Principles of Quantum Physics.

In this lecture we introduce the principles in axiomatic form. We formulate them in algebraic manner appropriate for quantum information and computation in finite dimensional Hilbert spaces.

Principle 1. State vectors.

The state of a physical system (isolated from the rest of the universe) is completely specified by a vector in a Hilbert space. The vector $|\psi\rangle \in \mathcal{H}$ must be normalised $\langle\psi|\psi\rangle = 1$.

Remark: The principle doesn't say how to choose $\mathcal{H}$ and $|\psi\rangle$. This depends on the underlying physics of the system.

Remark: For a non-isolated system we will have to generalize this principle. This will be done in third part of the course introducing the notion of Density Matrix.

Examples

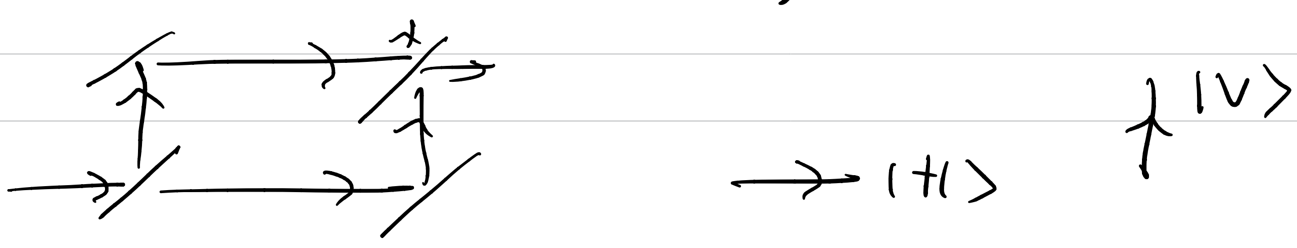
- $\mathcal{H} = \mathbb{C}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \ \alpha, \beta \in \mathbb{C} \right\}$.

$$= \left\{ \alpha \underbrace{|0\rangle}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \beta \underbrace{|1\rangle}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}, \ \alpha \in \mathbb{C}, \beta \in \mathbb{C} \right\}$$

State vectors satisfy $\alpha\alpha^* + \beta\beta^* = 1$

This is the Hilbert space of a qubit.

- In the Mach-Zehnder interferometer experiment we have two basis states $|H\rangle$, $|V\rangle$



- For a "qudit" we take $\mathcal{H} = \mathbb{C}^d$.

States have the form $|\psi\rangle = \sum_{i=1}^{d} \alpha_i |i\rangle$

with $|i\rangle \in \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$.

and $\alpha_i \in \mathbb{C}$. We have the normalisation condition $\sum_{i=1}^{d} |\alpha_i|^2 = 1$.

<u>Principle 2.</u> <u>Unitary evolution or time evolution</u>

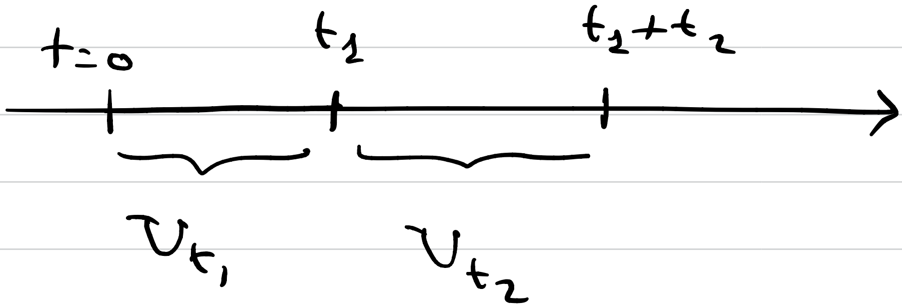
The state of an isolated system evolves (as a function of time) unitarily. This means that if $|\psi_0\rangle$ is a state at time $t=0$, the state $|\psi_t\rangle$ is given by

$$U_t |\psi_0\rangle = |\psi_t\rangle$$

where $U_t^\dagger U_t = U_t U_t^\dagger = \mathbb{1}$ i.e U_t is a unitary (usually time dependent) matrix.

Remark: time evolution forms a group in the sense that

$$U_{t_2} U_{t_1} = U_{t_1+t_2} \,, \quad U_{t=0} = \mathbb{1} .$$



evolution for a duration t_1 — evolution for a duration t_2.

total time evolution for a duration $U_{t_1+t_2}$.

