Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 4 – Solutions

Exercise 1: Let M, N and P be smooth manifolds, let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps, and let $p \in M$. Prove the following assertions:

(a) The map $dF_p\colon T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p\colon T_pM \to T_{(G\circ F)(p)}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p\colon T_pM \to T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Solution:

(a) Let $v, w \in T_pM$ and $\lambda, \mu \in \mathbb{R}$. For any $f \in C^\infty(N)$, we have

$$\begin{aligned} dF_p(\lambda v + \mu w)(f) &= (\lambda v + \mu w)(f \circ F) \\ &= \lambda\, v(f \circ F) + \mu\, w(f \circ F) \\ &= \lambda\, dF_p(v)(f) + \mu\, dF_p(w)(f) \\ &= \big(\lambda\, dF_p(v) + \mu\, dF_p(w)\big)(f), \end{aligned}$$

which implies

$$dF_p(\lambda v + \mu w) = \lambda\, dF_p(v) + \mu\, dF_p(w).$$

(b) For any $v \in T_pM$ and any $f \in C^\infty(P)$, we have

$$\begin{aligned} d(G \circ F)_p(v)(f) = v\big(f \circ (G \circ F)\big) &= v\big((f \circ G) \circ F\big) \\ &= dF_p(v)(f \circ G) \\ &= dG_{F(p)}\big(dF_p(v)\big)(f) \\ &= \big(dG_{F(p)} \circ dF_p\big)(v)(f), \end{aligned}$$

and thus

$$d(G \circ F)_p(v) = \big(dG_{F(p)} \circ dF_p\big)(v),$$

which yields the assertion.

(c) For any $v \in T_pM$ and any $f \in C^\infty(M)$, we have

$$d(\mathrm{Id}_M)_p(v)(f) = v\big(f \circ \mathrm{Id}_M\big) = v(f),$$

and hence

$$d(\mathrm{Id}_M)_p(v) = v = \mathrm{Id}_{T_pM}(v),$$

which proves the claim.

(d) Since F is a diffeomorphism, we have

$$F \circ F^{-1} = \mathrm{Id}_N \quad \text{and} \quad F^{-1} \circ F = \mathrm{Id}_M,$$

so by (b) and (c) we obtain

$$\mathrm{Id}_{T_pM} = d(\mathrm{Id}_M)_p = d\left(F^{-1} \circ F\right)_p = d\left(F^{-1}\right)_{F(p)} \circ dF_p$$

and

$$\mathrm{Id}_{T_{F(p)}N} = d(\mathrm{Id}_N)_{F(p)} = d\left(F \circ F^{-1}\right)_{F(p)} = dF_p \circ d\left(F^{-1}\right)_{F(p)}.$$

Hence, dF_p is an $\mathbb{R}$-linear isomorphism with inverse

$$(dF_p)^{-1} = d\left(F^{-1}\right)_{F(p)}.$$

Remark. For those familiar with categorical language, let us put *Exercise* 1 into context. Let $\mathbf{Man}_*^\infty$ be the category of pointed smooth manifolds, i.e., the category whose objects are pairs (M, p), where M is a smooth manifold and $p \in M$, and whose morphisms $F\colon (M,p) \to (N,q)$ are smooth maps $F\colon M \to N$ with $F(p) = q$. Denote by $\mathbf{Vect}_\mathbb{R}$ the category of $\mathbb{R}$-vector spaces. Parts (a), (b) and (c) of the above exercise show that the assignment $T\colon \mathbf{Man}_*^\infty \to \mathbf{Vect}_\mathbb{R}$, which to a pointed smooth manifold (M,p) assigns the tangent space $T(M,p) = T_pM$ and which to a smooth map $F\colon (M,p) \to (N,q)$ assigns the differential $T(F) = dF_p$ of F at p, is a covariant functor. It is a general fact that functors send isomorphisms to isomorphisms, and that $T(F^{-1}) = T(F)^{-1}$, which is why part (d) of *Exercise* 1 is a formal consequence of the previous parts.

