Homework 3 - Solution
Introduction to quantum information processing

Exercise 1 *Orthonormal basis and measurement principle*

1) It involves the following checking:

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \big(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big)\\
&= \cos^2\alpha + \sin^2\alpha = 1\\
\langle\alpha_\perp|\alpha_\perp\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(-\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle\,\big)\\
&= \cos^2\alpha + \sin^2\alpha = 1\\
\langle\alpha_\perp|\alpha\rangle &= \big(-\sin\alpha\,\langle x| + \cos\alpha\,\langle y|\,\big)\big(\cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle\,\big)\\
&= -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha = 0
\end{aligned}$$

$$\begin{aligned}
\langle R|R\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle + i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)i) = 1\\
\langle L|L\rangle &= \frac{1}{2}\big(\,\langle x| + i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + i(-i)) = 1\\
\langle R|L\rangle &= \frac{1}{2}\big(\,\langle x| - i\,\langle y|\,\big)\big(\,|x\rangle - i\,|y\rangle\,\big) = \frac{1}{2}(1^2 + (-i)(-i)) = 0
\end{aligned}$$

2) For each experiment, the possible states just after the measurement would be the corresponding measurement basis with the following probabilities:

$$\begin{aligned}
\text{Prob}(|x\rangle) &= |\langle x|\psi\rangle|^2 = \cos^2\theta\\
\text{Prob}(|y\rangle) &= |\langle y|\psi\rangle|^2 = |(\sin\theta)e^{i\varphi}|^2 = \sin^2\theta
\end{aligned}$$

where we use $e^{i\varphi} = \cos\varphi + i\sin\varphi$ so that $|e^{i\varphi}|^2 = \cos^2\varphi + \sin^2\varphi = 1$. For the other probabilities we have:

$$\begin{aligned}
\text{Prob}(|R\rangle) &= |\langle R|\psi\rangle|^2\\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| - i\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2\\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}i\sin\theta e^{i\varphi}\right|^2\\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi - \frac{1}{\sqrt{2}}i\sin\theta\cos\varphi\right|^2\\
&= \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2\\
&= \frac{1}{2} + \cos\theta\sin\theta\sin\varphi
\end{aligned}$$

$$\begin{aligned}
\text{Prob}(|L\rangle) &= |\langle L|\psi\rangle|^2 \\
&= \left|\frac{1}{\sqrt{2}}\left(\langle x| + i\,\langle y|\right)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}i\sin\theta e^{i\varphi}\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi + \frac{1}{\sqrt{2}}i\sin\theta\cos\varphi\right|^2 \\
&= \left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\sin\varphi\right)^2 + \left(\frac{1}{\sqrt{2}}\sin\theta\cos\varphi\right)^2 \\
&= \frac{1}{2} - \cos\theta\sin\theta\sin\varphi
\end{aligned}$$

$$\begin{aligned}
\text{Prob}(|\alpha\rangle) &= |\langle\alpha|\psi\rangle|^2 \\
&= \left|(\cos\alpha\,\langle x| + \sin\alpha\,\langle y|)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|\cos\alpha\cos\theta + \sin\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi + i\sin\alpha\sin\theta\sin\varphi|^2 \\
&= (\cos\alpha\cos\theta + \sin\alpha\sin\theta\cos\varphi)^2 + (\sin\alpha\sin\theta\sin\varphi)^2 \\
&= \cos^2\alpha\cos^2\theta + 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \sin^2\alpha\sin^2\theta \\
\text{Prob}(|\alpha_\perp\rangle) &= |\langle\alpha_\perp|\psi\rangle|^2 \\
&= \left|(\sin\alpha\,\langle x| + \cos\alpha\,\langle y|)\left(\cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle\right)\right|^2 \\
&= \left|-\sin\alpha\cos\theta + \cos\alpha\sin\theta e^{i\varphi}\right|^2 \\
&= |-\sin\alpha\cos\theta + \cos\alpha\sin\theta\cos\varphi + i\cos\alpha\sin\theta\sin\varphi|^2 \\
&= (\sin\alpha\cos\theta - \cos\alpha\sin\theta\cos\varphi)^2 + (\cos\alpha\sin\theta\sin\varphi)^2 \\
&= \sin^2\alpha\cos^2\theta - 2\cos\alpha\sin\alpha\cos\theta\sin\theta\cos\varphi + \cos^2\alpha\sin^2\theta
\end{aligned}$$

