Homework 3 - graded
Introduction to quantum information processing COM 309

Exercise 1 *Orthonormal basis and measurement principle*

Let $\{|x\rangle, |y\rangle\}$ an orthonormal basis of $\mathbb{C}^2$. This means that $\langle x|x\rangle = \langle y|y\rangle = 1$ and $\langle x|y\rangle = \langle y|x\rangle = 0$. Let $|\alpha\rangle = \cos\alpha\,|x\rangle + \sin\alpha\,|y\rangle$, $|\alpha_\perp\rangle = -\sin\alpha\,|x\rangle + \cos\alpha\,|y\rangle$, $|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle)$, $|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$.

1) Check that $\{|\alpha\rangle, |\alpha_\perp\rangle\}$ and $\{|R\rangle, |L\rangle\}$ are two orthonormal basis.

2) We measure the polarization with three different measurement apparatus. The first apparatus is modeled by the basis $\{|x\rangle, |y\rangle\}$; the second one is modeled by the basis $\{|R\rangle, |L\rangle\}$; and the third one by the basis $\{|\alpha\rangle, |\alpha_\perp\rangle\}$. Let

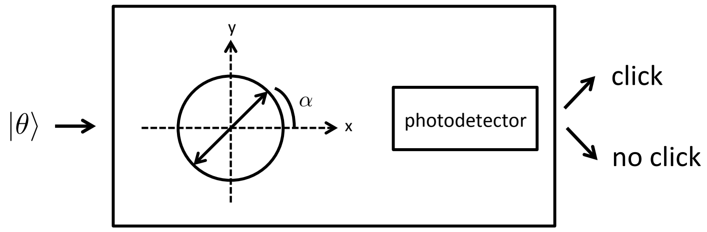
$$|\psi\rangle = \cos\theta\,|x\rangle + (\sin\theta)e^{i\varphi}\,|y\rangle$$

be the polarized state of a photon just before the measurement. For each of the three experiments, give the (two) possible outcoming states just after the measurement and their corresponding probabilities of outcome. *You are asked to give the probabilities in terms of real quantities (i.e., compute the complex modulus) and completely expand the squares of trigonometric expressions.*

Exercise 2 *Polarization observable and measurement principle*

Consider the "measurement apparatus" (in the below figure) constituted of "an analyzer and a detector". The incoming (initial) state of the photon is linearly polarized:

$$|\theta\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle\,.$$



When the photodetector clicks we record $+1$ and when it does not click we record -1. Thus the "polarization observable" is represented by the 2×2 matrix

$$P_\alpha = (+1) |\alpha\rangle \langle\alpha| + (-1) |\alpha_\perp\rangle \langle\alpha_\perp|$$

where $|\alpha\rangle = \cos\alpha |x\rangle + \sin\alpha |y\rangle$ and $|\alpha_\perp\rangle = -\sin\alpha |x\rangle + \cos\alpha |y\rangle$ are the two vectors of the measurement basis. Note that the two orthogonal projectors of the measurement basis are $\Pi_\alpha = |\alpha\rangle \langle\alpha|$ and $\Pi_{\alpha_\perp} = |\alpha_\perp\rangle \langle\alpha_\perp|$.

1) Show that $\Pi_\alpha^2 = \Pi_\alpha$, $\Pi_{\alpha_\perp}^2 = \Pi_{\alpha_\perp}$ and $\Pi_\alpha \Pi_{\alpha_\perp} = \Pi_{\alpha_\perp} \Pi_\alpha = 0$.

2) Check the following formulas:

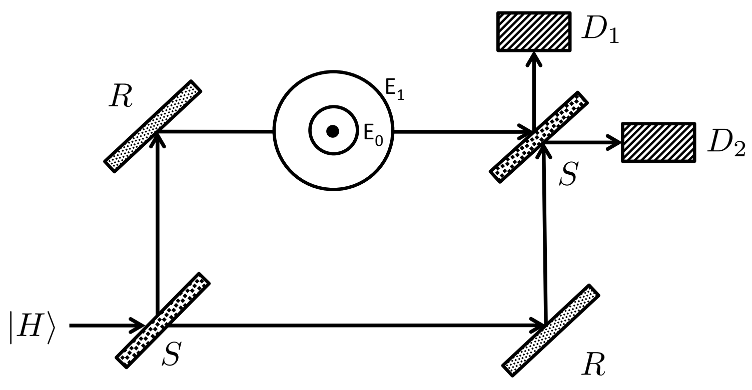
$$| \langle\theta|\alpha\rangle |^2 = \langle\theta| \Pi_\alpha |\theta\rangle \,,$$
$$| \langle\theta|\alpha_\perp\rangle |^2 = \langle\theta| \Pi_{\alpha_\perp} |\theta\rangle$$

3) Let $p = \pm 1$ the random variable corresponding to the event click / no-click of the detector. Express $\mathrm{Prob}(p = \pm 1)$ with simple trigonometric functions and check that the two probabilities sum to one.