Example :

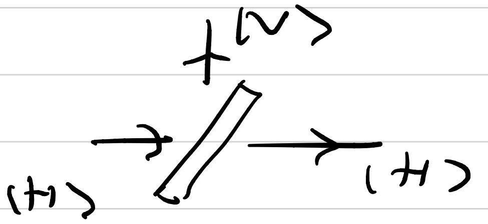
• Perfectly reflecting mirror.

The physical process which transforms the incident ray into the reflected ray is described by the unitary matrix :

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |H\rangle\langle V| + |V\rangle\langle H|$$

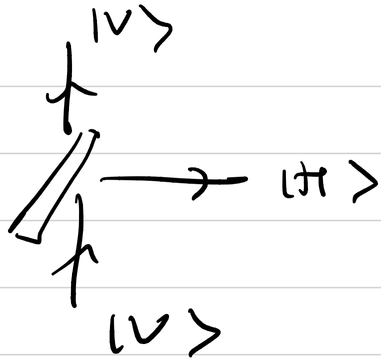
for $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

• Semi-transparent mirror.



initial state $|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

final state $\dfrac{|H\rangle + |V\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



initial state $|V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

final state $\dfrac{|H\rangle - |V\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The unitary matrix describing the process is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$ called the Hadamard matrix.

$$H = \frac{1}{\sqrt{2}} \Big\{ |V\rangle\langle V| + |V\rangle\langle H| + |H\rangle\langle V| - |V\rangle\langle V| \Big\}.$$

$$\begin{cases} H\,|H\rangle = \dfrac{1}{\sqrt{2}}\left(|H\rangle + |V\rangle\right) \text{ or } H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ H\,|V\rangle = \dfrac{1}{\sqrt{2}}\left(|H\rangle - |V\rangle\right) \text{ or } H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \dfrac{2}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases}$$

Principle 3. Observable quantities.

An observable (or measurable) quantity (such as energy, position, momentum, magnetic moment, polarization) is given by an Hermitian matrix A (satisfying $A^{+} = A$).

Example.

In the Hilbert space $\mathbb{C}^2$ (for a qubit) the observables are 2×2 matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \gamma \end{pmatrix}$$

$$= \alpha |0\rangle\langle 0| + \beta |0\rangle\langle 1| + \bar{\beta} |1\rangle\langle 0| + \gamma |1\rangle\langle 1|$$

These satisfy $A = A^{+}$ i.e are Hermitian

Such 2×2 matrices can be represented as linear combinations of the so-called Pauli matrices

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and $\sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(see exercises for their properties).

Any $A = A^{+}$ can be written like

$$A = a_0 \mathbb{1} + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$$

with $a_0, a_x, a_y, a_z \in \mathbb{R}$.

Spectral theorem for Hermitian matrices:

Let A a $d \times d$ matrix satisfying $A = A^{+}$ (Hermitian). Let $|v_1\rangle \dots |v_d\rangle$ and $\lambda_1, \lambda_2, \dots, \lambda_d$ the eigenvectors and eigenvalues i.e

$$A|v_i\rangle = \lambda_i |v_i\rangle , \quad i = 1 \dots d$$

The eigenvectors can always be chosen to form an orthonormal basis of $\mathbb{C}^d$ and the eigenvalues $\lambda_1, \lambda_2 \dots \lambda_d$ are real (in $\mathbb{R}$).

Moreover we have the "spectral decomposition"

$$A = \sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i|$$

Remark: This last formula just says that in the basis $|v_1\rangle, |v_2\rangle \dots |v_d\rangle$ of eigenvectors A is diagonal $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$.

Example

In the exercises you will show that the eigenvalues and eigenvectors of the Pauli matrices are

- for $\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \pm 1$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$
- for $\sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \rightarrow \pm 1$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$
- for $\sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \pm 1$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Principle 4 : Measurement postulate.

Let $|\psi\rangle$ be the state of a system. Let A be an observable that we measure.

The result of a measurement is a random outcome where the state becomes

$$|v_i\rangle \quad \text{for some} \quad i = 1 \dots d$$

(eigenvector of A)

the "value of A" is

$$\lambda_i \qquad (\text{associated to } |v_i\rangle)$$

and the probability law is

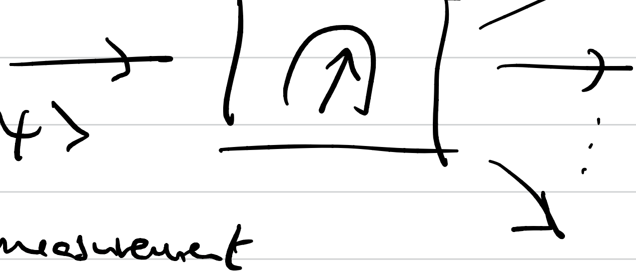
$$\text{prob}(i) = |\langle v_i | \psi \rangle|^2$$

given by the squared bracket between $|\psi\rangle$ & $|v_i\rangle$.