Exercise 2 (*The tangent space to a vector space*)**:** Let V be a finite-dimensional $\mathbb{R}$-vector space with its standard smooth manifold structure, see [*Exercise Sheet* 2*, Exercise* 2]. Fix a point $a \in V$.

(a) For each $v \in V$ define a map

$$D_v\big|_a \colon C^\infty(V) \longrightarrow \mathbb{R},\ f \mapsto \left.\frac{d}{dt}\right|_{t=0} f(a + tv).$$

Show that $D_v\big|_a$ is a derivation at a.

(b) Show that the map

$$V \to T_aV,\ v \mapsto D_v\big|_a$$

is a canonical isomorphism, such that for any linear map $L\colon V \to W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle dL_a} \\ W & \xrightarrow{\cong} & T_{La}W. \end{array}$$

Solution:

(a) Choose a basis $E_1, \dots, E_n$ of V and let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Let $\varphi\colon \mathbb{R}^n \to V$ be the induced isomorphism, which is a diffeomorphism by definition of the standard smooth structure and by *Example 2.14*(2). Let $\vec{a} := \varphi^{-1}(a)$ and $\vec{v} := \varphi^{-1}(v)$. By *Exercise* 1(d) the differential $d\varphi_{\vec{a}}\colon T_{\vec{a}}\mathbb{R}^n \to T_aV$ is an $\mathbb{R}$-linear isomorphism.

As shown in the lecture, the map

$$\widehat{D}_{\vec{v}}\big|_{\vec{a}}\colon C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R},\ f \mapsto \frac{d}{dt}\bigg|_{t=0} f(\vec{a} + t\vec{v})$$

is a derivation of $C^\infty(\mathbb{R}^n)$ at $\vec{a}$. Let us now prove that $d\varphi_{\vec{a}}\big(\widehat{D}_{\vec{v}}\big|_{\vec{a}}\big) = D_v\big|_a$ as functions from $C^\infty(V)$ to $\mathbb{R}$, thereby proving that $D_v\big|_a$ is a derivation of $C^\infty(V)$ at a, as $d\varphi_{\vec{a}}\big(\widehat{D}_{\vec{v}}\big|_{\vec{a}}\big)$ is so. To this end, let $f \in C^\infty(V)$. Then

$$d\varphi_{\vec{a}}\big(\widehat{D}_{\vec{v}}\big|_{\vec{a}}\big)(f) = \widehat{D}_{\vec{v}}\big|_{\vec{a}}(f \circ \varphi) = \frac{d}{dt}\bigg|_{t=0} (f \circ \varphi)(\vec{a} + t\vec{v}) = \frac{d}{dt}\bigg|_{t=0} f(a + tv) = D_v\big|_a(f).$$

As f was arbitrary, we conclude that $d\varphi_{\vec{a}}\left(\widehat{D}_{\vec{v}}\big|_{\vec{a}}\right) = D_v\big|_a$, which yields the assertion.

(b) Denote by $\eta_{(V,a)}$ the map $V \to T_aV,\ v \mapsto D_v\big|_a$. In part (a) we proved that

$$d\varphi_{\vec{a}} \circ \eta_{(\mathbb{R}^n,\vec{a})}(\vec{v}) = \eta_{(V,\varphi(\vec{a}))} \circ \varphi(\vec{v})$$

for all $\vec{a}, \vec{v} \in \mathbb{R}^n$. In other words, we have $d\varphi_{\vec{a}} \circ \eta_{(\mathbb{R}^n,\vec{a})} = \eta_{(V,\varphi(\vec{a}))} \circ \varphi$. In particular, since in *Proposition 3.3*(b) we already saw that $\eta_{(\mathbb{R}^n,\vec{a})}$ is an isomorphism, and as $d\varphi_{\vec{a}}$ and φ are isomorphisms as well, we conclude that $\eta_{(V,\varphi(\vec{a}))}$ is an isomorphism.