One can verify that these probabilities are normalized to one,

$$\text{Prob}(|x\rangle) + \text{Prob}(|y\rangle) = \text{Prob}(|R\rangle) + \text{Prob}(|L\rangle) = \text{Prob}(|\alpha\rangle) + \text{Prob}(|\alpha_\perp\rangle) = 1.$$

Exercise 2 *Polarization observable and measurement principle*

1) In Homework 3, we have checked that $\langle\alpha|\alpha\rangle = \langle\alpha_\perp|\alpha_\perp\rangle = 1$ and $\langle\alpha|\alpha_\perp\rangle = \langle\alpha_\perp|\alpha\rangle = 0$. Therefore, we have

$$\begin{aligned}
\Pi_\alpha^2 &= |\alpha\rangle\,\langle\alpha|\alpha\rangle\,\langle\alpha| = |\alpha\rangle\,\langle\alpha| = \Pi_\alpha \\
\Pi_{\alpha_\perp}^2 &= |\alpha_\perp\rangle\,\langle\alpha_\perp|\alpha_\perp\rangle\,\langle\alpha_\perp| = |\alpha_\perp\rangle\,\langle\alpha_\perp| = \Pi_{\alpha_\perp} \\
\Pi_\alpha\Pi_{\alpha_\perp} &= |\alpha\rangle\,\langle\alpha|\alpha_\perp\rangle\,\langle\alpha_\perp| = 0 \\
\Pi_{\alpha_\perp}\Pi_\alpha &= |\alpha_\perp\rangle\,\langle\alpha_\perp|\alpha\rangle\,\langle\alpha| = 0
\end{aligned}$$

2)

$$| \langle\theta|\alpha\rangle |^2 = \langle\theta|\alpha\rangle \langle\theta|\alpha\rangle^* = \langle\theta|\alpha\rangle \langle\alpha|\theta\rangle = \langle\theta| \Pi_\alpha |\theta\rangle ,$$
$$| \langle\theta|\alpha_\perp\rangle |^2 = \langle\theta|\alpha_\perp\rangle \langle\theta|\alpha_\perp\rangle^* = \langle\theta|\alpha_\perp\rangle \langle\alpha_\perp|\theta\rangle = \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle$$

3) The probabilities are

$$\begin{aligned}\text{Prob}(p = +1) &= | \langle\alpha|\theta\rangle |^2 = | \cos\alpha\cos\theta + \sin\alpha\sin\theta|^2 = (\cos(\theta-\alpha))^2 \\ \text{Prob}(p = -1) &= | \langle\alpha_\perp|\theta\rangle |^2 = | -\sin\alpha\cos\theta + \cos\alpha\sin\theta|^2 = (\sin(\theta-\alpha))^2\end{aligned}$$

and they sum to 1,

$$\text{Prob}(p = +1) + \text{Prob}(p = -1) = (\cos(\theta-\alpha))^2 + (\sin(\theta-\alpha))^2 = 1$$

4) The expectation is

$$\begin{aligned}\text{E}[p] &= (+1)\text{Prob}(p = +1) + (-1)\text{Prob}(p = -1) \\ &= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\ &= \cos\big(2(\theta-\alpha)\big)\end{aligned}$$

and the variance is

$$\begin{aligned}\text{Var}(p) &= \text{E}[p^2] - (\text{E}[p])^2 \\ &= 1 - (\text{E}[p])^2 \\ &= (\cos(\theta-\alpha))^2 - (\sin(\theta-\alpha))^2 \\ &= 1 - \big(\cos(2(\theta-\alpha))\big)^2 \\ &= \big(\sin(2(\theta-\alpha))\big)^2\end{aligned}$$

