Introduction to Quantum Information Processing
COM 309 Week 2

Exercise 1

Properties of Pauli matrices

We collect useful properties of Pauli matrices. Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ a vector formed by the 3 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The identity matrix is denoted $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Show that all 2×2 matrices, A, can be written as a linear combination of I and σ_x, σ_y, σ_z:

   $$A = a_0 I + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z.$$

   This can also be written as $A = a_0 I + \vec{a} \cdot \vec{\sigma}$ where $\vec{a} \cdot \vec{\sigma}$ is an "inner product" between the "vectors" $\vec{a} = (a_1, a_2, a_3)$ et $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

   Check also that if $A = A^\dagger$ we have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

2. Check the following algebraic identities:

   $$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$$
   $$\sigma_x \sigma_y = i\sigma_z$$
   $$\sigma_y \sigma_z = i\sigma_x$$
   $$\sigma_z \sigma_x = i\sigma_y$$

   Deduce

   $$\sigma_x \sigma_y + \sigma_y \sigma_x = 0$$
   $$\sigma_y \sigma_z + \sigma_z \sigma_y = 0$$
   $$\sigma_z \sigma_x + \sigma_x \sigma_z = 0$$

3. Let $[A, B] = AB - BA$ be the "commutator". Show (you may use preceding results)

   $$[\sigma_x, \sigma_y] = 2i\sigma_z$$
   $$[\sigma_y, \sigma_z] = 2i\sigma_x$$
   $$[\sigma_z, \sigma_x] = 2i\sigma_y$$

These relations are called "commutation relations for spin".

4. Compute eigenvalues and eigenvectors of σ_x, σ_y, σ_z. Check that the eigenvalues satisfy $\mathrm{Tr}\,\sigma_x = \mathrm{Tr}\,\sigma_y = \mathrm{Tr}\,\sigma_z = 0$ et $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$.

5. Dirac notation: set
$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ et } |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$
Check that
$$\begin{aligned}\sigma_z &= |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\\ \sigma_x &= |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\\ \sigma_y &= i\,|\downarrow\rangle\langle\uparrow| - i\,|\uparrow\rangle\langle\downarrow|\end{aligned}$$

Exercise 2

Exponentials of Pauli matrices

1. We define the exponential of a matrix A by (for $t \in \mathbb{R}$)
$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + ...$$
We want to prove the identity:
$$e^{it\vec{n}\cdot\vec{\sigma}} = I\cos t + i\vec{n}\cdot\vec{\sigma}\sin t$$
where $\vec{n}$ is a unit vector and $t \in \mathbb{R}$. Remark that this is a generalization of Euler's identity:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
To show the identity show first that:
$$(\vec{n}\cdot\vec{\sigma})^2 = I$$
Use Taylor expansions of $\cos t$ and $\sin t$ to deduce the wanted identity above.

2. Explicitly write 2×2 matrices (in component/array form) $\exp(it\sigma_x)$; $\exp(it\sigma_y)$; $\exp(it\sigma_z)$ as well as $\exp(it\vec{n}\cdot\vec{\sigma})$.

Exercise 3

Inner products of tensor products

1. Which pairs of vectors are mutually orthogonal? Work in Dirac notation.

$$|0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$\left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \left(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle\right)$$

2. Given the canonical coordinate representation for the states $|0\rangle$ and $|1\rangle$ compute the component form (4 components of the above tensor products).

3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix}$$

Express the matrices in Dirac notation in the $|0\rangle, |1\rangle$ basis. Compute their tensor products $A \otimes B$ and $B \otimes A$ in Dirac notation.

4. Make the calculation of $A \otimes B$ and $B \otimes A$ in component form in the canonical basis. Check that the result is consistent with the one obtained above in Dirac notation.

Exercise 4

Product versus entangled states

Prove whether the following states are product or entangled states ? (check also they are correctly normalized)

1. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
2. $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$
3. $\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{6}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$
4. $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, .
5. $\frac{1}{\sqrt{1+\epsilon^2}}(|00\rangle + \epsilon|11\rangle)$, for $0 \leq \epsilon \leq 1$
6. $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$
7. $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
8. $\frac{1}{2\sqrt{2}}(|000\rangle + |100\rangle + |010\rangle + |001\rangle + |110\rangle + |101\rangle + |011\rangle + |111\rangle)$

Exercise 5

Unitary transformations

Verify that the following transformations are unitary (check also the identities between matrix tables and Dirac notation):

1. Simple time evolution of the type $|\psi_t\rangle = e^{i\omega t}|\psi_0\rangle$. This is for example the time evolution of a free photon of frequency $\nu = \omega/2\pi$ or energy $E = h\nu = \hbar\omega$.