It remains to check the above diagram commutes. Firstly, since L is linear, it is in particular smooth (all first order partial derivatives with respect to some basis exist and are constant, and all higher order partial derivatives vanish). Now, let $v \in V$ and $f \in C^\infty(W)$ be arbitrary. We have

$$\begin{aligned}\big(dL_a \circ \eta_{(V,a)}(v)\big)(f) &= dL_a\left(D_v\big|_a\right)(f) = D_v\big|_a(f \circ L) \\ &= \frac{d}{dt}\bigg|_{t=0} f\big(L(a + tv)\big) = \frac{d}{dt}\bigg|_{t=0} f(La + tLv) = D_{Lv}\big|_{La}(f) \\ &= \eta_{(W,La)}(Lv)(f) = \big(\eta_{(W,La)} \circ L(v)\big)(f).\end{aligned}$$

As v and f were arbitrary, we conclude that

$$dL_a \circ \eta_{(V,a)} = \eta_{(W,La)} \circ L;$$

in other words, the diagram in part (b) is commutative.

Remark. It is important to understand that each isomorphism $V \cong T_a V$ is canonically defined, independently of any choice of basis (notwithstanding the fact that we used a choice of basis to prove that it is an isomorphism). Because of this result, we can routinely *identify* tangent vectors to a finite-dimensional vector space with elements of the space itself.

More generally, if M is an open submanifold of an $\mathbb{R}$-vector space V, we can combine our identifications $T_pM \leftrightarrow T_pV \leftrightarrow V$ to obtain a canonical identification of each tangent space to M with V. For example, since $\mathrm{GL}(n,\mathbb{R})$ is an open submanifold of the $\mathbb{R}$-vector space $M(n,\mathbb{R})$, see [*Exercise Sheet* 2, *Exercise* 3], we can identify its tangent space at each point $X \in \mathrm{GL}(n,\mathbb{R})$ with the full space of matrices $M(n,\mathbb{R})$.

Remark. For those familiar with categorical language, let us put *Exercise* 2 into context. The category $\mathbf{Man}_*^\infty$ of pointed smooth manifolds described in the previous remark has the category $\mathbf{Vect}_{\mathbb{R},*}$ of pointed vector spaces as a subcategory (but not as a *full* subcategory, since only linear maps between pointed vector spaces are considered). Therefore, the tangent space yields a functor $T\colon \mathbf{Vect}_{\mathbb{R},*} \to \mathbf{Vect}_{\mathbb{R}}$ by restricting to this subcategory. But there is also another natural functor between these two categories, namely the forgetful functor $U\colon \mathbf{Vect}_{\mathbb{R},*} \to \mathbf{Vect}_{\mathbb{R}}$ which to a pointed vector space (V,a) associates the underlying vector space V, and to a linear map $L\colon (V,a) \to (W,b)$ (i.e., a linear map with $La = b$) associates the linear map $L\colon V \to W$. In the preceding exercise, we showed that $\eta_\bullet$ is a natural transformation from U to T (by showing that the given diagram commutes), and in fact that it is a natural isomorphism (by showing that each individual map $\eta_{(V,a)}\colon U(V,a) \to T(V,a)$ is an isomorphism).

Exercise 3 (*The tangent space to a product manifold*)**:** Let $M_1, \ldots, M_k$ be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \ldots, k\}$, let

$$\pi_j\colon M_1 \times \ldots \times M_k \to M_j$$

be the projection onto the j-th factor M_j. Show that for any point $p = (p_1, \ldots, p_k) \in M_1 \times \ldots \times M_k$, the map

$$\begin{aligned} \alpha\colon T_p\big(M_1 \times \ldots \times M_k\big) &\longrightarrow T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k \\ v &\mapsto \big(d(\pi_1)_p(v), \ldots, d(\pi_k)_p(v)\big) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism.