Second form of measurement postulate often use in information theory:

A measurement process is performed thanks to a "measurement apparatus". The measurement apparatus is modelled by an orthonormal basis of the Hilbert space. $\{|v_1\rangle, |v_2\rangle \dots |v_d\rangle\}$ and the measurement outcome is a random state $|v_i\rangle$ with $prob(i) = |\langle v_i|\psi\rangle|^2$.

Pictorially:



$|\psi\rangle$

before the measurement

$|v_i\rangle$ with

$prob(i) = |\langle v_i|\psi\rangle|^2$

after measurement.

Remark: The second form of the measurement postulate is used when we do not need to specify the observable being measured. Only the "measurement apparatus" is specified i.e an orthonormal basis.

Property. The probabilities in the measurement postulate sum to one. Here is the proof:

$$\sum_{i=1}^{d} \text{prob}(i) = \sum_{i=1}^{d} |\langle v_i | \psi \rangle|^2$$

$$= \sum_{i=1}^{d} \overline{\langle v_i | \psi \rangle} \langle v_i | \psi \rangle$$

$$= \sum_{i=1}^{d} \langle \psi | v_i \rangle \langle v_i | \psi \rangle$$

$$= \langle \psi | \Big(\sum_{i=1}^{d} |v_i\rangle\langle v_i| \Big) | \psi \rangle$$

$$= \langle \psi | \psi \rangle = 1.$$

Normalisation of quantum state in principle 1.

We used $\sum_{i=1}^{d} |v_i\rangle\langle v_i| = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$

$= I$

identity matrix.

Important property : average value of an observable

Let A be an observable being measured many times when the system is prepared always in state $|\psi\rangle$. Then the average value of A is

$$Av(A) = \underbrace{\langle \psi |}_{\text{bra}} \underbrace{A \overbrace{|\psi\rangle}^{\text{ket}}}_{\text{vector}} \in \mathbb{R}.$$

Proof :

$$\mathrm{Av}(A) = \sum_{i=1}^{d} \lambda_i \, \mathrm{prob}(i)$$

$$= \sum_{i=1}^{d} \lambda_i \, |\langle v_i | \psi \rangle|^2$$

$$= \sum_{i=1}^{d} \lambda_i \, \langle \psi | v_i \rangle \langle v_i | \psi \rangle$$

$$= \langle \psi | \underbrace{\left(\sum_{i=1}^{d} \lambda_i |v_i\rangle\langle v_i| \right)}_{A} | \psi \rangle$$

$$= \langle \psi | A | \psi \rangle .$$

Note this is a real value $\in \mathbb{R}$ since $\lambda_i \in \mathbb{R}$ for a Hermitian matrix.

Heisenberg uncertainty principle.

Define the mean square error (or square root of variance) of measurements for two observables A & B:

$$\Delta A \equiv \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}$$

$$\Delta B \equiv \sqrt{\langle\psi|B^2|\psi\rangle - \langle\psi|B|\psi\rangle^2}$$

We have

$$(\Delta A)(\Delta B) \geqslant \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|$$

where $[A,B] \equiv AB - BA$ (the commutator)

Proof: see exercises.

This property says that fluctuations of A & B cannot simultaneously vanish if they do not commute $[A,B] \neq 0$.

Principle 5 : Composition of systems.

Let $\mathcal{H}_A$ be the Hilbert space of system A

Let $\mathcal{H}_B$ be the Hilbert space of system B.

Then $\mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space of the composed system $A \cup B$. The state of the composed system is some vector in $\mathcal{H}_A \otimes \mathcal{H}_B$.

Example

$\mathcal{H}_A = \mathbb{C}^2$ first qubit (first photon polarization say)

$\mathcal{H}_B = \mathbb{C}^2$ second qubit (second photon polarization say)

$\mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^2 \otimes \mathbb{C}^2$

states are thus $4 = 2 \times 2$ dimensional vectors.

Important definitions.

Product states: Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$.