2. The Hadamard gate.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|) \tag{1}$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interferometers models for example a semi-transparent mirror.

3. The X or NOT gate

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

   Check how the basis $|0\rangle$, $|1\rangle$ is transformed. Remark: in interfreometers it models for example a reflecting mirror.

4. $U_1 \otimes U_2$ if U_1 and U_2 are unitary. Remark: if U_i, $i = 1, 2$ act each on a one-qubit Hilbert space $\mathbb{C}^2$ then the tensor product acts on the two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

5. The control-NOT gate. This gate flips the control bit (the second) if the target bit (the first) is 1.

$$\begin{aligned} CNOT &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|) \\ &= (|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X) \end{aligned}$$

Compute the following states:

$$|\beta_{ij}\rangle = CNOT \otimes H|i\rangle \otimes |j\rangle, \quad i, j = 0, 1$$

and show that they are equal to the four Bell states introduced in class. Deduce that $\{|\beta_{ij}\rangle, i, j = 0, 1\}$ is an orthonormal basis. This identity shows that $CNOT$ entangles the two qubits.

Solution

Exercise 1

1. We have:

$$A = a_0 I + a_1\sigma_x + a_2\sigma_y + a_3\sigma_z = \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

   Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. One must have

$$\begin{cases} a_0 + a_3 = a_{11} \\ a_0 - a_3 = a_{22} \end{cases}$$

   which implies $a_0 = \dfrac{a_{11} + a_{22}}{2}$ et $a_3 = \dfrac{a_{11} - a_{22}}{2}$. On the other hand one must have:

$$\begin{cases} a_1 - ia_2 = a_{12} \\ a_1 + ia_2 = a_{21} \end{cases}$$

   which implies $a_1 = \dfrac{a_{12} + a_{21}}{2}$ et $a_2 = \dfrac{a_{21} - a_{12}}{2i}$.
   Thus 2×2 matrices A can be written as:

$$A = \frac{a_{11} + a_{22}}{2} I + \frac{a_{12} + a_{21}}{2}\sigma_x + \frac{a_{21} - a_{12}}{2i}\sigma_y + \frac{a_{11} - a_{22}}{2}\sigma_z.$$

   Note that if $A = A^\dagger$, since $\sigma_x = \sigma_x^\dagger$, $\sigma_y = \sigma_y^\dagger$, $\sigma_z = \sigma_z^\dagger$, one must also have a_0, a_1, a_2, $a_3 \in \mathbb{R}$.

2. These relations are checked by explicit calculation. Note that they are related by cyclic permutations of xyz.

3. Idem

4. Diagonalization of σ_x.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Longrightarrow \begin{cases} v_1 = \lambda v_2 \\ v_2 = \lambda v_1 \end{cases}$$

   $\Rightarrow v_1 = \lambda^2 v_1$ and $v_2 = \lambda^2 v_2$. To have v_1, $v_2 \neq 0$ it must be that $\lambda^2 = +1$ and thus $\lambda = \pm 1$. The eigenvalues are ± 1.

   The eigenvector corresponding to $\lambda = +1$ satisfies

$$v_1 = v_2 \qquad \text{et} \qquad v_2 = v_1.$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{is a normalized eigenvector}$$

The eigenvector associated to $\lambda = -1$ satisfies:

$$v_1 = -v_2 \qquad \text{et} \qquad v_2 = -v_1.$$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \quad \text{is a normalized eigenvector}$$

Diagonalization of σ_y.

We proceed as above:

$$\det\begin{pmatrix}-\lambda & -i\\ i & -\lambda\end{pmatrix} = \lambda^2 - (-i)(i) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

The eigenvector associated to $\lambda = +1$ satisfies

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = +1\begin{pmatrix}v_1\\v_2\end{pmatrix} \Rightarrow \begin{cases} -iv_2 = v_1 \\ iv_1 = v_2\end{cases}$$

One can choose $v_1 = 1$ et $v_2 = i$

$$\Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \quad \text{is a normalized eigenvector.}$$

For the eigenvector associated to -1 we have:

$$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = -1\begin{pmatrix}v_1\\v_2\end{pmatrix} \Rightarrow \begin{cases} -iv_2 = -v_1 \\ iv_1 = -v_2\end{cases}$$

We choose $v_1 = 1$ et $v_2 = -i$

$$\text{Thus} \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \quad \text{is a normalized eigenvector.}$$

Diagonalization of σ_z.

$$\sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{in the basis} \quad \begin{pmatrix}1\\0\end{pmatrix} \quad \text{et} \quad \begin{pmatrix}0\\1\end{pmatrix}.$$

Thus 1 is the eigenvalue associated to $\begin{pmatrix}1\\0\end{pmatrix}$ and -1 the eigenvalue associated to $\begin{pmatrix}0\\1\end{pmatrix}$.