Solution: The map α is $\mathbb{R}$-linear; indeed, this follows readily from the fact that every component $d(\pi_j)_p$ is $\mathbb{R}$-linear. Note also that both vector spaces have dimension $\sum_i \dim M_i$. Thus, to show that α is an isomorphism, it suffices to prove that it is surjective. We will achieve this by constructing a right-inverse to α.

To this end, for each $1 \leq j \leq k$, define the map

$$\begin{aligned} \iota_j\colon M_j &\to M_1 \times \ldots \times M_k \\ m_j &\mapsto (p_1, \ldots, p_{j-1}, m_j, p_{j+1}, \ldots, p_k). \end{aligned}$$

By part (b) of [*Exercise Sheet* 3, *Exercise* 4] we infer that ι_j is smooth, because $\pi_{j'} \circ \iota_j$ is either constant or the identity (so in particular smooth) for all $1 \leq j' \leq k$, with $\iota_j(p_j) = p$, so we obtain a map

$$d(\iota_j)_{p_j}\colon T_{p_j}M_j \to T_p(M_1 \times \ldots \times M_k).$$

We now define the following map:

$$\begin{aligned}\beta\colon T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k &\to T_p\big(M_1 \times \ldots \times M_k\big)\\ (v_1, \ldots, v_k) &\mapsto d(\iota_1)_{p_1}(v_1) + \ldots + d(\iota_k)_{p_k}(v_k).\end{aligned}$$

We will show that β is a right-inverse for α. To this end, let

$$(v_1, \ldots, v_k) \in T_{p_1}M_1 \oplus \ldots \oplus T_{p_k}M_k.$$

Then

$$\alpha \circ \beta(v_1, \ldots, v_k) = \alpha\left(\sum_j d(\iota_j)_{p_j}(v_j)\right) = \sum_j \alpha\left(d(\iota_j)_{p_j}(v_j)\right). \tag{$*$}$$

Now, let $1 \leq i, j \leq k$ be arbitrary. Note that

$$d(\pi_i)_p\left(d(\iota_j)_{p_j}(v_j)\right) = d(\pi_i \circ \iota_j)_{p_j}(v_j) = \delta_{ij}v_j, \tag{$**$}$$

because if $i \neq j$, then $\pi_i \circ \iota_j$ is constant and thus has 0 differential by *Lemma 3.5*(a) (see also [*Exercise Sheet* 5, *Exercise* 5]), and if $i = j$, then $\pi_i \circ \iota_j = \mathrm{Id}_{M_j}$ and thus its differential is the identity by *Exercise* 1(c). Thus, by $(*)$ and $(**)$ we obtain

$$(\alpha \circ \beta)(v_1, \ldots, v_k) = \sum_j (\delta_{1j}v_1, \ldots, \delta_{kj}v_k) = (v_1, \ldots, v_k),$$

and since $(v_1, \ldots, v_k)$ was arbitrary, we conclude that $\alpha \circ \beta = \mathrm{Id}$. It follows that α is surjective, and hence an isomorphism, as explained above.

Remark. Since the isomorphism α in *Exercise* 3 is canonically defined, independently of any choice of coordinates, we can consider it as a canonical identification, and we will always do so. Thus, for example, we identify $T_{(p,q)}(M \times N)$ with $T_pM \oplus T_qN$, and treat both T_pM and T_qN as subspaces of $T_{(p,q)}(M \times N)$.

Exercise 4 (*Tangent vectors as derivations of the space of germs*)**:** Let M be a smooth manifold and let p be a point of M.

(a) Consider the set $\mathcal{S}$ of ordered pairs (U, f), where U is an open subset of M containing p and $f\colon U \to \mathbb{R}$ is a smooth function. Define on $\mathcal{S}$ the following relation:

$$(U, f) \sim (V, g) \quad \text{if } f \equiv g \text{ on some open neighborhood of } p.$$

Show that $\sim$ is an equivalence relation on $\mathcal{S}$. The equivalence class of an ordered pair (U, f) is denoted by $[(U, f)]$ or simply by $[f]_p$ and is called *the germ of f at p*.