It is said to be a product state if one can find $|\varphi_A\rangle \in \mathcal{H}_A$, $|\varphi_B\rangle \in \mathcal{H}_B$ such that $|\psi\rangle = |\varphi_A\rangle \otimes |\varphi_B\rangle$.

Entangled states: a state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is said to be entangled (intriqué) if it not a product state.

As we will see this classification of states plays a very important role in quantum information processing.

Example.

- $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$

are product states.

- $\frac{1}{2}\left(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

is a product state.

- $\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)$ is entangled.

(the Bell state)

Proof: assume

$$\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right) = \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \left(\gamma|0\rangle + \delta|1\rangle\right)$$

$$\Rightarrow \quad \alpha\gamma = \frac{1}{\sqrt{2}} \ ; \ \alpha\delta = 0 \ ; \ \beta\gamma = 0 \ ; \ \beta\delta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{cases} \alpha = 0 \Rightarrow \text{contradiction with } \alpha\gamma = \frac{1}{\sqrt{2}} \\ \delta = 0 \Rightarrow \text{contradiction with } \beta\delta = \frac{1}{\sqrt{2}} \end{cases}$$

∎

Geometrical representation of qubits and the Bloch sphere

Quantum state vectors belong to Hilbert space which consist of vectors with complex components. These are NOT vectors in usual Euclidean space and thus difficult to represent geometrically.

For the particular case of one qubit there is a convenient geometrical representation which we will often use an gives good intuition (for two qubits the situation is already more complicated, and for n-qubit this constitutes an open problem) [" large "]

For one qubit $\mathcal{H} = \mathbb{C}^2$

$$= \{ \alpha|0\rangle + \beta|1\rangle \;;\; \alpha, \beta \in \mathbb{C} \}$$

but we must also have $|\alpha|^2 + |\beta|^2 = 1$.

$\Rightarrow$ This makes 4 real parameters $-$ 1 real par

$= 3$ real parameters.

In fact one should still remove one more parameter. The reason being that:

$$\boxed{|\psi\rangle \quad \text{and} \quad e^{i\gamma}|\psi\rangle \;,\; \gamma \in \mathbb{R} \quad \text{are equivalent states physically.}}$$

Indeed γ can never be observed! To get an intuition about this recall that the probabilities in the Measurement Postulate satisfy:

$$|\langle v_i | \psi \rangle|^2 = |\langle v_i | e^{i\gamma} |\psi\rangle|^2$$

since $|e^{i\gamma}|^2 = \cos^2\gamma + \sin^2\gamma = \underline{1}$.

In reality we should state $\boxed{\text{principle 1}}$ in a more fundamental form stating that:

$\boxed{\text{state vectors of } \underbrace{\text{a system}}^{\text{isolated}} \text{ are "rays" in the Hilbert space i.e } e^{i\gamma}|\psi\rangle,\ \gamma \in \mathbb{R}}$

<u>Remark</u>: in $e^{i\gamma}|\psi\rangle$ γ is called a <u>global phase</u>. <u>It cannot be observed</u>

But if you have $|0\rangle + e^{i\delta}|1\rangle$ for example this is a <u>local phase δ</u>. And a local phase <u>can be observed</u> (e.g. in interference experiments)

$\Rightarrow$ For $\alpha|0\rangle + \beta|1\rangle$ by multiplying by a global phase we can make α real.

$$\Rightarrow \boxed{\begin{aligned} &4 \text{ param} - 1 \text{ param} - 1 \text{ param} \\ &= 2 \text{ parameters eventually.} \end{aligned}}$$