Trace. The trace of a matrix is the sum of diagonal elements and is invariant under change of basis. It is also the sum of eigenvalues. One can check that every thing is consistent $\operatorname{Tr}\sigma_x = \operatorname{Tr}\sigma_y = \operatorname{Tr}\sigma_z = 0$.

Determinant. The determinant equals $a_{11}a_{22} - a_{12}a_{21}$ and is also invariant under change of basis. It is also the product of eigenvalues. One can again check that everything is consistent $\det\sigma_x = \det\sigma_y = \det\sigma_z = -1$.

5. In Dirac notation a 2×2 matrix becomes:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}\left|\uparrow\right\rangle\left\langle\uparrow\right| + a_{12}\left|\uparrow\right\rangle\left\langle\downarrow\right| + a_{21}\left|\downarrow\right\rangle\left\langle\uparrow\right| + a_{22}\left|\downarrow\right\rangle\left\langle\downarrow\right|.$$

We also remark the important relations:

$$\left\langle\uparrow\right| A \left|\uparrow\right\rangle = a_{11};\ \left\langle\uparrow\right| A \left|\downarrow\right\rangle = a_{12};\ \left\langle\downarrow\right| A \left|\uparrow\right\rangle = a_{21}\ \text{ et }\ \left\langle\downarrow\right| A \left|\downarrow\right\rangle = a_{22}.$$

We note

$$\left|\uparrow\right\rangle\left\langle\uparrow\right| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};\ \left|\uparrow\right\rangle\left\langle\downarrow\right| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};\ \left|\downarrow\right\rangle\left\langle\uparrow\right| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ \text{ et }\ \left|\downarrow\right\rangle\left\langle\downarrow\right| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

From which the expressions of Pauli matrices in Dirac notation follow.

Exercise 2

1.

$$\vec{n}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z \qquad \text{by definition}$$

$$\begin{aligned}
(\vec{n}\cdot\vec{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) \\
&= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 \\
&+ n_xn_y\sigma_x\sigma_y + n_yn_x\sigma_y\sigma_x \\
&+ n_xn_z\sigma_x\sigma_z + n_zn_x\sigma_z\sigma_x \\
&+ n_yn_z\sigma_y\sigma_z + n_zn_y\sigma_z\sigma_y \\
&= \left(n_x^2 + n_y^2 + n_z^2\right) I = I
\end{aligned}$$

In the second equality we were careful to note that Pauli matrices do not commute.. In the third one we used the relation in point 2. In the last one we used that $\vec{n}$ is a unit norm vector.

This identity implies $(\vec{n}\cdot\vec{\sigma})^3 = \vec{n}\cdot\vec{\sigma}; (\vec{n}\cdot\vec{\sigma})^4 = I;$ etc...

Thus

$$\begin{aligned}\exp(it\vec{n}\cdot\vec{\sigma}) &= \sum_{k=0}^{+\infty}\frac{(it)^k}{k!}\,(\vec{n}\cdot\vec{\sigma})^k \\ &= \sum_{k \text{ even}}\frac{(it)^k}{k!}I + \left\{\sum_{k \text{ odd}}\frac{(it)^k}{k!}\right\}(\vec{n}\cdot\vec{\sigma})\end{aligned}$$

Moreover

$$\cos t = \sum_{k \text{ even}}\frac{(it)^k}{k!} \qquad \text{et} \qquad i\sin t = \sum_{k \text{ odd}}\frac{(it)^k}{k!} \tag{*}$$

(*Parenthesis:* note that

$$e^{it} = \cos t + i\sin t \qquad \text{thus:}$$

$$\sum_{k \text{ pairs}}\frac{(it)^k}{k!} + \sum_{k \text{ impairs}}\frac{(it)^k}{k!} = \cos t + i\sin t,$$

changing $t \to -t$ we also have

$$\sum_{k \text{ pairs}}\frac{(it)^k}{k!} - \sum_{k \text{ impairs}}\frac{(it)^k}{k!} = \cos t - i\sin t,$$

and adding or subtracting we find $(*)$.)
Finally we proved:

$$\exp(it\vec{n}\cdot\vec{\sigma}) = (\cos t)I + (i\sin t)\vec{n}\cdot\vec{\sigma}$$