(b) The set of all germs of smooth functions at p is denoted by $C^\infty_p(M)$. Show that $C^\infty_p(M)$ is an $\mathbb{R}$-vector space and an associative $\mathbb{R}$-algebra under the operations

$$\begin{aligned} c[(U, f)] &= [(U, cf)], \text{ where } c \in \mathbb{R},\\ [(U, f)] + [(V, g)] &= [(U \cap V, f + g)],\\ [(U, f)][(V, g)] &= [(U \cap V, fg)].\end{aligned}$$

(c) A *derivation* of $C_p^\infty(M)$ is an $\mathbb{R}$-linear map $v\colon C_p^\infty(M) \to \mathbb{R}$ satisfying the following product rule:

$$v[fg]_p = f(p)v[g]_p + g(p)v[f]_p.$$

The set of derivations of $C_p^\infty(M)$ is denoted by $\mathcal{D}_pM$.

(i) Show that $\mathcal{D}_pM$ is an $\mathbb{R}$-vector space.

(ii) Show that the map

$$\Phi\colon \mathcal{D}_pM \to T_pM,\ \Phi(v)(f) = v[f]_p$$

is an isomorphism.

Solution:

(a) Straightforward.

(b) Straightforward. Note that the zero element of the $\mathbb{R}$-vector space (or the associative and commutative $\mathbb{R}$-algebra) $C_p^\infty(M)$ is the class $[(M,\mathbb{O})]$, where

$$\mathbb{O}\colon M \to \mathbb{R},\ x \mapsto 0$$

is the constant function with value 0 on M, which is clearly smooth by [*Exercise Sheet* 3, *Exercise* 3], and the unit of the $\mathbb{R}$-algebra $C_p^\infty(M)$ is the class $[(M,\mathbb{I})]$, where

$$\mathbb{I}\colon M \to \mathbb{R},\ x \mapsto 1$$

is the constant function with value 1 on M, which is smooth again by [*Exercise Sheet* 3, *Exercise* 3].

(c) We first prove (i). Clearly, it suffices to show that $\mathcal{D}_pM$ is a vector subspace of the vector space of linear maps $C_p^\infty(M) \to \mathbb{R}$ (the dual of $C_p^\infty(M)$). In other words, if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathcal{D}_pM$, we have to show that $\lambda_1 v_1 + \lambda_2 v_2$ satisfies the product rule. To this end, let $[f]_p, [g]_p \in C_p^\infty(M)$ be arbitrary. Then

$$\begin{aligned}(\lambda_1 v_1 + \lambda_2 v_2)\big([fg]_p\big) &= \lambda_1 v_1\big([fg]_p\big) + \lambda_2 v_2\big([fg]_p\big)\\ &= \lambda_1\big(f(p)v_1[g]_p + g(p)v_1[f]_p\big) + \lambda_2\big(f(p)v_2[g]_p + g(p)v_2[f]_p\big)\\ &= f(p)(\lambda_1 v_1 + \lambda_2 v_2)\big([g]_p\big) + g(p)(\lambda_1 v_1 + \lambda_2 v_2)\big([f]_p\big).\end{aligned}$$

Hence, $\lambda_1 v_1 + \lambda_2 v_2 \in \mathcal{D}_pM$.