4) Deduce from 3) $\mathbb{E}(p)$ and $\mathrm{Var}(p)$ and check that you find the same expressions by directly computing $\langle\theta| P_\alpha |\theta\rangle$ and $\langle\theta| P_\alpha^2 |\theta\rangle - \langle\theta| P_\alpha |\theta\rangle^2$ in Dirac notation.

Exercise 3 *Interferometer with an atom on the upper path*

Consider the following set-up where an atom may absorb the photon on the upper arm of the interferometer. We want to show that teh usual interference phenomenon is destroyed in this setting.



The Hilbert space of the photon is here $\mathbb{C}^3$ with basis states

$$|H\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |V\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |\mathrm{abs}\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

the semi-transparent and reflecting mirrors are modeled by the unitary matrices

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the "absorption-reemission" process[1] is modeled by the unitary matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that in Dirac notation

$$A = |H\rangle \langle \text{abs}| + |V\rangle \langle V| + |\text{abs}\rangle \langle H|$$

This models three possible transitions: $A\,|H\rangle = |\text{abs}\rangle$ (absorption); $A\,|\text{abs}\rangle = |H\rangle$ (emission); and $A\,|V\rangle = |V\rangle$ (nothing happens).

1) Write down all matrices in Dirac notation and then compute the unitary operator $U = SARS$ representing the total evolution process of this interferometer.

2) Given that the initial state is $|H\rangle$, what is the state after the second semi-transparent mirror? What are the probabilities of the following three events: click in D_1; or click in D_2; or no clicks in D_1 nor D_2? Verify the probabilities sum to to 1.

3) Suppose the photon-atom interaction is not absorption-reemission but some other process modeled by a matrix. Which of the two following matrices would be legitimate in QM?

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

and why?

Exercise 4 *No-cloning theorem*

The no cloning theorem says that in quantum mechanics there doesnt exist a "universal machine" which copies states. This is mathematically formulated as follows:

Theorem: Let $\mathcal{H}$ a Hilbert space of states $|\Psi\rangle$ to be copied and a copy of this Hilbert space where copies should be stored. The total Hilbert space is thus $\mathcal{H} \otimes \mathcal{H}$. Let U be an operation (the universal machine) such that

$$U|\Psi\rangle \otimes |\text{blank}\rangle = |\Psi\rangle \otimes |\Psi\rangle$$

[1] On the picture E_0 and E_1 are two energy levels of the atom corresponding to ground state and excited state; but you can ignore this aspect in this problem.

The theorem states that an operation U independent of $|\Psi\rangle$ does not exist, if it is linear or if it is unitary.

Important remark: If we consider a set of orthonormal states (so not any states) the theorem is evaded. In other words it is possible to construct a unitary U which copies the elements of an orthonormal basis. But note that the U depends on the basis.

In this exercise we guide you through two different proofs. One that uses unitarity and the other one that uses only linearity of U.

- *First proof using unitarity*

  Use unitarity $U^\dagger U = I$ to prove that two non-orthogonal states $|\Psi_1\rangle$, $|\Psi_2\rangle$ cannot be copied by the same U. *Hint*: proceed by contradiction and consider the inner product between the above equation and its Dirac conjugate.

- *Second proof using linearity*

  For simplicity let us assume that $\mathcal{H} = \mathbf{C}^2$ and $|\Psi_1\rangle = |0\rangle$, $|\Psi_2\rangle = |1\rangle$ (some thought shows that this is not loss of generality for the argument). Suppose a common copying machine U exists for all inputs $|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. Show that on one hand:

  $$U\,|\Psi\rangle \otimes |\text{blank}\rangle = \alpha\,|0\rangle \otimes |0\rangle + \beta\,|1\rangle \otimes |1\rangle\,.$$

  and show that on the other hand:

  $$U\,|\Psi\rangle \otimes |\text{blank}\rangle = \alpha^2\,|0\rangle \otimes |0\rangle + \alpha\beta\,|0\rangle \otimes |1\rangle + \alpha\beta\,|1\rangle \otimes |0\rangle + \beta^2\,|1\rangle \otimes |1\rangle\,.$$

  What are the only possible values of α and β for which there is no contradiction ? Conclude.