Exercise 3

1. Which pairs of vectors are mutually orthogonal? Work in Dirac notation.
   - $|a\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   - $|b\rangle = |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$
   - $|c\rangle = \left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$
   - $|d\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \left(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle\right)$

$|a\rangle$ and $|b\rangle$ are orthogonal :

$$\begin{aligned}
\langle a|b\rangle &= \left(\langle 0| \otimes \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right)\left(|1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) \\
&= \langle 0|1\rangle \cdot \left(\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\right)\left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) \\
&= \langle 0|1\rangle \cdot \left(\frac{1}{2}(\langle 0|0\rangle + \langle 1|0\rangle + i\langle 0|1\rangle + i\langle 1|1\rangle)\right) \\
&= 0 \cdot \left(\frac{1}{2}(1 + 0 + i0 + i1)\right) = 0
\end{aligned}$$

$|c\rangle$ and $|d\rangle$ are orthogonal :

$$\begin{aligned}
\langle c|d\rangle &= \left((\frac{1}{\sqrt{3}}\langle 0| + \sqrt{\frac{2}{3}}\langle 1|) \otimes (\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|) \right)\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes (\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle) \right) \\
&= \left(\frac{1}{\sqrt{3}}\langle 0| + \sqrt{\frac{2}{3}}\langle 1|\right)\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \cdot \left(\frac{1}{\sqrt{5}}\langle 0| + \frac{2}{\sqrt{5}}\langle 1|\right)\left(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{6}}\langle 0|0\rangle + \frac{1}{\sqrt{6}}\langle 0|1\rangle + \frac{1}{\sqrt{3}}\langle 1|0\rangle + \frac{1}{\sqrt{3}}\langle 1|1\rangle\right) \\
&\quad \cdot \left(\frac{2}{5}\langle 0|0\rangle - \frac{1}{5}\langle 0|1\rangle + \frac{4}{5}\langle 1|0\rangle - \frac{2}{5}\langle 1|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}}\right) \cdot \left(\frac{2}{5} - \frac{2}{5}\right) = 0
\end{aligned}$$

2. The canonical representation of $|0\rangle$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and for $|1\rangle$ it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- $|a\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)^T$
- $|b\rangle = (0, 0, \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}})^T$
- $|c\rangle = (\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{\sqrt{2}}{\sqrt{15}}, \frac{2\sqrt{2}}{\sqrt{15}})^T$
- $|d\rangle = (\frac{2}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{-1}{\sqrt{10}})^T$

3.

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = |0\rangle\langle 0| + 2|0\rangle\langle 1| + 3|1\rangle\langle 0| + 4|1\rangle\langle 1| \\
B &= \begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix} = 5|0\rangle\langle 1| + 8|1\rangle\langle 0| + 10|1\rangle\langle 1| \\
A \otimes B &= 5|00\rangle\langle 01| + 8|01\rangle\langle 00| + 10|01\rangle\langle 01| \\
&\quad + 10|00\rangle\langle 11| + 16|01\rangle\langle 10| + 20|01\rangle\langle 11| \\
&\quad + 15|10\rangle\langle 01| + 24|11\rangle\langle 00| + 30|11\rangle\langle 01| \\
&\quad + 20|10\rangle\langle 11| + 32|11\rangle\langle 10| + 40|11\rangle\langle 11| \\
B \otimes A &= 5|00\rangle\langle 10| + 10|00\rangle\langle 11| + 15|01\rangle\langle 10| + 20|01\rangle\langle 11| \\
&\quad + 8|10\rangle\langle 00| + 16|10\rangle\langle 01| + 24|11\rangle\langle 00| + 32|11\rangle\langle 01| \\
&\quad + 10|10\rangle\langle 10| + 20|10\rangle\langle 11| + 30|11\rangle\langle 10| + 40|11\rangle\langle 11|
\end{aligned}
$$

4.

$$
A \otimes B = \begin{pmatrix} 1\begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix} & 2\begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix} \\ 3\begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix} & 4\begin{pmatrix} 0 & 5 \\ 8 & 10 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 5 & 0 & 10 \\ 8 & 10 & 16 & 20 \\ 0 & 15 & 0 & 20 \\ 24 & 30 & 32 & 40 \end{pmatrix}
$$

$$
B \otimes A = \begin{pmatrix} 0\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ 8\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & 10\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 15 & 20 \\ 8 & 16 & 10 & 20 \\ 24 & 32 & 30 & 40 \end{pmatrix}
$$