In fact they should match with the computation in Dirac notation because

$$\begin{aligned}\langle\theta| P_\alpha |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big) |\theta\rangle \\ &= (+1) \langle\theta| \Pi_\alpha |\theta\rangle + (-1) \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\ &= (+1)\text{Prob}(p = +1) + (-1)\text{Prob}(p = -1) \\ &= \text{E}[p]\end{aligned}$$

and

$$\begin{aligned}\langle\theta| P_\alpha^2 |\theta\rangle &= \langle\theta| \big((+1)\Pi_\alpha + (-1)\Pi_{\alpha_\perp}\big)^2 |\theta\rangle \\ &= \langle\theta| \big(\Pi_\alpha^2 - \Pi_\alpha\Pi_{\alpha_\perp} - \Pi_{\alpha_\perp}\Pi_\alpha + \Pi_{\alpha_\perp}^2\big) |\theta\rangle \\ &= \langle\theta| \big(\Pi_\alpha + \Pi_{\alpha_\perp}\big) |\theta\rangle \\ &= (+1)^2 \langle\theta| \Pi_\alpha |\theta\rangle + (-1)^2 \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle \\ &= (+1)^2\text{Prob}(p = +1) + (-1)^2\text{Prob}(p = -1) \\ &= \text{E}[p^2]\end{aligned}$$

thereby giving $\text{E}[p] = \langle\theta| P_\alpha |\theta\rangle$ and $\text{Var}(p) = \text{E}[p^2] - (\text{E}[p])^2 = \langle\theta| P_\alpha^2 |\theta\rangle - \langle\theta| P_\alpha |\theta\rangle^2$.

Exercise 3 *Interferometer with an atom on one path*

1) The matrices in Dirac notation are

$$S = \frac{1}{\sqrt{2}} |H\rangle \langle H| + \frac{1}{\sqrt{2}} |H\rangle \langle V| + \frac{1}{\sqrt{2}} |V\rangle \langle H| - \frac{1}{\sqrt{2}} |V\rangle \langle V| + |\text{abs}\rangle \langle \text{abs}|$$
$$R = |H\rangle \langle V| + |V\rangle \langle H| + |\text{abs}\rangle \langle \text{abs}| .$$

To find $U = SARS$ we proceed by steps:

$$RS = \frac{1}{\sqrt{2}} |H\rangle \langle H| - \frac{1}{\sqrt{2}} |H\rangle \langle V| + \frac{1}{\sqrt{2}} |V\rangle \langle H| + \frac{1}{\sqrt{2}} |V\rangle \langle V| + |\text{abs}\rangle \langle \text{abs}| ,$$
$$ARS = |H\rangle \langle \text{abs}| + \frac{1}{\sqrt{2}} |V\rangle \langle H| + \frac{1}{\sqrt{2}} |V\rangle \langle V| + \frac{1}{\sqrt{2}} |\text{abs}\rangle \langle H| - \frac{1}{\sqrt{2}} |\text{abs}\rangle \langle V|$$

and finally

$$\begin{aligned} U = SARS = & \frac{1}{2} |H\rangle \langle H| + \frac{1}{2} |H\rangle \langle V| + \frac{1}{\sqrt{2}} |H\rangle \langle \text{abs}| \\ & - \frac{1}{2} |V\rangle \langle H| - \frac{1}{2} |V\rangle \langle V| + \frac{1}{\sqrt{2}} |V\rangle \langle \text{abs}| \\ & + \frac{1}{\sqrt{2}} |\text{abs}\rangle \langle H| - \frac{1}{\sqrt{2}} |\text{abs}\rangle \langle V| . \end{aligned}$$