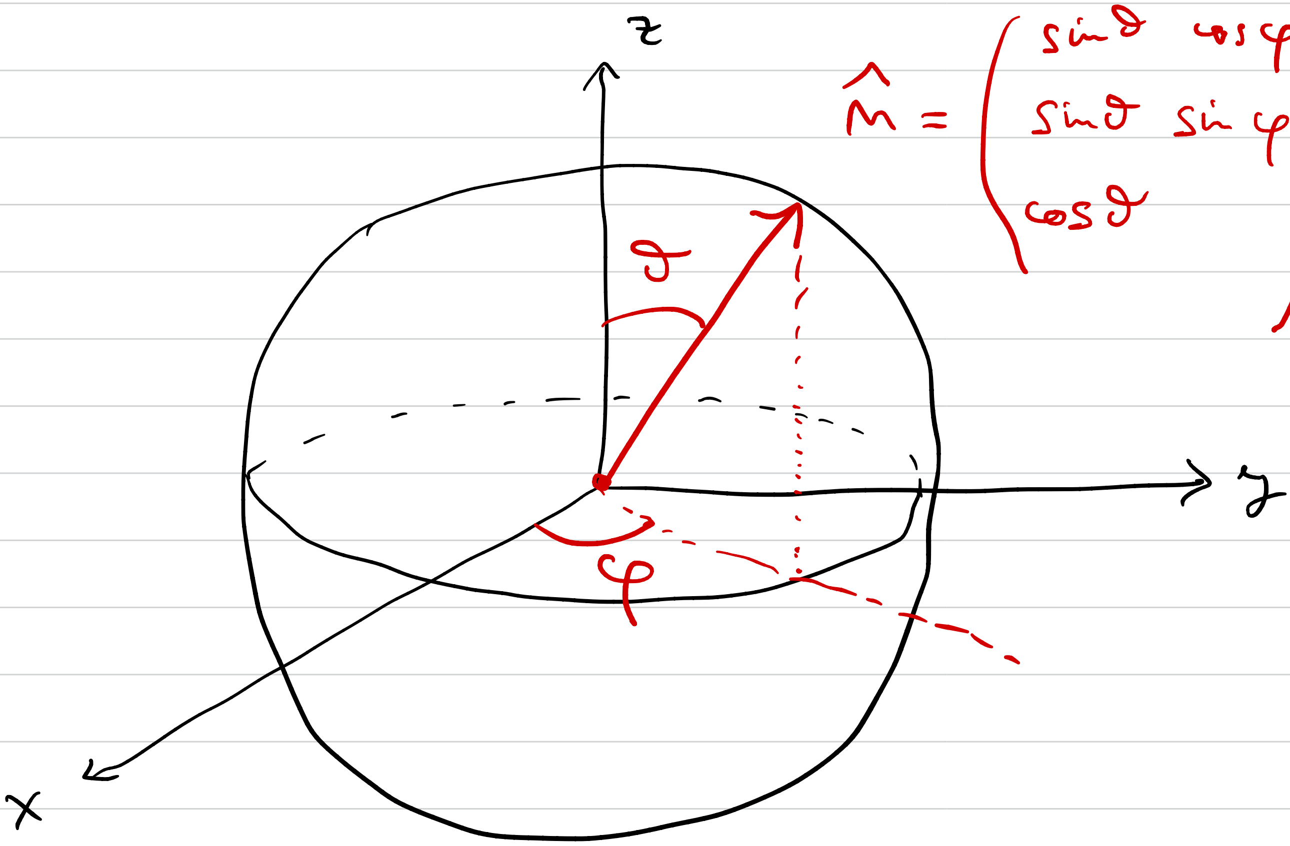
The canonical and useful parametrisation is:

$$\alpha = \cos\frac{\theta}{2} \quad ; \quad \beta = \left(\sin\frac{\theta}{2}\right) e^{i\varphi}$$

$$0 \leq \theta \leq \pi \quad , \quad 0 \leq \varphi \leq 2\pi$$

This is the parametrisation of a sphere.

The Bloch sphere:



$$\hat{n} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

We have representation of

$$|\psi\rangle = \cos\frac{\vartheta}{2}|0\rangle + \left(\sin\frac{\vartheta}{2}\right)e^{i\varphi}|1\rangle$$