We now prove (ii). First of all, the assertion that $\Phi(v)\colon C^\infty(M) \to \mathbb{R}$ is a derivation follows from the fact that

$$\begin{aligned}[\bullet]_p\colon C^\infty(M) &\to C_p^\infty(M)\\ f &\mapsto [f]_p\end{aligned}$$

is a homomorphism of $\mathbb{R}$-algebras, and thus if $v \in \mathcal{D}_pM$ is a derivation of $C_p^\infty(M)$, then $\Phi(v) = v \circ [\bullet]_p$ is a derivation of $C^\infty(M)$. Furthermore, Φ is $\mathbb{R}$-linear because it is given by precomposition with $[\bullet]_p$ (so pointwise addition and scalar multiplication are obviously preserved). Therefore, it remains to show that Φ is an isomorphism. To this end, define the map

$$\begin{aligned} \Psi\colon T_pM &\to \mathcal{D}_pM \\ v &\mapsto \Big([f]_p \in C_p^\infty(M) \mapsto \Psi(v)\big([f]_p\big) := v(\widetilde{f}) \in \mathbb{R} \Big) \end{aligned}$$

where for $[f]_p \in C_p^\infty(M)$ we denote by $\widetilde{f} \in C^\infty(M)$ some smooth function defined on all of M such that $[f]_p = [\widetilde{f}]_p$, which exists due to the *extension lemma*. Note that the value $v(\widetilde{f})$ is well-defined for $[f]_p$ thanks to *Proposition 3.8*. Moreover, one readily checks that $\Psi(v)$ is indeed a derivation of $C_p^\infty(M)$. Now, let us show that Φ and Ψ are mutually inverse. Indeed, given $v \in T_pM$ and $f \in C^\infty(M)$, we have

$$\big(\Phi \circ \Psi(v)\big)(f) = \Psi(v)\big([f]_p\big) = v(\widetilde{f}) = v(f),$$

and thus $\Phi \circ \Psi = \mathrm{Id}$; conversely, given $v \in \mathcal{D}_pM$ and $[f]_p \in C_p^\infty(M)$, we have

$$\big(\Psi \circ \Phi(v)\big)\big([f]_p\big) = \Phi(v)(\widetilde{f}) = v[\widetilde{f}]_p = v[f]_p,$$

and hence $\Psi \circ \Phi = \mathrm{Id}$. In conclusion, Φ is an isomorphism with inverse Ψ.

Exercise 5: Prove the following assertions:

(a) *Tangent vectors as velocity vectors of smooth curves*: Let M be a smooth manifold. If $p \in M$, then for any $v \in T_pM$ there exists a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(b) *The velocity of a composite curve*: If $F\colon M \to N$ is a smooth map and if $\gamma\colon J \to M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma\colon J \to N$ is given by

$$(F \circ \gamma)'(t_0) = dF\big(\gamma'(t_0)\big).$$

(c) *Computing the differential using a velocity vector*: If $F\colon M \to N$ is a smooth map, $p \in M$ and $v \in T_pM$, then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma\colon J \to M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.

Solution:

(a) Let (U, φ) be a smooth coordinate chart for M centered at p with components functions $(x^1, \dots, x^n)$, and write $v = v^i \frac{\partial}{\partial x^i}\big|_p$ in terms of the coordinate basis. For sufficiently small $\varepsilon > 0$, let $\gamma\colon (-\varepsilon, \varepsilon) \to U$ be the curve whose coordinate representation is

$$\gamma(t) = (tv^1, \dots, tv^n).$$

This is a smooth curve with $\gamma(0) = p$ and

$$\gamma'(0) = \frac{d\gamma^i}{dt}(0) \frac{\partial}{\partial x^i}\bigg|_p = v^i \frac{\partial}{\partial x^i}\bigg|_p = v.$$

(b) By definition and by *Exercise* 1(b) we obtain

$$
\begin{aligned}
(F \circ \gamma)'(t_0) &= d(F \circ \gamma)\left(\left.\frac{d}{dt}\right|_{t_0}\right) = (dF \circ d\gamma)\left(\left.\frac{d}{dt}\right|_{t_0}\right) \\
&= dF\left(d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right)\right) = dF\big(\gamma'(t_0)\big).
\end{aligned}
$$

(c) Follows immediately from (a) and (b).