2) As $SARS |H\rangle = \frac{1}{2} |H\rangle - \frac{1}{2} |V\rangle + \frac{1}{\sqrt{2}} |\text{abs}\rangle$, the probabilities of the three events are

$$\text{Prob}(D_1) = | \langle V| SARS |H\rangle |^2 = \frac{1}{4},$$
$$\text{Prob}(D_2) = | \langle H| SARS |H\rangle |^2 = \frac{1}{4},$$
$$\text{Prob}(\text{abs}) = | \langle \text{abs}| SARS |H\rangle |^2 = \frac{1}{2},$$

which sum to 1. Importantly, we see that the "interference effects" at the detectors have disappeared since we have equal probability of detection in the two detectors. This is due to the presence of the atom on one of the possible paths.

3) A legitimate matrix has to be unitary. The first matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is not unitary because

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I.$$

The second matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is unitary because

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Thus the second matrix may model the absorption and reemission of the photon. Note also that this matrix acts like a Hadamard matrix on the subspace $\{|H\rangle, |\text{abs}\rangle\}$.

Exercise 4 *No-cloning theorem*

Proof using unitarity of U

We have

$$U|\Psi_1\rangle \otimes |\text{blank}\rangle = |\Psi_1\rangle \otimes |\Psi_1\rangle, \quad U|\Psi_2\rangle \otimes |\text{blank}\rangle = |\Psi_2\rangle \otimes |\Psi_2\rangle$$

Taking the Dirac conjugate of the second equation

$$\langle\Psi_2| \otimes \langle\text{blank}|U^\dagger = |\Psi_2\rangle \otimes |\Psi_2\rangle$$

Now we make the inner product between the two equations, or other words form the Dirac bracket:

$$\langle\Psi_2| \otimes \langle\text{blank}|U^\dagger U|\Psi_1\rangle \otimes |\text{blank}\rangle = |\Psi_1\rangle \otimes |\Psi_1\rangle|\Psi_2\rangle \otimes |\Psi_2\rangle$$

Using $U^\dagger U = I$ and the rules of the inner product in the tensor product space we get

$$\langle\Psi_2|\Psi_1\rangle\langle\text{blank}|\text{blank}\rangle = (\langle\Psi_2|\Psi_1\rangle)^2$$

Thus

$$\langle\Psi_2|\Psi_1\rangle(1 - \langle\Psi_2|\Psi_1\rangle) = 0$$

This can only be true if $|\Psi\rangle_1 = |\Psi_2\rangle$ or $|\Psi_1\rangle \perp |\Psi_1\rangle$.

Proof using linearity of U

For $|\Psi\rangle = |0\rangle$, the machine should give $U|0\rangle \otimes |\text{blank}\rangle = |0\rangle \otimes |0\rangle$. For $|\Psi\rangle = |1\rangle$, the machine should give $U|1\rangle \otimes |\text{blank}\rangle = |1\rangle \otimes |1\rangle$. The first claim follows by linearity,

$$\begin{aligned} U(\alpha|0\rangle + \beta|1\rangle) \otimes |\text{blank}\rangle &= \alpha U|0\rangle \otimes |\text{blank}\rangle + \beta U|1\rangle \otimes |\text{blank}\rangle \\ &= \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle\,. \end{aligned}$$

The second claim is based on applying directly the definition to $|\Psi\rangle$,

$$\begin{aligned} U(\alpha|0\rangle + \beta|1\rangle) \otimes |\text{blank}\rangle &= (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) \\ &= \alpha^2|0\rangle \otimes |0\rangle + \alpha\beta|0\rangle \otimes |1\rangle + \alpha\beta|1\rangle \otimes |0\rangle + \beta^2|1\rangle \otimes |1\rangle\,. \end{aligned}$$

The two equations are equivalent when $(\alpha, \beta) = (0, 1)$ or $(\alpha, \beta) = (1, 0)$, which corresponds to two orthogonal input states $|\Psi\rangle = |0\rangle$ and $|1\rangle$. This means that it is possible to copy two orthogonal states with an appropriate machine U but no cloning is possible when the set of input states is not orthogonal.