



Differential Geometry II - Smooth Manifolds
Winter Term 2024/2025
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Exercise Sheet 4

Exercise 1 (to be submitted by Thursday, 10.10.2024, 16:00):

Let M , N and P be smooth manifolds, let $F: M \rightarrow N$ and $G: N \rightarrow P$ be smooth maps, and let $p \in M$. Prove the following assertions:

- (a) The map $dF_p: T_pM \rightarrow T_{F(p)}N$ is \mathbb{R} -linear.
- (b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p: T_pM \rightarrow T_{(G \circ F)(p)}P$.
- (c) $d(\text{Id}_M)_p = \text{Id}_{T_pM}: T_pM \rightarrow T_pM$.
- (d) If F is a diffeomorphism, then $dF_p: T_pM \rightarrow T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Exercise 2 (The tangent space to a vector space):

Let V be a finite-dimensional \mathbb{R} -vector space with its standard smooth manifold structure, see [Exercise Sheet 2, Exercise 2]. Fix a point $a \in V$.

- (a) For each $v \in V$ define a map

$$D_v|_a: C^\infty(V) \longrightarrow \mathbb{R}, f \mapsto \left. \frac{d}{dt} \right|_{t=0} f(a + tv).$$

Show that $D_v|_a$ is a derivation at a .

- (b) Show that the map

$$V \rightarrow T_aV, v \mapsto D_v|_a$$

is a canonical isomorphism, such that for any linear map $L: V \rightarrow W$ the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\cong} & T_aV \\ \downarrow L & & \downarrow dL_a \\ W & \xrightarrow{\cong} & T_{L_a}W. \end{array}$$

Exercise 3 (*The tangent space to a product manifold*):

Let M_1, \dots, M_k be smooth manifolds, where $k \geq 2$. For each $j \in \{1, \dots, k\}$, let

$$\pi_j: M_1 \times \dots \times M_k \rightarrow M_j$$

be the projection onto the j -th factor M_j . Show that for any point $p = (p_1, \dots, p_k) \in M_1 \times \dots \times M_k$, the map

$$\begin{aligned} \alpha: T_p(M_1 \times \dots \times M_k) &\longrightarrow T_{p_1}M_1 \oplus \dots \oplus T_{p_k}M_k \\ v &\longmapsto (d(\pi_1)_p(v), \dots, d(\pi_k)_p(v)) \end{aligned}$$

is an \mathbb{R} -linear isomorphism.

Exercise 4 (*Tangent vectors as derivations of the space of germs*):

Let M be a smooth manifold and let p be a point of M .

- (a) Consider the set \mathcal{S} of ordered pairs (U, f) , where U is an open subset of M containing p and $f: U \rightarrow \mathbb{R}$ is a smooth function. Define on \mathcal{S} the following relation:

$$(U, f) \sim (V, g) \quad \text{if } f \equiv g \text{ on some open neighborhood of } p.$$

Show that \sim is an equivalence relation on \mathcal{S} . The equivalence class of an ordered pair (U, f) is denoted by $[(U, f)]$ or simply by $[f]_p$ and is called *the germ of f at p* .

- (b) The set of all germs of smooth functions at p is denoted by $C_p^\infty(M)$. Show that $C_p^\infty(M)$ is an \mathbb{R} -vector space and an associative \mathbb{R} -algebra under the operations

$$\begin{aligned} c[(U, f)] &= [(U, cf)], \text{ where } c \in \mathbb{R}, \\ [(U, f)] + [(V, g)] &= [(U \cap V, f + g)], \\ [(U, f)][(V, g)] &= [(U \cap V, fg)]. \end{aligned}$$

- (c) A *derivation* of $C_p^\infty(M)$ is an \mathbb{R} -linear map $v: C_p^\infty(M) \rightarrow \mathbb{R}$ satisfying the following product rule:

$$v[fg]_p = f(p)v[g]_p + g(p)v[f]_p.$$

The set of derivations of $C_p^\infty(M)$ is denoted by \mathcal{D}_pM .

- (i) Show that \mathcal{D}_pM is an \mathbb{R} -vector space.
(ii) Show that the map

$$\Phi: \mathcal{D}_pM \rightarrow T_pM, \quad \Phi(v)(f) = v[f]_p$$

is an isomorphism.

Exercise 5:

Prove the following assertions:

- (a) *Tangent vectors as velocity vectors of smooth curves*: Let M be a smooth manifold. If $p \in M$, then for any $v \in T_pM$ there exists a smooth curve $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

- (b) *The velocity of a composite curve:* If $F: M \rightarrow N$ is a smooth map and if $\gamma: J \rightarrow M$ is a smooth curve, then for any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma: J \rightarrow N$ is given by

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0)).$$

- (c) *Computing the differential using a velocity vector:* If $F: M \rightarrow N$ is a smooth map, $p \in M$ and $v \in T_pM$, then

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma: J \rightarrow M$ such that $0 \in J$, $\gamma(0) = p$ and $\gamma'(0) = v$.