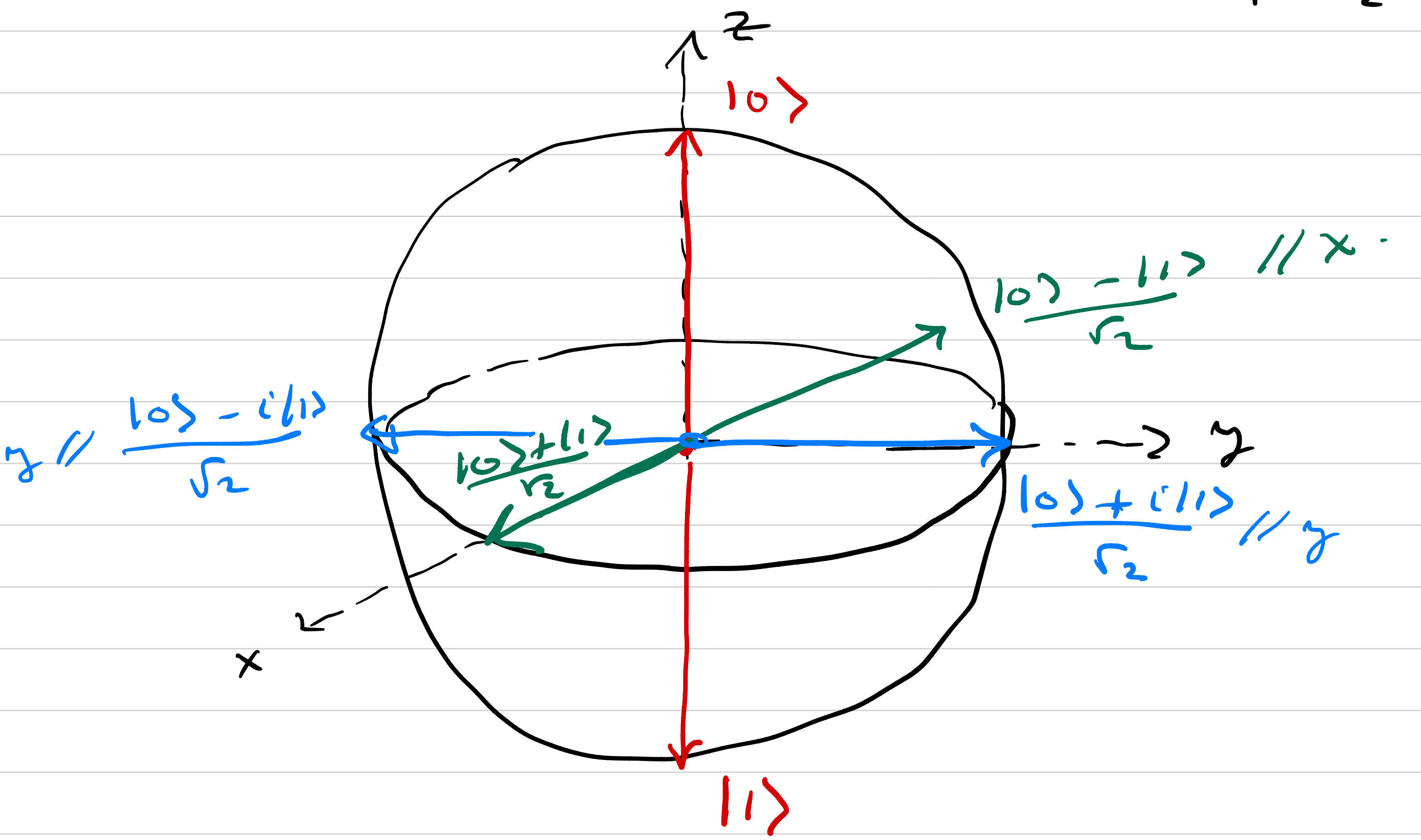
is given by $\hat{n} = \begin{pmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{pmatrix}$ on sphere

Special cases:

$|0\rangle \to \vartheta = 0$ $\qquad$ $|1\rangle \to \vartheta = \pi$

$\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \to \vartheta = \dfrac{\pi}{2} \quad \varphi = 0$ $\qquad$ $\dfrac{|0\rangle - |1\rangle}{\sqrt{2}} \to \vartheta = \dfrac{\pi}{2} \quad \varphi = \pi$

$\dfrac{|0\rangle + i|1\rangle}{\sqrt{2}} \to \vartheta = \dfrac{\pi}{2} \quad \varphi = \dfrac{\pi}{2}$ $\qquad$ $\dfrac{|0\rangle - i|1\rangle}{\sqrt{2}} \to \vartheta = \dfrac{\pi}{2}$, $\varphi = \dfrac{3\pi}{2}$.



Remark: Other usual notation for spin interpretation later in class

$|0\rangle = |\uparrow\rangle$ $\qquad$ $|1\rangle = |\downarrow\rangle$

"up state" $\qquad$ "down state"

Property : be careful here!

Vectors that are orthogonal in $\mathbb{C}^2$ appear opposite on the Bloch sphere.

For example $\langle 1 | 0 \rangle = 0$ (see picture)

$$\left(\frac{\langle 0 | + \langle 1 |}{\sqrt{2}} \right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \quad \text{(see picture)}$$

ect.

Property : recall that a unitary $\overbrace{\text{evolution}}^{\text{time}}$ preserves the norm of vectors. This implies that on the Bloch sphere unitary evolution will appear as rotations of vectors on this sphere (more on this in later chapters).