Exercise 4

We use the conventional correspondance $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then we have

$$
(\alpha|0\rangle + \beta|1\rangle) \otimes (x|0\rangle + y|1\rangle) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} \alpha x & \alpha y \\ \beta x & \beta y \end{pmatrix}
$$

Thus a product state of two qubits is a rank-one matrix. So for a general state $|\psi\rangle = \sum_{ij} \alpha_{ij}|ij\rangle$, a simple condition is to check if the matrix $A = (\alpha_{ij})_{0\leq i,j\leq 1}$ is of rank one, that is to say (and because $A \neq 0$), if its determinant is 0: $\det(A) = 0$, that is $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$

1. product state ($\det(A) = 0$), normalized
2. entangled ($\det(A) = -2$), normalized
3. entangled $\det(A) = -\frac{1}{\sqrt{3\cdot 6}} - \frac{1}{\sqrt{36}} \neq 0$, not normalized

4. all entangled ($\det(A) \neq 0$ for all of them), normalized

5. we find $\det(A) = \epsilon$, so only a product state for $\epsilon = 0$, entangled otherwise, normalized in all cases

6. Let's assume $|\psi\rangle = (x|0\rangle + y|1\rangle) \otimes (u|0\rangle + v|1\rangle) \otimes (s|0\rangle + t|1\rangle)$, then we should have $xut = xvs = yuv = 1$ and all the other products are 0 otherwise. For instance: $yut = 0$. However, because $xut = 1$, then $ut \neq 0$, and because $yuv = 1$ then $y \neq 0$, therefore $yut \neq 0$. So our assumption is wrong and the state is entangled, and also normalized

7. entangled for similar reasons, normalized

8. product state as $|\psi\rangle = \frac{1}{\sqrt{2}^3}\left(|0\rangle + |1\rangle\right)^{\otimes 3}$, and normalized

Exercise 5

1. The operator is $U = e^{i\omega t}$, thus $U^\dagger = e^{-i\omega t}$ and it is straightforward to check that $U^\dagger U = UU^\dagger = 1$. In this (trivial) case the unitary time evolution matrix is a $1x1$ matrix.

2. We find $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, so in fact $H|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^i|1\rangle)$ for $i \in \{0, 1\}$. Therefore, we have:

$$\langle j|H^\dagger H|i\rangle = \frac{1}{2}\left((\langle 0| + (-1)^j\langle 1|)(|0\rangle + (-1)^i|1\rangle)\right) = \frac{1}{2}\left(1 + (-1)^{i+j}\right) = \delta_{ij}$$

   with δ_{ij} the kroenecker symbol.

3. Similarly, we find $X|i\rangle = |i \oplus 1\rangle$ thus $\langle j|X^\dagger X|i\rangle = \langle j \oplus 1|i \oplus 1\rangle = \delta_{ij}$

4. Step by step:

$$\begin{aligned}
(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) &= (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) && (2)\\
&= (U_1^\dagger U_1) \otimes (U_2^\dagger U_2) && (3)\\
&= I \otimes I && (4)\\
&= I && (5)
\end{aligned}$$

5. It is straightforward to check that: $\text{CNOT}(|i, j\rangle) = |i, j \oplus i\rangle$. Thus:

$$\langle k, l|\text{CNOT}^\dagger\text{CNOT}|i, j\rangle = \langle k, l \oplus k|i, i \oplus j\rangle = \delta_{i,k}\delta_{(l\oplus k),(i\oplus j)} = \delta_{i,k}\delta_{l,j}$$

First let $|\psi_1\rangle = (H|i\rangle) \otimes |j\rangle$, we have using question 2:

$$|\psi_1\rangle = (H|i\rangle) \otimes |j\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^i|1\rangle) \otimes |j\rangle = \frac{1}{\sqrt{2}}(|0, j\rangle + (-1)^i|1, j\rangle)$$

Therefore using question 5:

$$\text{CNOT}|\psi_1\rangle = \frac{1}{\sqrt{2}}\big(|0, j\rangle + (-1)^i|1, j \oplus 1\rangle\big) = |\beta_{ij}\rangle$$

Because H and I are both unitary using question 2, then $U \otimes I$ is unitary using question 4. Then because CNOT is unitary (question 5), using the fact that the set of unitary matrices equipped with the product of matrices is a group, then $O = \text{CNOT} \cdot (H \otimes I)$ is also unitary, hence β_{ij} forms an orthonormal basis as it is the image of an orthonormal basis with the unitary operator O.