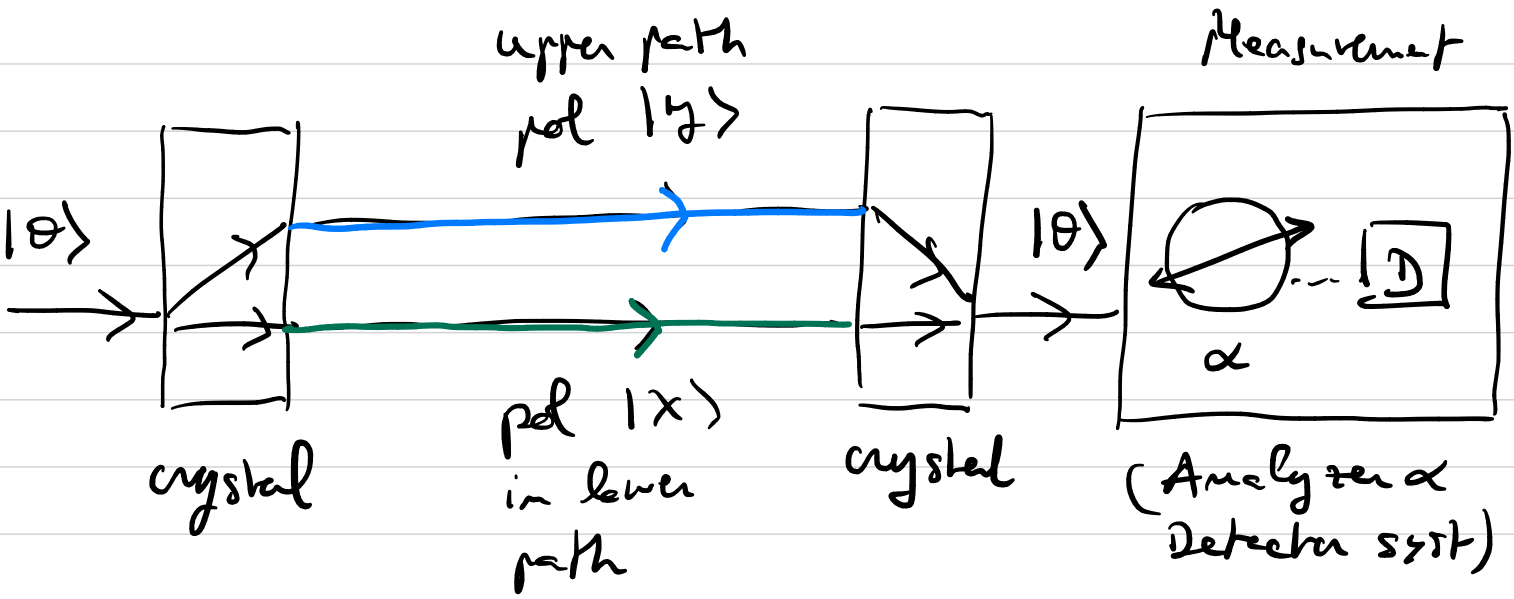
$$= \left| \cos\vartheta \underbrace{\langle y|x\rangle}_{0} + \sin\vartheta \underbrace{\langle y|y\rangle}_{1} \right|^2$$

$$= (\sin\vartheta)^2 .$$

$$\mathrm{Prob}\,(\text{state } |x\rangle \text{ is observed}) = \left| \langle x | \cos\vartheta\, |x\rangle + \sin\vartheta\, |y\rangle \right|^2$$

$$= (\cos\vartheta)^2 .$$

(V.3) Interference experiment by combining the previous two.



If we do this experiment we obsreve in D a sequence of clics/no clics:

$$(+1)\ (-1)\ (-1)\ (+1)\ (+1)\ (+1)\ \ldots (+1)\ (-1)\ \ldots$$

no clic.   clic

The observed experimental statistics is

$$\text{Prob}(\text{clic}) = (\cos(\theta-\alpha))^2 \, ; \; \text{Prob}(\text{No clic}) = (\sin(\theta-\alpha))^2$$

Note; an experiment with classical electromagnetic waves gives that the intensity of electric field collected in D is $\propto (\cos(\theta-\alpha))^2$.

Quantum analysis of experiment;

According to the postulates we are measuring the observable;

$$P_\alpha = (+1) \underbrace{|\alpha\rangle\langle\alpha|}_{\text{clic}} + (-1) \underbrace{|\alpha_\perp\rangle\langle\alpha_\perp|}_{\text{no clic.}}$$

- Initial state before crystal $|\theta\rangle$
- State in between $\cos\theta|x\rangle + \sin\theta|y\rangle$ with split paths
- Final state after second crystal $|\theta\rangle$

- Measurement: applying the Born rule,

$$\text{Prob}(+1) = \text{Prob}(|\alpha\rangle \text{ is observed})$$
$$= |\langle\alpha|\theta\rangle|^2 = (\cos(\theta-\alpha))^2$$

$$\text{Prob}(-1) = \text{Prob}(|\alpha_\perp\rangle \text{ is observed})$$
$$= |\langle\alpha_\perp|\theta\rangle|^2 = (\sin(\theta-\alpha))^2$$

#.

Classical analysis with classical "balls"

Classical balls would choose either

* the upper path $\rightarrow$ prob $(\sin\theta)^2$ say

or

* the lower path $\rightarrow$ prob $(\cos\theta)^2$ say.

Thus:

$$\text{Prob}(D\text{ clics}) = \text{Prob}(D\text{ clics} \mid \text{upper path})\ \text{Prob}(\text{upper path})$$
$$+$$
$$\text{Prob}(D\text{ clics} \mid \text{lower path})\ \text{Prob}(\text{lower path})$$

We have
$$\begin{cases} \text{Prob}(D\text{ clics} \mid \text{upper path}) = |\langle\alpha|y\rangle|^2 = (\sin\alpha)^2 \\ \text{Prob}(D\text{ clics} \mid \text{lower path}) = |\langle\alpha|x\rangle|^2 = (\cos\alpha)^2 \end{cases}$$

$$\Rightarrow \text{Prob}(D \text{ clics}) = \underset{\substack{\uparrow \\ \underline{\text{classical prediction}}}}{(\sin\alpha)^2(\sin\theta)^2 + (\cos\alpha)^2(\cos\theta)^2}$$

But the <u>quantum prediction</u> is

$$\text{Prob}(D \text{ clics}) = (\cos(\theta-\alpha))^2$$

$$= (\cos\theta\cos\alpha - \sin\theta\sin\alpha)^2$$

$$= (\cos\theta)^2(\cos\alpha)^2 + (\sin\theta)^2(\sin\alpha)^2$$

$$\underbrace{- 2\cos\theta\sin\theta\cos\alpha\sin\alpha}$$

interference term missed by the classical prediction.

The classical description misses the interference between the two crystals.

Exercise:

Do the same analysis with a dephaser on the upper path. Assume the dephaser acts as:

$$|y\rangle \to e^{i\lambda}|y\rangle \; ; \; |x\rangle \to |x\rangle$$

1) What is the state here?

2) Prob (D clics) = ?

Answer: Prob (D clics)

$$= \cos^2\theta \cos^2\alpha + \sin^2\theta \sin^2\alpha + 2\cos\theta\cos\alpha\sin\theta\sin\alpha\cos\lambda$$